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Decomposition of cartesian product of complete graphs into sunlet graphs of order eight*

Research Article

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Abstract: For any integer $k \geq 3$, we define the sunlet graph of order $2k$, denoted by L_{2k}, as the graph consisting of a cycle of length k together with k pendant vertices such that, each pendant vertex adjacent to exactly one vertex of the cycle so that the degree of each vertex in the cycle is 3. In this paper, we establish necessary and sufficient conditions for the existence of decomposition of the Cartesian product of complete graphs into sunlet graphs of order eight.

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Keywords: Graph decomposition, Cartesian product, Corona graph, Sunlet graph

1. Introduction

All graphs considered here are finite, simple and undirected. A *cycle* of length k is called k-cycle and it is denoted by C_k. K_m denotes the *complete* graph on m vertices and $K_{m,n}$ denotes the *complete bipartite* graph with m and n vertices in the parts. We denote the *complete* m*-partite* graph with $n_1, n_2, \dots, n_m$ vertices in the parts by $K_{n_1,n_2,\dots,n_m}$. For any integer $\lambda > 0$, λG denotes the graph consisting of λ edge-disjoint copies of G.

Let G and H be two graphs of orders m and n, respectively. The *corona product* $G \odot H$ is the graph obtained by taking one copy of G and m copies of H such that the ith vertex of G is connected to every vertex in the ith copy of H. We define the *sunlet graph* L_{2k} with $V(L_{2k}) = \{x_1, x_2, \dots, x_k, x_{k+1}, x_{k+2}, \dots, x_{2k}\}$ and $E(L_{2k}) = \{x_i x_{i+1} \cup x_i x_{k+i} \mid i = 1, 2, ..., k$ and subscripts of the first term is taken addition modulo $k\}$. We denote it by $L_{2k} = \begin{pmatrix} x_1 & x_2 & \dots & x_k \\ x_{k+1} & x_{k+2} & \dots & x_{2k} \end{pmatrix}$. Clearly, $L_{2k} \cong C_k \odot K_1$.

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The *Cartesian product* of two graphs, G and H, denoted by $G\Box H$, has the vertex set $V(G)\times V(H)$ and two vertices (g,h) and (g',h') are adjacent if and only if either $g=g'$ and h is adjacent to h' in H or $h=h'$ and g is adjacent to g' in G. It is well known that Cartesian product is commutative, associative and distributive over edge-disjoint union of graphs.

We shall use the following notation throughout the paper. Let G and H be simple graphs with vertex sets $V(G)=\{x_1,x_2,\ldots,x_n\}$ and $V(H)=\{y_1,y_2,\ldots,y_m\}$. Then for our convenience, we write $V(G)\times V(H)=\bigcup_{i=1}^{n} X_i$, where X_i stands for $x_i\times V(H)$. Further, in the sequel, we shall denote the vertices of X_i as $\left\{x_i^j | 1\leq j\leq m\right\}$, where x_i^j stands for the vertex $(x_i,y_j)\in V(G)\times V(H)$.

By a *decomposition* of a graph G, we mean a list of edge-disjoint subgraphs whose union is G. For a graph G, if $E(G)$ can be partitioned into $E_1,E_2,...,E_k$ such that the subgraph induced by E_i is H_i, for all i, $1\leq i\leq k$, then we say that $H_1,H_2,...,H_k$ decompose G and we write $G=H_1\oplus H_2\oplus\ldots\oplus H_k$, since $H_1,H_2,...,H_k$ are edge-disjoint subgraphs of G. For $1\leq i\leq k$, if $H_i=H$, we say that G has a H-*decomposition*.

Study of H-decomposition of graphs is not new. Many authors around the world are working in the field of cycle decomposition [4, 7, 8, 18, 19], path decomposition [22, 23], star decompositon [17, 21, 24, 25] and Hamilton cycle decomposition [2, 3, 13, 14] problems in graphs. Here we consider the sunlet decomposition of product graphs. Anitha and Lekshmi [5, 6] proved that n-sun decomposition of complete graph, complete bipartite graph and the Harary graphs. Liang and Guo [15, 16] gave the existence spectrum of a k-sun system of order v as $k=2,4,5,6,8$. Fu et. al. [10, 11] obtained that 3-sun decompositions of $K_{p,p,r}$, $K_n$and embed a cyclic steiner triple system of order n into a 3-sun system of order $2n-1$, for $n\equiv 1 \pmod 6$. Further they obtained k-sun system when $k=6,10,14,2^t$, for $t>1$. Fu et. al. [9] obtained the existence of a 5-sun system of order v. Gionfriddo et.al. [12] obtained the spectrum for uniformly resolvable decompositions of K_v into 1-factor and h-suns. Akwu and Ajayi [1] obtained the necessary and sufficient conditions for the existence of decomposition of $K_n\otimes\overline{K_m}$ and $(K_n-I)\otimes\overline{K_m}$, where I denote the 1-factor of a complete graph into sunlet graph of order $2p$, p is a prime. Sowndhariya and Muthusamy [20] obtained necessary and sufficient conditions for the existence of decomposition of $K_m\times K_n$ and $K_m\otimes\overline{K_n}$ into sunlet graph of order eight.

In this paper, we prove the existence of an L_8-decomposition of $K_m\Box K_n$. In fact, we establish necessary and sufficient conditions for the existence of an L_8-decomposition of $K_m\Box K_n$. To prove our results, we state the following:

Theorem 1.1. *[11] Let $t\geq 2$ be an integer. An $L_{2.2^t}$-decomposition of K_n exists if and only if $n\equiv 0$ (or) $1 \pmod{2^{t+2}}$.*

Theorem 1.2. *[20] For any $m,n\geq 4$, $K_{m,n}$ has an L_8- decomposition if and only if $mn\equiv 0 \pmod 8$ except $(m,n)=(4,2 \pmod 4))$ & $(8,5)$.*

2. Decomposition of $K_m\Box K_n$ into sunlet graph of order 8

Necessary conditions:

Lemma 2.1. *If $K_m\Box K_n$ has an L_8-decomposition, then either*

1. $m,n\equiv\ 0 \pmod 4$
2. $m\equiv\ 0 \pmod 8$, $n\equiv\ 0 \pmod 2$
3. $m\equiv\ 4 \pmod 8$, $n\equiv\ 2 \pmod 4$
4. $m\equiv\ 0 \pmod{16}$
5. $m\equiv\ 1 \pmod{16}$, $n\equiv\ 1 \pmod{16}$
6. $m\equiv\ 15 \pmod{16}$, $n\equiv\ 3 \pmod{16}$

7. $m \equiv 13 \ (mod\ 16), \ n \equiv 5 \ (mod\ 16)$

8. $m \equiv 11 \ (mod\ 16), \ n \equiv 7 \ (mod\ 16)$

9. $m \equiv 9 \ (mod\ 16), \ n \equiv 9 \ (mod\ 16)$

Proof. The graph $K_m \Box K_n$ has mn vertices, each having degree $m+n-2$ and hence has $\frac{mn(m+n-2)}{2}$ edges. Assume that $K_m \Box K_n$ admits an L_8- decomposition. Then the number of edges in the graph must be divisible by 8. i.e., $16|mn(m+n-2)$. Hence these conditions are met in each of the above nine cases and only in these cases. □

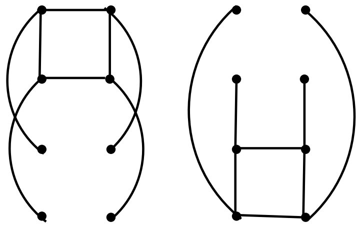

Figure 1. **L_8-decomposition of $K_4 \Box K_2$.**

Sufficient conditions:

We now prove the above necessary conditions are also sufficient by proving the following Lemmas:

Lemma 2.2. *If $m \equiv 0 \pmod 4$ and $n \equiv 0 \pmod 4$, then the graph $K_m \Box K_n$ has an L_8-decomposition.*

Proof. Let $m = 4s$ and $n = 4t$ for some $s, t > 0$. We can divide the graph $K_m \Box K_n$ into $st\,(K_4 \Box K_4)$, the L_8-decomposition of $K_4 \Box K_4$ is shown in Fig. 2 and the remaining edges are viewed in the following manner; for each row we have a 't' set of four vertices and each set is adjacent to other. Therefore we get $t(t-1)/2$ complete bipartite graph $K_{4,4}$. Then by Theorem 1.2, $K_{4,4}$ has an L_8-decomposititon. Similarly we can use the same procedure to column vertices. Hence the graph $K_m \Box K_n$ has the desired decomposition. □

Lemma 2.3. *If $m \equiv 0 \pmod 8$ and $n \equiv 0 \pmod 2$, then the graph $K_m \Box K_n$ has an L_8-decomposition.*

Proof. For $m = 8$ and $n = 2$, the graph $K_8 \Box K_2$ has an L_8- decomposition by Fig. 1 and by Theorem 1.2. Let $m \equiv 0 \pmod 8$ and $n \equiv 0$ (or) $4 \pmod 8$, then the proof follows from Lemma 2.2. Set $m \equiv 0 \pmod 8$ and consider two cases for n.

Case (1) $n \equiv 2 \pmod 8$.

Let $m = 8s$ and $n = 8t+2$ for some $s, t > 0$. The graph $K_m \Box K_n$ can be viewed as $s(t-1)\,(K_8 \Box K_8) \oplus s\,(K_8 \Box K_{10})$ and the remaining edges viewed as follows; in each row, we have $(t-1)$ set of eight vertices and one set of ten vertices which are form the complete bipartite graph $K_{8,8}$ and $K_{8,10}$. Similarly each column can be viewed as 's' set of eight vertices and each set is adjacent with each other (i.e. $K_{8,8}$). The L_8-decomposition of $K_8 \Box K_{10}$ is given in Appendix 3.1.1 and the L_8-decomposition of the graphs $K_8 \Box K_8$, $K_{8,8}$ and $K_{8,10}$ follows from Lemma 2.2 and Theorem 1.2.

Case (2) $n \equiv 6 \pmod 8$.

Let $m = 8s$ and $n = 8t+6$ for some $s, t > 0$. Then we can view $K_m \Box K_n$ as $st\,(K_8 \Box K_8) \oplus s\,(K_8 \Box K_6)$ and the remaining edges form the complete bipartite graph $K_{8,8}$ and $K_{8,6}$ which are obtained by using the above procedure. The L_8-decomposition of $K_8 \Box K_6$ is given in Appendix 3.1.2 and the L_8-decomposition of the remaining graphs follows from Lemma 2.2 and Theorem 1.2. Hence the graph $K_m \Box K_n$ has the desired decomposition. □

Lemma 2.4. *If $m \equiv 4 \pmod 8$ and $n \equiv 2 \pmod 4$, then the graph $K_m \Box K_n$ has an L_8-decomposition.*

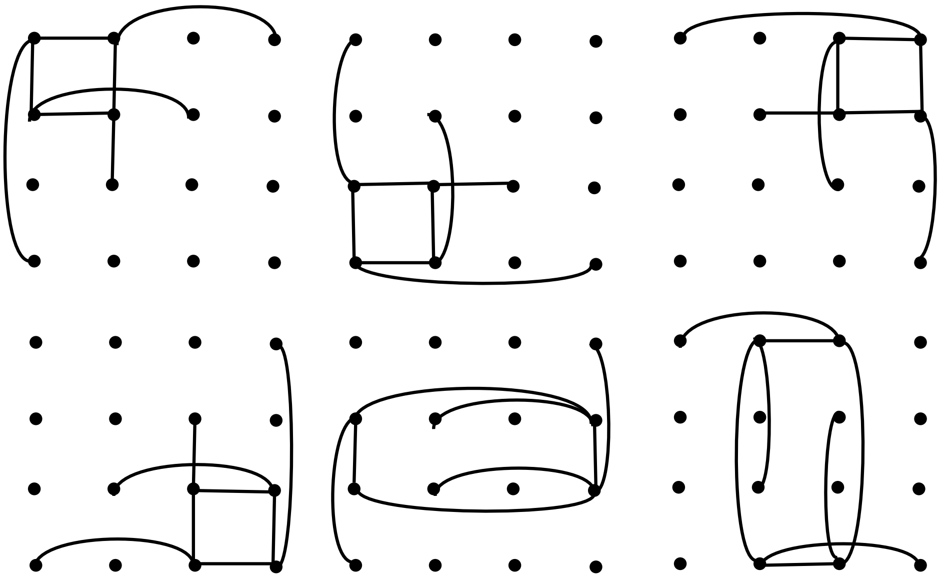

Figure 2. **L_8-decomposition of $K_4 \square K_4$.**

Proof. Let $m = 8s + 4$ and $n = 4t + 2$ for some $s, t > 0$. Now, we divide the proof into three cases.
Case (1) $m = 4$ and $n = 4t + 2$ for some $t > 0$.
If $t = 1$, then the graph $K_4 \square K_6$ has an L_8-decomposition, see Appendix 3.2.1. If $t = 2$, then the graph $K_4 \square K_{10}$ has an L_8-decomposition, see Appendix 3.2.2. Further, for $t > 2$, the graph $K_4 \square K_{4t+2}$ can be viewed as $K_4 \square K_6 \oplus K_4 \square K_{4(t-1)} \oplus 4K_{6,4(t-1)}$. Then by Appendix 3.2.1, Lemma 2.2 and Theorem 1.2, we get the desired decomposition.
Case (2) $m = 12$ and $n = 4t + 2$ for some $t > 0$.
If $t = 1$, then the graph $K_{12} \square K_6$ has an L_8-decomposition see Appendix 3.2.3. If $t = 2$, then the graph $K_{12} \square K_{10}$ has an L_8-decomposition see Appendix 3.2.4. Further, for $t > 2$, the graph $K_{12} \square K_{4t+2}$ can be viewed as $K_{12} \square K_6 \oplus K_{12} \square K_{4(t-1)} \oplus 12K_{6,4(t-1)}$. Then by Appendix 3.2.3, Lemma 2.2 and Theorem 1.2, we get the desired decomposition.
Case (3) $m > 12$ and $n = 4t + 2$ for some $t > 0$.
The graph $K_m \square K_n$ can be viewed as $K_{8(s-1)} \square K_{4t+2} \oplus K_{12} \square K_{4t+2} \oplus (4t + 2)K_{12,8(s-1)}$. Then by Lemma 2.3, Theorem 1.2 and the above Case (2), we get the desired decomposition. □

Lemma 2.5. *If $m \equiv 0 \pmod{16}$, then the graph $K_m \square K_n$ has an L_8-decomposition.*

Proof. If $n = 1$, the graph K_m has an L_8-decomposition by Theorem 1.1. Let $m \equiv 0 \pmod{16}$ and $n \equiv 0, 2, 4, 6 \pmod{8}$, then the proof follows from Lemma 2.3. Set $m = 0 \pmod{16}$ and consider four cases for odd n.
Case (1) $n \equiv 1 \pmod{8}$.
Let $m = 16s$ and $n = 8t + 1$ for some $s, t > 0$. The graph $K_m \square K_n$ can be viewed as $s(t-1)(K_{16} \square K_8) \oplus s(K_{16} \square K_9)$. Then the remaining edges are viewed as follows; each row contains $(t-1)$ set of eight vertices and one set of nine vertices. Then each set is adjacent to each other and these forms the complete bipartite graphs $K_{8,8}$, $K_{8,9}$. Similarly each column can be viewed as 's' set of sixteen vertices and each set is adjacent with each other (i.e. we have $K_{16,16}$). Finally theL_8-decomposition of $K_{16} \square K_9$ is shown in Appendix 3.3.1 and the L_8-decomposition of the remaining graphs follows from Lemma 2.2 and Theorem 1.2.
Case (2) $n \equiv 3 \pmod{8}$.
If $n = 3$, then the graph $K_m \square K_3$ can be viewed as copies of $K_{16} \square K_3$ and $K_{16,16}$ which has an L_8-decomposition, see Appendix 3.3.2 and Theorem 1.2. Let $m = 16s$ and $n = 8t + 3$ for some $s, t > 0$.

Apply the same procedure as in Case (1) and we write $K_m \square K_n$ as $s(t-1)\,(K_{16} \square K_8) \oplus s\,(K_{16} \square K_{11}) \oplus 8s(t-1)(t-2)K_{8,8} \oplus \frac{s(8t+3)(s-1)}{2}K_{16,16} \oplus 16s(t-1)K_{8,11}$. An L_8-decomposition of $K_{16} \square K_{11}$ is shown in Appendix 3.3.3 and the L_8-decomposition of the remaining graphs follows from Lemma 2.2 and Theorem 1.2.

Case (3) $n \equiv 5 \pmod 8$.

If $n = 5$, then the graph $K_m \square K_5$ can be viewed as $K_{16} \square K_5 \oplus K_{16,16}$ which has an L_8-decomposition, see Appendix 3.3.4 and Theorem 1.2. Let $m = 16s$ and $n = 8t + 5$ for some $s, t > 0$. Then we can write $K_m \square K_n$ as $s(t-1)\,(K_{16} \square K_8) \oplus s\,(K_{16} \square K_{13}) \oplus 8s(t-1)(t-2)K_{8,8} \oplus \frac{s(8t+5)(s-1)}{2}K_{16,16} \oplus 16s(t-1)K_{8,13}$. An L_8-decomposition of $K_{16} \square K_{13}$ is presented in Appendix 3.3.5 and the L_8-decomposition of the remaining graphs follows from Lemma 2.2 and Theorem 1.2.

Case (4) $n \equiv 7 \pmod 8$.

Let $m = 16s$ and $n = 8t+7$ for some $s, t > 0$. Then we can write $K_m \square K_n$ as $st\,(K_{16} \square K_8) \oplus s\,(K_{16} \square K_7) \oplus 8st(t-1)K_{8,8} \oplus \frac{s(8t+7)(s-1)}{2}K_{16,16} \oplus 16stK_{8,7}$. An L_8-decomposition of $K_{16} \square K_7$ is presented in Appendix 3.3.6 and the L_8-decomposition of the remaining graphs follows from Lemma 2.2 and Theorem 1.2.

Hence the graph $K_m \square K_n$ has the desired decomposition. □

Lemma 2.6. *If* $m \equiv 1 \pmod{16}$ *and* $n \equiv 1 \pmod{16}$, *then the graph* $K_m \square K_n$ *has an* L_8*-decomposition.*

Proof. Let $m = 16s + 1$ and $n = 16t + 1$ for some $s, t > 0$. Then we can write $K_m \square K_n = nK_m \oplus mK_n$. i.e, $(16t + 1)K_{16s+1} \oplus (16s + 1)K_{16t+1}$. By Theorem 1.1, the graph $K_m \square K_n$ has the desired decomposition. □

Lemma 2.7. *If* $m \equiv 15 \pmod{16}$ *and* $n \equiv 3 \pmod{16}$, *then the graph* $K_m \square K_n$ *has an* L_8*-decomposition.*

Proof. Let $m = 16s+15$ and $n = 16t+3$ for some $s, t > 0$. We can write $K_m \square K_n$ as $(16t+3)K_{16s+15} \oplus (16s + 15)K_{16t+3}$. Now the first $16t$ columns can be viewed as $K_{16s} \oplus K_{15} \oplus sK_{16,15}$ and the first $16s$ rows can be viewed as $K_{16(t-1)} \oplus K_{19} \oplus (t-1)K_{16,19}$. Then $K_{16s}(= sK_{16} \oplus \frac{s(s-1)}{2}K_{16,16})$, $K_{16(t-1)}(= (t-1)K_{16} \oplus \frac{(t-1)(t-2)}{2}K_{16,16})$, $K_{16,15}$ and $K_{16,19}$ have L_8-decompositions by Theorems 1.1, 1.2. The graph K_{19} can be viewed as $K_{19} \backslash K_3 \oplus K_3$. The L_8-decomposition of $K_{19} \backslash K_3$ follows from Appendix 3.4.1. Then $16s\,(K_{19} \backslash K_3)$ has an L_8-decomposition. The remaining graph can be viewed as $s(K_{16} \square K_3) \oplus t(K_{15} \square K_{16}) \oplus K_{15} \square K_3$. Hence the desired decomposition follows from Appendixes 3.3.2, 3.4.2 and Lemma 2.5 □

Lemma 2.8. *If* $m \equiv 13 \pmod{16}$ *and* $n \equiv 5 \pmod{16}$, *then the graph* $K_m \square K_n$ *has an* L_8*-decomposition.*

Proof. Let $m = 16s+13$ and $n = 16t+5$ for some $s, t > 0$. Then we can write $K_m \square K_n$ as $st\,(K_{16} \square K_{16}) \oplus t\,(K_{16} \square K_{13}) \oplus s\,(K_{16} \square K_5) \oplus K_{13} \square K_5 \oplus \frac{t(t-1)(16s+13)+s(s-1)(16t+5)}{2}K_{16,16} \oplus s(16t + 5)K_{16,13} \oplus t(16s+13)K_{16,5}$. An L_8-decomposition of $K_{13} \square K_5$ is given in Appendix 3.5.1. and the L_8-decomposition of the remaining graphs follows from Lemma 2.5 and Theorem 1.2. Hence the graph $K_m \square K_n$ has the desired decomposition. □

Lemma 2.9. *If* $m \equiv 11 \pmod{16}$ *and* $n \equiv 7 \pmod{16}$, *then the graph* $K_m \square K_n$ *has an* L_8*-decomposition.*

Proof. Let $m = 16s+11$ and $n = 16t+7$ for some $s, t > 0$. Then we can write $K_m \square K_n$ as $st\,(K_{16} \square K_{16}) \oplus t\,(K_{16} \square K_{11}) \oplus s\,(K_{16} \square K_7) \oplus K_{11} \square K_7 \oplus \frac{s(s-1)(16t+7)+t(t-1)(16s+11)}{2}K_{16,16} \oplus (16t + 7)sK_{16,11} \oplus (16s+11)tK_{16,7}$. An L_8-decomposition of $K_{11} \square K_7$ is given in Appendix 3.6.1 and the L_8-decomposition of the remaining graphs follows from Lemma 2.5 and Theorem 1.2. Hence the graph $K_m \square K_n$ has the desired decomposition. □

Lemma 2.10. *If* $m \equiv 9 \pmod{16}$ *and* $n \equiv 9 \pmod{16}$, *then the graph* $K_m \square K_n$ *has an* L_8*-decomposition.*

Proof. Let $m = 16s+9$ and $n = 16t+9$ for some $s, t > 0$. Then we can write $K_m \square K_n$ as $st\,(K_{16} \square K_{16}) \oplus (s+t)\,(K_{16} \square K_9) \oplus K_9 \square K_9 \oplus \frac{s(s-1)(16t+9)+t(t-1)(16s+9)}{2} K_{16,16} \oplus [s(16t+9)+t(16s+9)]\,K_{16,9}$. An L_8-decomposition of $K_{11} \square K_7$ is given in Appendix 3.7.1 and the L_8-decomposition of the remaining graphs follows from Lemma 2.5 and Theorem 1.2. Hence the graph $K_m \square K_n$ has the desired decomposition. □

2.1. Main theorem

Combining the results from Lemma 2.1 to Lemma 2.10, we get the following main result.

Theorem 2.11. *The graph $K_m \square K_n$ admits an L_8- decomposition if and only if one of the following holds:*

1. $m, n \equiv 0 \ (mod\ 4)$
2. $m \equiv 0 \ (mod\ 8),\ n \equiv 0 \ (mod\ 2)$
3. $m \equiv 4 \ (mod\ 8),\ n \equiv 2 \ (mod\ 4)$
4. $m \equiv 0 \ (mod\ 16)$
5. $m \equiv 1 \ (mod\ 16),\ n \equiv 1 \ (mod\ 16)$
6. $m \equiv 15 \ (mod\ 16),\ n \equiv 3 \ (mod\ 16)$
7. $m \equiv 13 \ (mod\ 16),\ n \equiv 5 \ (mod\ 16)$
8. $m \equiv 11 \ (mod\ 16),\ n \equiv 7 \ (mod\ 16)$
9. $m \equiv 9 \ (mod\ 16),\ n \equiv 9 \ (mod\ 16)$

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3. Appendix

3.1. L_8-decomposition required for Lemma 2.3

3.1.1. An L_8- decomposition of $K_8 \Box K_{10}$

$\begin{pmatrix} x_i^1 & x_i^6 & x_i^2 & x_i^7 \\ x_i^3 & x_i^8 & x_i^4 & x_i^9 \end{pmatrix}$, $\begin{pmatrix} x_i^3 & x_i^8 & x_i^4 & x_i^9 \\ x_i^{10} & x_i^5 & x_i^7 & x_i^2 \end{pmatrix}$, $\begin{pmatrix} x_i^2 & x_i^3 & x_i^4 & x_i^5 \\ x_i^1 & x_i^7 & x_i^6 & x_i^9 \end{pmatrix}$ for $i = 1, 2, ..., 8$;
$\begin{pmatrix} x_1^j & x_5^j & x_2^j & x_6^j \\ x_3^j & x_7^j & x_4^j & x_8^j \end{pmatrix}$, $\begin{pmatrix} x_3^j & x_7^j & x_4^j & x_8^j \\ x_5^j & x_1^j & x_6^j & x_2^j \end{pmatrix}$ for $j = 1, 2, ..., 10$;
$\begin{pmatrix} x_1^j & x_2^j & x_3^j & x_4^j \\ x_8^j & x_7^j & x_6^j & x_5^j \end{pmatrix}$ for $j = 1, 3, 5, 7, 8, 9$; $\begin{pmatrix} x_5^j & x_6^j & x_7^j & x_8^j \\ x_4^j & x_3^j & x_2^j & x_1^j \end{pmatrix}$ for $j = 2, 4, 6, 10$;
$\begin{pmatrix} x_i^1 & x_i^5 & x_i^6 & x_i^{10} \\ x_i^8 & x_i^{k_1} & x_i^{k_2} & x_i^{k_3} \end{pmatrix}$ for $i = 1, 2, 3, 4,\ (k_1, k_2, k_3) = (3, 9, 4)$ & $i = 5, 6, 7, 8$, $(k_1, k_2, k_3) = (7, 3, 2)$;
$\begin{pmatrix} x_i^7 & x_i^8 & x_i^9 & x_i^{10} \\ x_i^6 & x_i^2 & x_i^1 & x_i^k \end{pmatrix}$ for $i = 1, 2, 3, 4,\ k = 5$ & $i = 5, 6, 7, 8,\ k = 4$;

$\begin{pmatrix} x_1^j & x_2^j & x_3^j & x_4^j \\ x_1^k & x_2^k & x_3^k & x_4^k \end{pmatrix}$ for $(j,k) = (2,10), (4,1), (6,3), (10,8)$;
$\begin{pmatrix} x_5^j & x_6^j & x_7^j & x_8^j \\ x_5^k & x_6^k & x_7^k & x_8^k \end{pmatrix}$ for $(j,k) = (1,4), (3,5), (5,10), (7,5), (8,10), (9,6)$.

3.1.2. An L_8- decomposition of $K_8 \square K_6$

$\begin{pmatrix} x_1^j & x_5^j & x_2^j & x_6^j \\ x_1^k & x_7^j & x_2^k & x_8^j \end{pmatrix}, \begin{pmatrix} x_3^j & x_7^j & x_4^j & x_8^j \\ x_3^k & x_1^j & x_4^k & x_2^j \end{pmatrix}$ for $(j,k) = (1,3), (5,2), (6,4)$;
$\begin{pmatrix} x_1^j & x_5^j & x_2^j & x_6^j \\ x_3^j & x_5^k & x_4^j & x_6^k \end{pmatrix}; \begin{pmatrix} x_3^j & x_7^j & x_4^j & x_8^j \\ x_5^j & x_7^k & x_6^j & x_8^k \end{pmatrix}$ for $(j,k) = (2,5), (3,1), (4,6)$;
$\begin{pmatrix} x_i^1 & x_i^2 & x_i^3 & x_i^4 \\ x_k^1 & x_{9-i}^2 & x_{9-i}^3 & x_{9-i}^4 \end{pmatrix}$ for $(i,k) = (1,3), (2,4), (3,6)$;
$\begin{pmatrix} x_i^1 & x_i^5 & x_i^3 & x_i^6 \\ x_{i+1}^1 & x_k^5 & x_{i+1}^3 & x_k^6 \end{pmatrix}$ for $(i,k) = (1,8), (2,4), (3,5)$;
$\begin{pmatrix} x_i^1 & x_i^2 & x_i^3 & x_i^4 \\ x_{i+1}^1 & x_{i+1}^2 & x_{i+1}^3 & x_{i+1}^4 \end{pmatrix}$ for $i = 4,5,6,7$; $\begin{pmatrix} x_i^2 & x_i^4 & x_i^5 & x_i^6 \\ x_{i+1}^2 & x_{i+1}^4 & x_{i+1}^5 & x_{i+1}^6 \end{pmatrix}$ for $i = 1,2,3$;
$\begin{pmatrix} x_8^1 & x_8^2 & x_8^3 & x_8^4 \\ x_5^1 & x_5^2 & x_5^3 & x_5^4 \end{pmatrix}, \begin{pmatrix} x_4^1 & x_4^5 & x_4^3 & x_4^6 \\ x_1^1 & x_6^5 & x_1^3 & x_6^6 \end{pmatrix}, \begin{pmatrix} x_5^1 & x_5^5 & x_5^3 & x_5^6 \\ x_3^1 & x_6^5 & x_7^3 & x_6^6 \end{pmatrix}, \begin{pmatrix} x_6^1 & x_6^5 & x_6^3 & x_6^6 \\ x_4^1 & x_7^5 & x_8^3 & x_7^6 \end{pmatrix}$,
$\begin{pmatrix} x_7^1 & x_7^5 & x_7^3 & x_7^6 \\ x_2^1 & x_8^5 & x_1^3 & x_8^6 \end{pmatrix}, \begin{pmatrix} x_8^1 & x_8^5 & x_8^3 & x_8^6 \\ x_1^1 & x_5^5 & x_2^3 & x_5^6 \end{pmatrix}, \begin{pmatrix} x_4^2 & x_4^4 & x_4^5 & x_4^6 \\ x_1^2 & x_1^4 & x_1^5 & x_1^6 \end{pmatrix}, \begin{pmatrix} x_5^2 & x_5^4 & x_5^5 & x_5^6 \\ x_7^2 & x_7^4 & x_4^5 & x_4^6 \end{pmatrix}$,
$\begin{pmatrix} x_6^2 & x_6^4 & x_6^5 & x_6^6 \\ x_8^2 & x_8^4 & x_3^5 & x_3^6 \end{pmatrix}, \begin{pmatrix} x_7^2 & x_7^4 & x_7^5 & x_7^6 \\ x_1^2 & x_1^4 & x_2^5 & x_2^6 \end{pmatrix}, \begin{pmatrix} x_8^2 & x_8^4 & x_8^5 & x_8^6 \\ x_2^2 & x_2^4 & x_1^5 & x_1^6 \end{pmatrix}$.

3.2. L_8-decomposition required for Lemma 2.4

3.2.1. An L_8- decomposition of $K_4 \square K_6$

$\begin{pmatrix} x_i^2 & x_i^3 & x_i^6 & x_i^4 \\ x_k^2 & x_k^3 & x_i^1 & x_i^5 \end{pmatrix}$ for $(i,k) = (1,3), (2,4)$; $\begin{pmatrix} x_i^2 & x_i^3 & x_i^6 & x_i^4 \\ x_i^5 & x_i^1 & x_k^6 & x_k^4 \end{pmatrix}$ for $(i,k) = (3,1), (4,2)$;
$\begin{pmatrix} x_i^1 & x_i^5 & x_k^5 & x_k^1 \\ x_i^2 & x_i^6 & x_k^6 & x_k^2 \end{pmatrix}$ for $(i,k) = (1,3), (2,4)$; $\begin{pmatrix} x_1^j & x_2^j & x_3^j & x_4^j \\ x_1^k & x_2^k & x_3^k & x_4^k \end{pmatrix}$ for $(j,k) = (1,4), (4,3)$;
$\begin{pmatrix} x_1^j & x_2^j & x_3^j & x_4^j \\ x_1^{k_1} & x_2^{k_1} & x_3^{k_2} & x_4^{k_2} \end{pmatrix}$ for $(j, k_1, k_2) = (2,5,6), (3,1,5), (5,3,4), (6,2,1)$.

3.2.2. An L_8- decomposition of $K_4 \square K_{10}$

$\begin{pmatrix} x_i^1 & x_i^9 & x_i^2 & x_i^{10} \\ x_k^1 & x_i^3 & x_k^2 & x_i^4 \end{pmatrix}, \begin{pmatrix} x_i^7 & x_i^9 & x_i^8 & x_i^{10} \\ x_i^4 & x_i^5 & x_i^6 & x_k^{10} \end{pmatrix}, \begin{pmatrix} x_i^3 & x_i^5 & x_k^5 & x_k^3 \\ x_i^1 & x_i^{10} & x_k^2 & x_k^1 \end{pmatrix}$ for $(i,k) = (1,3), (2,4)$;
$\begin{pmatrix} x_i^7 & x_i^9 & x_i^8 & x_i^{10} \\ x_k^7 & x_k^9 & x_k^8 & x_i^3 \end{pmatrix}$ for $(i,k) = (3,1), (4,2)$; $\begin{pmatrix} x_i^4 & x_i^6 & x_k^6 & x_k^4 \\ x_i^5 & x_i^1 & x_k^8 & x_k^5 \end{pmatrix}$ for $(i,k) = (1,3), (2,4)$;
$\begin{pmatrix} x_i^1 & x_i^2 & x_i^3 & x_i^4 \\ x_i^7 & x_i^6 & x_i^8 & x_i^9 \end{pmatrix}$ for $i = 1,2,3,4$; $\begin{pmatrix} x_i^5 & x_i^6 & x_i^7 & x_i^8 \\ x_i^2 & x_i^{10} & x_i^3 & x_i^4 \end{pmatrix}$ for $i = 1,2$;
$\begin{pmatrix} x_i^5 & x_i^6 & x_i^7 & x_i^8 \\ x_i^9 & x_i^{10} & x_i^3 & x_i^4 \end{pmatrix}, \begin{pmatrix} x_i^1 & x_i^9 & x_i^2 & x_i^{10} \\ x_i^6 & x_i^3 & x_i^4 & x_i^5 \end{pmatrix}$ for $i = 3,4$;
$\begin{pmatrix} x_1^j & x_2^j & x_3^j & x_4^j \\ x_1^k & x_2^k & x_3^k & x_4^k \end{pmatrix}$ for $(j,k) = (1,5), (2,8), (3,6), (5,7), (6,9), (7,2), (8,1), (9,10)$;
$\begin{pmatrix} x_1^4 & x_2^4 & x_3^4 & x_4^4 \\ x_1^2 & x_2^2 & x_3^7 & x_4^7 \end{pmatrix}, \begin{pmatrix} x_1^{10} & x_2^{10} & x_3^{10} & x_4^{10} \\ x_1^3 & x_2^3 & x_3^4 & x_4^4 \end{pmatrix}$.

3.2.3. An L_8- decomposition of $K_{12} \square K_6$

$\begin{pmatrix} x_1^j & x_2^j & x_3^j & x_4^j \\ x_1^k & x_2^k & x_6^j & x_5^j \end{pmatrix}, \begin{pmatrix} x_5^j & x_6^j & x_7^j & x_8^j \\ x_5^k & x_6^k & x_{10}^j & x_9^j \end{pmatrix}, \begin{pmatrix} x_9^j & x_{10}^j & x_{11}^j & x_{12}^j \\ x_9^k & x_{10}^k & x_4^j & x_3^j \end{pmatrix}$ for $(j,k) = (1,3), (2,4)$;

$\begin{pmatrix} x_1^j & x_2^j & x_3^j & x_4^j \\ x_1^k & x_2^k & x_3^k & x_4^k \end{pmatrix}$, $\begin{pmatrix} x_5^j & x_6^j & x_7^j & x_8^j \\ x_5^k & x_6^k & x_7^k & x_{48}^k \end{pmatrix}$, $\begin{pmatrix} x_9^j & x_{10}^j & x_{11}^j & x_{12}^j \\ x_9^k & x_{10}^k & x_{11}^k & x_{12}^k \end{pmatrix}$ for $(j,k) = (3,5), (4,6),$ $(5,4), (6,3)$; $\begin{pmatrix} x_5^j & x_9^j & x_6^j & x_{10}^j \\ x_{12}^k & x_2^k & x_{11}^j & x_1^j \end{pmatrix}$ for $j = 1, ..., 6$;
$\begin{pmatrix} x_1^j & x_5^j & x_2^j & x_6^j \\ x_7^j & x_{11}^j & x_8^j & x_{12}^j \end{pmatrix}$, $\begin{pmatrix} x_3^j & x_9^j & x_4^j & x_{10}^j \\ x_5^j & x_1^j & x_6^j & x_2^j \end{pmatrix}$, $\begin{pmatrix} x_7^j & x_{11}^j & x_8^j & x_{12}^j \\ x_9^j & x_3^j & x_{10}^j & x_4^j \end{pmatrix}$ for $j = 1, ..., 6$;
$\begin{pmatrix} x_1^j & x_{11}^j & x_2^j & x_{12}^j \\ x_8^j & x_9^j & x_7^j & x_{10}^j \end{pmatrix}$, $\begin{pmatrix} x_3^j & x_7^j & x_4^j & x_8^j \\ x_1^j & x_5^j & x_2^j & x_6^j \end{pmatrix}$ for $j = 1, 2$;
$\begin{pmatrix} x_1^j & x_{11}^j & x_2^j & x_{12}^j \\ x_8^j & x_4^j & x_7^j & x_3^j \end{pmatrix}$, $\begin{pmatrix} x_3^j & x_7^j & x_4^j & x_8^j \\ x_6^j & x_{10}^j & x_5^j & x_9^j \end{pmatrix}$ for $j = 5, 6$;
$\begin{pmatrix} x_1^j & x_{11}^j & x_2^j & x_{12}^j \\ x_8^j & x_{11}^k & x_7^j & x_{12}^k \end{pmatrix}$, $\begin{pmatrix} x_3^j & x_7^j & x_4^j & x_8^j \\ x_3^k & x_7^k & x_4^k & x_8^k \end{pmatrix}$ for $(j,k) = (3,1), (4,2)$;
$\begin{pmatrix} x_i^5 & x_i^6 & x_k^6 & x_k^5 \\ x_i^2 & x_i^1 & x_k^1 & x_k^2 \end{pmatrix}$ for $(i,k) = (1,3), (2,4), (5,7), (6,8), (9,11), (10,12)$;
$\begin{pmatrix} x_i^1 & x_i^2 & x_i^3 & x_i^4 \\ x_i^5 & x_i^6 & x_k^3 & x_k^4 \end{pmatrix}$ for $(i,k) = (1,3), (2,4), (3,6), (4,5), (5,7), (6,8), (7,10), (8,9), (9,11),$ $(10,12), (11,4), (12,3)$.

3.2.4. An L_8- decomposition of $K_{12}\square K_{10}$

$\begin{pmatrix} x_i^1 & x_i^2 & x_i^3 & x_i^4 \\ x_k^1 & x_k^2 & x_k^3 & x_k^4 \end{pmatrix}$, $\begin{pmatrix} x_i^5 & x_i^6 & x_i^7 & x_i^8 \\ x_k^5 & x_k^6 & x_k^7 & x_k^8 \end{pmatrix}$, $\begin{pmatrix} x_i^1 & x_i^9 & x_i^2 & x_i^{10} \\ x_i^6 & x_k^9 & x_i^5 & x_k^{10} \end{pmatrix}$ for $(i,k) = (1,7), (2,8), (3,6),$ $(4,5), (5,11), (6,12), (7,10), (8,9), (9,2), (10,1), (11,3), (12,4)$;
$\begin{pmatrix} x_i^7 & x_i^9 & x_i^8 & x_i^{10} \\ x_k^7 & x_k^9 & x_k^8 & x_k^{10} \end{pmatrix}$ for $(i,k) = (1,8), (2,7), (3,12), (4,11), (5,12), (6,11)$;
c$\begin{pmatrix} x_i^7 & x_i^9 & x_i^8 & x_i^{10} \\ x_i^3 & x_i^5 & x_i^6 & x_i^4 \end{pmatrix}$ for $i = 7, ..., 12$; $\begin{pmatrix} x_i^3 & x_i^5 & x_i^4 & x_i^6 \\ x_i^7 & x_i^{10} & x_i^2 & x_i^8 \end{pmatrix}$ for $i = 1, ..., 6$;
$\begin{pmatrix} x_i^3 & x_i^5 & x_i^4 & x_i^6 \\ x_k^3 & x_k^5 & x_k^4 & x_k^6 \end{pmatrix}$ for $(i,k) = (7,2), (8,1), (9,11), (10,12), (11,6), (12,5)$;
$\begin{pmatrix} x_1^j & x_2^j & x_3^j & x_4^j \\ x_1^k & x_2^k & x_3^k & x_4^k \end{pmatrix}$ for $(j,k) = (1,3), (3,10), (4,7), (5,7), (6,10), (7,1)(8,2), (9,3), (10,4)$;
$\begin{pmatrix} x_5^j & x_6^j & x_7^j & x_8^j \\ x_5^k & x_6^k & x_7^k & x_8^k \end{pmatrix}$ for $(j,k) = (1,3), (3,7), (4,7), (5,7), (6,8), (7,1)(8,2), (9,3)$;
$\begin{pmatrix} x_9^j & x_{10}^j & x_{11}^j & x_{12}^j \\ x_9^k & x_{10}^k & x_{11}^k & x_{12}^k \end{pmatrix}$ for $(j,k) = (1,3), (2,4), (3,7), (4,7), (5,7), (6,8), (7,1)(8,2), (9,3), (10,5)$;
$\begin{pmatrix} x_1^j & x_5^j & x_2^j & x_6^j \\ x_3^i & x_5^k & x_4^i & x_6^k \end{pmatrix}$ for $(j,k) = (3,8), (4,8), (5,9), (6,9)$;
$\begin{pmatrix} x_1^j & x_5^j & x_2^j & x_6^j \\ x_1^k & x_5^k & x_2^k & x_6^k \end{pmatrix}$ for $(j,k) = (1,5), (2,6), (7,2), (8,1), (9,4), (10,9)$;
$\begin{pmatrix} x_3^j & x_9^j & x_4^j & x_{10}^j \\ x_3^k & x_9^k & x_4^k & x_{10}^k \end{pmatrix}$ for $(j,k) = (2,6), (3,8), (4,8), (6,9), (7,2), (8,1), (9,4), (10,9)$;
$\begin{pmatrix} x_7^j & x_{11}^j & x_8^j & x_{12}^j \\ x_7^k & x_{11}^k & x_8^k & x_{12}^k \end{pmatrix}$ for $(j,k) = (2,6), (3,8), (4,8), (5,1), (6,9), (7,2), (8,1), (9,4), (10,9)$;
$\begin{pmatrix} x_1^j & x_{11}^j & x_2^j & x_{12}^j \\ x_3^j & x_9^j & x_4^j & x_{10}^j \end{pmatrix}$ for $i = 2, 7, 8, 9, 10$;
$\begin{pmatrix} x_1^j & x_{11}^j & x_2^j & x_{12}^j \\ x_1^k & x_4^j & x_2^k & x_3^j \end{pmatrix}$ for $(j,k) = (3,9), (4,8), (5,9), (6,9)$;
$\begin{pmatrix} x_5^j & x_9^j & x_6^j & x_{10}^j \\ x_7^j & x_1^j & x_8^j & x_2^j \end{pmatrix}$, $\begin{pmatrix} x_3^j & x_7^j & x_4^j & x_8^j \\ x_5^j & x_9^j & x_6^j & x_{10}^j \end{pmatrix}$, for $j = 1, ..., 10$;
$\begin{pmatrix} x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_8^2 & x_7^2 & x_{12}^2 & x_{11}^2 \end{pmatrix}$, $\begin{pmatrix} x_5^2 & x_6^2 & x_7^2 & x_8^2 \\ x_{12}^2 & x_{11}^2 & x_7^4 & x_8^4 \end{pmatrix}$, $\begin{pmatrix} x_5^{10} & x_6^{10} & x_7^{10} & x_8^{10} \\ x_5^4 & x_6^4 & x_7^4 & x_8^4 \end{pmatrix}$, $\begin{pmatrix} x_3^1 & x_9^1 & x_4^1 & x_{10}^1 \\ x_3^5 & x_{11}^1 & x_4^5 & x_{12}^1 \end{pmatrix}$,
$\begin{pmatrix} x_3^5 & x_9^5 & x_4^5 & x_{10}^5 \\ x_3^9 & x_9^1 & x_4^9 & x_{10}^1 \end{pmatrix}$, $\begin{pmatrix} x_7^1 & x_{11}^1 & x_8^1 & x_{12}^1 \\ x_2^1 & x_4^1 & x_1^1 & x_3^1 \end{pmatrix}$, $\begin{pmatrix} x_1^1 & x_{11}^1 & x_2^1 & x_{12}^1 \\ x_3^1 & x_6^1 & x_4^1 & x_5^1 \end{pmatrix}$.

3.3. L_8-decomposition required for Lemma 2.5

3.3.1. An L_8- decomposition of $K_{16}\Box K_9$

$\begin{pmatrix} x_1^j & x_2^j & x_{15}^j & x_{16}^j \\ x_{12}^j & x_4^j & x_6^j & x_5^j \end{pmatrix}, \begin{pmatrix} x_3^j & x_4^j & x_{13}^j & x_{14}^j \\ x_{12}^j & x_{11}^j & x_5^j & x_{15}^j \end{pmatrix}$ for $j = 1, 4, 5, 7, 8$;

$\begin{pmatrix} x_5^j & x_6^j & x_{11}^j & x_{12}^j \\ x_{15}^j & x_7^j & x_2^j & x_{16}^j \end{pmatrix}, \begin{pmatrix} x_7^j & x_8^j & x_9^j & x_{10}^j \\ x_{13}^j & x_{15}^j & x_{11}^j & x_1^j \end{pmatrix}$ for $j = 1, 4, 5, 7, 8$;

$\begin{pmatrix} x_1^j & x_{13}^j & x_2^j & x_{14}^j \\ x_{11}^j & x_6^j & x_{10}^j & x_5^j \end{pmatrix}, \begin{pmatrix} x_3^j & x_{15}^j & x_4^j & x_{16}^j \\ x_{11}^j & x_1^j & x_{10}^j & x_2^j \end{pmatrix}$ for $j = 1, 3, 4, 5, 6, 7, 8$;

$\begin{pmatrix} x_1^j & x_7^j & x_2^j & x_8^j \\ x_4^j & x_{16}^j & x_9^j & x_{13}^j \end{pmatrix}, \begin{pmatrix} x_9^j & x_{15}^j & x_{10}^j & x_{16}^j \\ x_1^j & x_{12}^j & x_3^j & x_{14}^j \end{pmatrix}$ for $j = 1, 3, 4, 5, 6, 7, 8, 9$;

$\begin{pmatrix} x_1^j & x_5^j & x_2^j & x_6^j \\ x_3^j & x_8^j & x_{12}^j & x_{16}^j \end{pmatrix}, \begin{pmatrix} x_9^j & x_{13}^j & x_{10}^j & x_{14}^j \\ x_3^j & x_{16}^j & x_{12}^j & x_7^j \end{pmatrix}$ for $j = 1, 3, 4, 5, 7, 8$;

$\begin{pmatrix} x_7^j & x_{11}^j & x_8^j & x_{12}^j \\ x_9^j & x_{15}^j & x_{10}^j & x_6^j \end{pmatrix}$ for $j = 1, 3, 4, 5, 6, 7, 8$; $\begin{pmatrix} x_{11}^j & x_{13}^j & x_{12}^j & x_{14}^j \\ x_{16}^j & x_{15}^j & x_4^j & x_8^j \end{pmatrix}$ for $j = 1, 3, 4, 5, 7, 8$;

$\begin{pmatrix} x_5^j & x_9^j & x_6^j & x_{10}^j \\ x_7^j & x_{12}^j & x_8^j & x_{11}^j \end{pmatrix}$ for $j = 1, 2, 3, 4, 5, 6, 7, 8$; $\begin{pmatrix} x_1^j & x_5^j & x_2^j & x_6^j \\ x_3^j & x_{16}^j & x_{12}^j & x_{14}^j \end{pmatrix}$ for $j = 6, 9$;

$\begin{pmatrix} x_3^j & x_7^j & x_4^j & x_8^j \\ x_{13}^j & x_{15}^j & x_{14}^j & x_{16}^j \end{pmatrix}$ for $j = 1, 2, ..., 8, 9$; $\begin{pmatrix} x_i^3 & x_i^7 & x_i^4 & x_i^9 \\ x_i^1 & x_i^5 & x_i^6 & x_i^8 \end{pmatrix}$ for $i = 1, 2, ..., 15, 16$;

$\begin{pmatrix} x_i^9 & x_{i+4}^9 & x_{i+1}^9 & x_{i+5}^9 \\ x_{i+2}^9 & x_{i+4}^5 & x_{i+3}^9 & x_{i+5}^5 \end{pmatrix}$ for $i = 5, 7$; $\begin{pmatrix} x_i^1 & x_i^8 & x_i^2 & x_i^9 \\ x_i^5 & x_i^6 & x_i^4 & x_i^7 \end{pmatrix}$ for $i = 1, 2, ..., 7, 8$;

$\begin{pmatrix} x_i^j & x_{12+i}^j & x_{i+1}^j & x_{13+i}^j \\ x_{11}^j & x_{12+i}^5 & x_{10}^j & x_{13+i}^5 \end{pmatrix}$ for $j = 2, 9$ & $i = 1, 3$;

$\begin{pmatrix} x_3^j & x_5^j & x_4^j & x_6^j \\ x_2^j & x_{11}^j & x_9^j & x_k^j \end{pmatrix}$ for $j = 1, 3, 4, 5, 7, 8,\ k = 14$ & $j = 2, 6, 9,\ k = 16$;

$\begin{pmatrix} x_i^5 & x_i^6 & x_i^7 & x_i^8 \\ x_i^4 & x_i^1 & x_i^k & x_i^3 \end{pmatrix}$ for $i = 1, 2, ..., 7, 8,\ k = 2,$ & $i = 9, ..., 16,\ k = 9$;

$\begin{pmatrix} x_5^j & x_6^j & x_{11}^j & x_{12}^j \\ x_{15}^j & x_7^j & x_{11}^k & x_{16}^j \end{pmatrix}, \begin{pmatrix} x_{11}^j & x_{13}^j & x_{12}^j & x_{14}^j \\ x_{16}^j & x_{15}^j & x_{12}^k & x_8^j \end{pmatrix}$ for $(j, k) = (6, 3), (9, 6)$;

$\begin{pmatrix} x_7^j & x_8^j & x_9^j & x_{10}^j \\ x_{13}^j & x_{15}^j & x_{11}^j & x_{10}^k \end{pmatrix}, \begin{pmatrix} x_3^j & x_4^j & x_{13}^j & x_{14}^j \\ x_{12}^j & x_{11}^j & x_{13}^k & x_{15}^j \end{pmatrix}$ for $(j, k) = (2, 7), (6, 3), (9, 6)$;

$\begin{pmatrix} x_1^j & x_2^j & x_{15}^j & x_{16}^j \\ x_{12}^j & x_4^j & x_{15}^k & x_{16}^k \end{pmatrix}$ for $(j, k) = (2, 7), (3, 5), (6, 3), (9, 6)$;

$\begin{pmatrix} x_i^3 & x_{i+1}^3 & x_{16-i}^3 & x_{17-i}^3 \\ x_{k_1}^3 & x_{k_2}^3 & x_{16-i}^5 & x_{17-i}^5 \end{pmatrix}$; for $(i, k_1, k_2) = (3, 12, 11)(5, 15, 7), (7, 13, 15)$;

$\begin{pmatrix} x_9^j & x_{13}^j & x_{10}^j & x_{14}^j \\ x_9^k & x_{16}^j & x_{12}^j & x_{14}^k \end{pmatrix}$ for $(j, k) = (2, 7), (6, 3), (9, 6)$;

$\begin{pmatrix} x_i^1 & x_i^2 & x_i^3 & x_i^4 \\ x_i^7 & x_k^2 & x_i^6 & x_i^8 \end{pmatrix}$ for $(i, k) = (1, 10), (2, 11), (3, 9), (4, 12), (5, 13), (6, 15), (7, 14), (8, 5)$;

$\begin{pmatrix} x_i^1 & x_i^2 & x_i^3 & x_i^4 \\ x_i^7 & x_i^6 & x_k^3 & x_i^8 \end{pmatrix}$ for $(i, k) = (9, 11), (10, 1), (11, 2), (12, 16), (13, 5), (14, 15),$
$(15, 6), (16, 5)$;

$\begin{pmatrix} x_i^1 & x_i^8 & x_i^2 & x_i^9 \\ x_i^5 & x_i^6 & x_i^4 & x_k^9 \end{pmatrix}$ for $(i, k) = (9, 12), (10, 11), (11, 15), (12, 6), (13, 6), (14, 5),$
$(15, 1), (16, 2)$;

$\begin{pmatrix} x_i^2 & x_i^5 & x_i^9 & x_i^6 \\ x_{k_1}^2 & x_i^3 & x_{k_2}^9 & x_{k_2}^6 \end{pmatrix}$ for $(i, k_1, k_2) = (1, 15, 10), (2, 16, 11), (3, 10, 9), (4, 1, 12), (5, 14, 13),$
$(6, 13, 15), (7, 6, 14), (8, 14, 5)$;

$\begin{pmatrix} x_7^2 & x_{11}^2 & x_8^2 & x_{12}^2 \\ x_9^2 & x_{15}^2 & x_{10}^2 & x_{12}^5 \end{pmatrix}, \begin{pmatrix} x_1^2 & x_7^2 & x_2^2 & x_8^2 \\ x_9^2 & x_{16}^2 & x_{12}^2 & x_{13}^2 \end{pmatrix}, \begin{pmatrix} x_9^2 & x_{15}^2 & x_{10}^2 & x_{16}^2 \\ x_9^5 & x_{12}^2 & x_{10}^5 & x_{11}^2 \end{pmatrix}, \begin{pmatrix} x_{11}^2 & x_{13}^2 & x_{12}^2 & x_{14}^2 \\ x_{11}^5 & x_{15}^2 & x_{12}^7 & x_{16}^2 \end{pmatrix},$
$\begin{pmatrix} x_1^2 & x_5^2 & x_2^2 & x_6^2 \\ x_3^2 & x_{16}^2 & x_9^2 & x_{12}^2 \end{pmatrix}, \begin{pmatrix} x_5^2 & x_6^2 & x_{11}^2 & x_{12}^2 \\ x_{15}^2 & x_{14}^2 & x_{11}^7 & x_{16}^2 \end{pmatrix}.$

3.3.2. An L_8- decomposition of $K_{16}\Box K_3$

$$\begin{pmatrix} x_1^j & x_2^j & x_{15}^j & x_{16}^j \\ x_{12}^j & x_4^j & x_6^j & x_5^j \end{pmatrix}, \begin{pmatrix} x_3^j & x_{15}^j & x_4^j & x_{16}^j \\ x_{11}^j & x_1^j & x_{10}^j & x_2^j \end{pmatrix}, \begin{pmatrix} x_{11}^j & x_{13}^j & x_{12}^j & x_{14}^j \\ x_{16}^j & x_{15}^j & x_4^j & x_8^j \end{pmatrix} \text{ for } j = 1, 2, 3;$$
$$\begin{pmatrix} x_5^j & x_6^j & x_{11}^j & x_{12}^j \\ x_{15}^j & x_7^j & x_2^j & x_{16}^j \end{pmatrix}, \begin{pmatrix} x_1^j & x_{13}^j & x_2^j & x_{14}^j \\ x_{11}^j & x_6^j & x_{10}^j & x_5^j \end{pmatrix} \text{ for } j = 1, 3;$$
$$\begin{pmatrix} x_5^j & x_9^j & x_6^j & x_{10}^j \\ x_7^j & x_{12}^j & x_8^j & x_{10}^1 \end{pmatrix} \text{ for } j = 2, 3;$$
$$\begin{pmatrix} x_7^j & x_8^j & x_9^j & x_{10}^j \\ x_{13}^j & x_{15}^j & x_{k_1}^{k_2} & x_1^j \end{pmatrix} \text{ for } (j, k_1, k_2) = (1, 9, 3), (2, 9, 3), (3, 11, 3);$$
$$\begin{pmatrix} x_7^j & x_{11}^j & x_8^j & x_{12}^j \\ x_9^j & x_{15}^j & x_{k_1}^{k_2} & x_6^j \end{pmatrix} \text{ for } (j, k_1, k_2) = (1, 10, 1), (2, 10, 2), (3, 8, 1);$$
$$\begin{pmatrix} x_3^j & x_5^j & x_4^j & x_6^j \\ x_2^j & x_{k_1}^{k_2} & x_9^j & x_{14}^j \end{pmatrix} \text{ for } (j, k_1, k_2) = (1, 11, 1), (2, 11, 2), (3, 5, 2);$$
$$\begin{pmatrix} x_9^j & x_{15}^j & x_{10}^j & x_{16}^j \\ x_1^j & x_{k_1}^{k_2} & x_3^j & x_{14}^j \end{pmatrix} \text{ for } (j, k_1, k_2) = (1, 12, 1), (2, 15, 3), (3, 12, 3);$$
$$\begin{pmatrix} x_3^j & x_7^j & x_4^j & x_8^j \\ x_{13}^j & x_{15}^j & x_{k_1}^{k_2} & x_{16}^j \end{pmatrix} (j, k_1, k_2) = (1, 4, 3), (2, 4, 1), (3, 14, 3);$$
$$\begin{pmatrix} x_9^j & x_{13}^j & x_{10}^j & x_{14}^j \\ x_3^j & x_{k_1}^{k_2} & x_{12}^j & x_7^j \end{pmatrix} \text{ for } (j, k_1, k_2) = (1, 16, 1), (2, 13, 1), (3, 13, 1);$$
$$\begin{pmatrix} x_3^1 & x_4^1 & x_{13}^1 & x_{14}^1 \\ x_3^3 & x_{11}^1 & x_5^1 & x_{14}^3 \end{pmatrix}, \begin{pmatrix} x_3^2 & x_4^2 & x_{13}^2 & x_{14}^2 \\ x_3^3 & x_4^3 & x_5^2 & x_{14}^3 \end{pmatrix}, \begin{pmatrix} x_3^3 & x_4^3 & x_{13}^3 & x_{14}^3 \\ x_{12}^3 & x_{11}^3 & x_5^3 & x_{15}^3 \end{pmatrix}, \begin{pmatrix} x_5^2 & x_6^2 & x_{11}^2 & x_{12}^2 \\ x_5^1 & x_6^1 & x_2^2 & x_{12}^3 \end{pmatrix},$$
$$\begin{pmatrix} x_1^2 & x_{13}^2 & x_2^2 & x_{14}^2 \\ x_1^1 & x_6^2 & x_2^1 & x_5^2 \end{pmatrix}, \begin{pmatrix} x_5^1 & x_9^1 & x_6^1 & x_{10}^1 \\ x_5^3 & x_{12}^1 & x_8^1 & x_{11}^1 \end{pmatrix}, \begin{pmatrix} x_1^1 & x_7^1 & x_2^1 & x_8^1 \\ x_4^1 & x_7^3 & x_2^3 & x_8^2 \end{pmatrix}, \begin{pmatrix} x_1^2 & x_7^2 & x_2^2 & x_8^2 \\ x_4^2 & x_7^3 & x_2^3 & x_8^3 \end{pmatrix},$$
$$\begin{pmatrix} x_1^3 & x_7^3 & x_2^3 & x_8^3 \\ x_4^3 & x_{16}^3 & x_9^3 & x_{13}^3 \end{pmatrix}, \begin{pmatrix} x_1^1 & x_5^1 & x_2^1 & x_6^1 \\ x_1^3 & x_8^1 & x_{12}^1 & x_{16}^1 \end{pmatrix}, \begin{pmatrix} x_1^2 & x_5^2 & x_2^2 & x_6^2 \\ x_1^3 & x_8^2 & x_{12}^2 & x_6^3 \end{pmatrix}, \begin{pmatrix} x_1^3 & x_5^3 & x_2^3 & x_6^3 \\ x_3^3 & x_8^3 & x_{12}^3 & x_6^1 \end{pmatrix},$$
$$\begin{pmatrix} x_3^1 & x_{12}^1 & x_{12}^2 & x_3^2 \\ x_1^1 & x_{12}^3 & x_{15}^2 & x_1^2 \end{pmatrix}, \begin{pmatrix} x_{14}^1 & x_{15}^1 & x_{15}^2 & x_{14}^2 \\ x_4^1 & x_{15}^3 & x_5^2 & x_4^2 \end{pmatrix}, \begin{pmatrix} x_9^1 & x_{11}^1 & x_{11}^2 & x_9^2 \\ x_2^1 & x_{11}^3 & x_1^2 & x_2^2 \end{pmatrix}, \begin{pmatrix} x_{10}^2 & x_{11}^2 & x_{11}^3 & x_{10}^3 \\ x_2^2 & x_4^2 & x_5^3 & x_8^3 \end{pmatrix},$$
$$\begin{pmatrix} x_7^1 & x_{16}^1 & x_{16}^2 & x_7^2 \\ x_5^1 & x_{16}^3 & x_{12}^2 & x_6^2 \end{pmatrix}, \begin{pmatrix} x_{13}^2 & x_{16}^2 & x_{16}^3 & x_{13}^3 \\ x_8^2 & x_6^2 & x_6^3 & x_8^3 \end{pmatrix}.$$

3.3.3. An L_8- decomposition of $K_{16}\Box K_{11}$

$$\begin{pmatrix} x_5^j & x_6^j & x_{11}^j & x_{12}^j \\ x_{15}^j & x_7^j & x_2^j & x_{16}^j \end{pmatrix}, \begin{pmatrix} x_7^j & x_8^j & x_9^j & x_{10}^j \\ x_{13}^j & x_{15}^j & x_{11}^j & x_1^j \end{pmatrix} \text{ for } j = 1, 2, 4, 5, 7, 8, 10;$$
$$\begin{pmatrix} x_1^j & x_{13}^j & x_2^j & x_{14}^j \\ x_{11}^j & x_6^j & x_{10}^j & x_5^j \end{pmatrix}, \begin{pmatrix} x_3^j & x_{15}^j & x_4^j & x_{16}^j \\ x_{11}^j & x_1^j & x_{10}^j & x_2^j \end{pmatrix} \text{ for } j = 2, 5, 6, 7, 8, 9;$$
$$\begin{pmatrix} x_1^j & x_7^j & x_2^j & x_8^j \\ x_4^j & x_{16}^j & x_9^j & x_{13}^j \end{pmatrix}, \begin{pmatrix} x_1^j & x_5^j & x_2^j & x_6^j \\ x_3^j & x_8^j & x_{12}^j & x_{16}^j \end{pmatrix} \text{ for } j = 1, 2, 4, 5, ..., 10, 11;$$
$$\begin{pmatrix} x_3^j & x_7^j & x_4^j & x_8^j \\ x_{13}^j & x_{15}^j & x_{14}^j & x_{16}^j \end{pmatrix}, \begin{pmatrix} x_9^j & x_{13}^j & x_{10}^j & x_{14}^j \\ x_3^j & x_{16}^j & x_{12}^j & x_7^j \end{pmatrix} \text{ for } j = 1, 2, 3, 5, ..., 11;$$
$$\begin{pmatrix} x_i^3 & x_i^7 & x_i^4 & x_i^{10} \\ x_i^{11} & x_i^6 & x_i^8 & x_i^9 \end{pmatrix}, \begin{pmatrix} x_i^2 & x_i^5 & x_i^7 & x_i^{11} \\ x_i^3 & x_i^9 & x_i^8 & x_i^{10} \end{pmatrix} \text{ for } i = 1, 2, 5, 6, 7, 8, 10, 11, 12, 14;$$
$$\begin{pmatrix} x_i^2 & x_i^5 & x_i^7 & x_i^{11} \\ x_i^1 & x_i^9 & x_i^8 & x_i^{10} \end{pmatrix}, \begin{pmatrix} x_i^1 & x_i^6 & x_i^2 & x_i^7 \\ x_i^3 & x_i^5 & x_i^4 & x_i^{10} \end{pmatrix} \text{ for } i = 15, 16;$$
$$\begin{pmatrix} x_i^3 & x_i^4 & x_i^9 & x_i^8 \\ x_i^5 & x_i^6 & x_i^{11} & x_i^{10} \end{pmatrix}, \begin{pmatrix} x_i^3 & x_i^7 & x_i^4 & x_i^{10} \\ x_i^{11} & x_i^6 & x_i^8 & x_k^9 \end{pmatrix} \text{ for } i = 15, 16;$$
$$\begin{pmatrix} x_1^j & x_2^j & x_{15}^j & x_{16}^j \\ x_1^k & x_2^k & x_6^j & x_5^j \end{pmatrix}, \begin{pmatrix} x_3^j & x_4^j & x_{13}^j & x_{14}^j \\ x_3^k & x_4^k & x_{13}^k & x_{14}^k \end{pmatrix} \text{ for } (j, k) = (6, 3), (9, 3), (11, 8);$$
$$\begin{pmatrix} x_5^j & x_6^j & x_{11}^j & x_{12}^j \\ x_5^k & x_6^k & x_{11}^k & x_{12}^k \end{pmatrix}, \begin{pmatrix} x_7^j & x_8^j & x_9^j & x_{10}^j \\ x_7^k & x_8^k & x_9^k & x_{10}^k \end{pmatrix} \text{ for } (j, k) = (6, 3), (9, 3), (11, 8);$$
$$\begin{pmatrix} x_1^j & x_2^j & x_{15}^j & x_{16}^j \\ x_{12}^j & x_4^j & x_6^j & x_5^j \end{pmatrix} \text{ for } j = 2, 5, 7, 8, 10; \begin{pmatrix} x_3^j & x_5^j & x_4^j & x_6^j \\ x_2^j & x_{11}^j & x_9^j & x_{14}^j \end{pmatrix} \text{ for } j = 1, 2, 4, 5, ..., 10, 11;$$

$\begin{pmatrix} x_5^j & x_9^j & x_6^j & x_{10}^j \\ x_7^j & x_{12}^j & x_8^j & x_{11}^j \end{pmatrix}$ for $j = 2, 3, ..., 10, 11$; $\begin{pmatrix} x_9^j & x_{15}^j & x_{10}^j & x_{16}^j \\ x_1^j & x_{12}^j & x_3^j & x_{14}^j \end{pmatrix}$ for $j = 6, 7, 8, 9$;
$\begin{pmatrix} x_3^j & x_4^j & x_{13}^j & x_{14}^j \\ x_{12}^j & x_{11}^j & x_5^j & x_{15}^j \end{pmatrix}$ for $j = 1, 2, 3, 4, 5, 7, 8, 10$;
$\begin{pmatrix} x_i^1 & x_i^8 & x_i^2 & x_i^9 \\ x_i^4 & x_i^6 & x_i^{10} & x_i^7 \end{pmatrix}$ for $i = 1, 2, 5, 6, 12, 14, 15, 16$;
$\begin{pmatrix} x_7^j & x_{11}^j & x_8^j & x_{12}^j \\ x_9^j & x_{15}^j & x_{10}^j & x_6^j \end{pmatrix}$ for $j = 1, 2, 4, 5, ..., 11$; $\begin{pmatrix} x_i^5 & x_i^{10} & x_i^6 & x_i^{11} \\ x_k^5 & x_i^1 & x_i^9 & x_i^8 \end{pmatrix}$ for $(i.k) = (15, 12), (16, 14)$;
$\begin{pmatrix} x_9^j & x_{15}^j & x_{10}^j & x_{16}^j \\ x_9^k & x_{12}^j & x_3^j & x_{14}^j \end{pmatrix}$ for $(j, k) = (1, 10), (4, 1), (10, 11), (11, 3)$;
$\begin{pmatrix} x_3^j & x_{15}^j & x_4^j & x_{16}^j \\ x_3^k & x_1^j & x_4^k & x_2^j \end{pmatrix}$ for $(j, k) = (10, 11), (11, 3)$; $\begin{pmatrix} x_{11}^j & x_{13}^j & x_{12}^j & x_{14}^j \\ x_{16}^j & x_{15}^j & x_4^j & x_8^j \end{pmatrix}$ for $j = 1, 2, 3, 5, ..., 11$;
$\begin{pmatrix} x_1^j & x_2^j & x_{15}^j & x_{16}^j \\ x_{12}^j & x_4^j & x_{15}^k & x_{16}^k \end{pmatrix}$ for $(j, k) = (1, 11), (3, 9), (4, 5)$;
$\begin{pmatrix} x_1^j & x_{13}^j & x_2^j & x_{14}^j \\ x_{11}^j & x_{13}^k & x_{10}^j & x_5^j \end{pmatrix}$ for $(j, k) = (4, 1), (10, 11), (11, 3)$;
$\begin{pmatrix} x_3^j & x_{15}^j & x_4^j & x_{16}^j \\ x_3^{k_1} & x_{15}^{k_2} & x_4^{k_1} & x_{16}^{k_2} \end{pmatrix}$ for $(j, k_1, k_2) = (1, 10, 5), (4, 1, 11)$;
$\begin{pmatrix} x_i^1 & x_i^6 & x_i^2 & x_i^7 \\ x_i^3 & x_k^6 & x_i^4 & x_i^{10} \end{pmatrix}$ for $(i, k) = (1, 12), (2, 4), (3, 12), (4, 11), (5, 15), (6, 7), (7, 13), (8, 15),$
$(9, 11), (10, 1), (11, 2), (13, 5)$;
$\begin{pmatrix} x_i^1 & x_i^6 & x_i^2 & x_i^7 \\ x_i^3 & x_{k_1}^6 & x_{k_2}^2 & x_i^{10} \end{pmatrix}$ for $(i, k_1, k_2) = (12, 16, 15), (14, 15, 16)$;
$\begin{pmatrix} x_i^3 & x_i^4 & x_i^9 & x_i^8 \\ x_i^5 & x_i^6 & x_k^9 & x_i^{10} \end{pmatrix}$ for $(i, k) = (1, 12), (2, 4), (3, 12), (4, 11), (5, 15), (6, 7), (7, 13), (8, 15),$
$(9, 11), (10, 1), (11, 2), (12, 16), (13, 5), (14, 15)$;
$\begin{pmatrix} x_i^5 & x_i^{10} & x_i^6 & x_i^{11} \\ x_i^8 & x_i^1 & x_i^9 & x_k^{11} \end{pmatrix}$ for $(i, k) = (1, 12), (2, 4), (5, 15), (6, 7), (7, 13), (8, 15), (10, 1), (11, 2),$
$(12, 16), (14, 15)$;
$\begin{pmatrix} x_i^5 & x_i^{10} & x_i^6 & x_i^{11} \\ x_i^8 & x_{k_1}^{10} & x_i^9 & x_{k_2}^{11} \end{pmatrix}$ for $(i, k_1, k_2) = (3, 11, 12), (4, 10, 11), (9, 1, 11), (13, 6, 5)$;
$\begin{pmatrix} x_i^1 & x_i^8 & x_i^2 & x_i^9 \\ x_k^1 & x_i^6 & x_i^{10} & x_i^7 \end{pmatrix}$ for $(i, k) = (3, 11), (4, 10), (7, 5), (8, 6), (9, 1), (10, 2), (11, 1), (13, 6)$;
$\begin{pmatrix} x_i^3 & x_i^7 & x_i^4 & x_i^{10} \\ x_k^3 & x_i^6 & x_i^8 & x_i^9 \end{pmatrix}$ for $(i, k) = (3, 10), (4, 1), (9, 2), (13, 7)$;
$\begin{pmatrix} x_i^2 & x_i^5 & x_i^7 & x_i^{11} \\ x_i^3 & x_i^9 & x_i^8 & x_k^{11} \end{pmatrix}$ for $(i, k) = (3, 11), (4, 10), (9, 1), (13, 6)$;
$\begin{pmatrix} x_i^1 & x_i^5 & x_i^4 & x_i^{11} \\ x_i^2 & x_i^6 & x_k^4 & x_i^9 \end{pmatrix}$ for $(i, k) = (1, 15), (2, 16), (3, 11), (4, 10), (5, 16), (6, 15), (7, 15),$
$(8, 16), (9, 1), (10, 12), (11, 16), (12, 4), (13, 6), (14, 8)$;
$\begin{pmatrix} x_5^3 & x_6^3 & x_{11}^3 & x_{12}^3 \\ x_{15}^3 & x_{14}^3 & x_2^3 & x_{16}^3 \end{pmatrix}, \begin{pmatrix} x_7^3 & x_8^3 & x_9^3 & x_{10}^3 \\ x_6^3 & x_{15}^3 & x_{11}^3 & x_1^3 \end{pmatrix}, \begin{pmatrix} x_1^1 & x_{13}^1 & x_2^1 & x_{14}^1 \\ x_{15}^1 & x_{13}^{10} & x_{16}^1 & x_5^1 \end{pmatrix}, \begin{pmatrix} x_1^3 & x_{13}^3 & x_2^3 & x_{14}^3 \\ x_{11}^3 & x_6^3 & x_{16}^3 & x_5^3 \end{pmatrix},$
$\begin{pmatrix} x_3^3 & x_{15}^3 & x_4^3 & x_{16}^3 \\ x_{11}^3 & x_{15}^6 & x_{10}^3 & x_{16}^6 \end{pmatrix}, \begin{pmatrix} x_7^3 & x_{11}^3 & x_8^3 & x_{12}^3 \\ x_9^3 & x_{15}^3 & x_5^3 & x_6^3 \end{pmatrix}, \begin{pmatrix} x_5^1 & x_9^1 & x_6^1 & x_{10}^1 \\ x_{16}^1 & x_{12}^1 & x_{15}^1 & x_{11}^1 \end{pmatrix}, \begin{pmatrix} x_3^3 & x_5^3 & x_4^3 & x_6^3 \\ x_2^3 & x_{11}^3 & x_9^3 & x_{16}^3 \end{pmatrix},$
$\begin{pmatrix} x_1^3 & x_7^3 & x_2^3 & x_8^3 \\ x_{15}^3 & x_{16}^3 & x_{10}^3 & x_{13}^3 \end{pmatrix}, \begin{pmatrix} x_9^2 & x_{15}^2 & x_{10}^2 & x_{16}^2 \\ x_1^2 & x_{15}^3 & x_3^2 & x_{16}^3 \end{pmatrix}, \begin{pmatrix} x_9^3 & x_{15}^3 & x_{10}^3 & x_{16}^3 \\ x_1^3 & x_{12}^3 & x_8^3 & x_{14}^3 \end{pmatrix}, \begin{pmatrix} x_{11}^4 & x_{13}^4 & x_{12}^4 & x_{14}^4 \\ x_{11}^1 & x_{15}^4 & x_{12}^2 & x_{14}^2 \end{pmatrix},$
$\begin{pmatrix} x_3^4 & x_7^4 & x_4^4 & x_8^4 \\ x_{13}^4 & x_7^1 & x_{14}^4 & x_8^1 \end{pmatrix}, \begin{pmatrix} x_1^3 & x_5^3 & x_2^3 & x_6^3 \\ x_3^3 & x_{16}^3 & x_{12}^3 & x_{15}^3 \end{pmatrix}, \begin{pmatrix} x_9^4 & x_{13}^4 & x_{10}^4 & x_{14}^4 \\ x_3^4 & x_{16}^4 & x_{10}^1 & x_7^4 \end{pmatrix}, \begin{pmatrix} x_9^5 & x_{15}^5 & x_{10}^5 & x_{16}^5 \\ x_1^5 & x_{15}^8 & x_3^5 & x_{16}^8 \end{pmatrix}.$

3.3.4. An L_8- decomposition of $K_{16} \square K_5$

$\begin{pmatrix} x_1^j & x_2^j & x_{15}^j & x_{16}^j \\ x_1^5 & x_2^5 & x_{15}^5 & x_{16}^5 \end{pmatrix}, \begin{pmatrix} x_3^j & x_4^j & x_{13}^j & x_{14}^j \\ x_3^5 & x_4^5 & x_{13}^5 & x_{14}^5 \end{pmatrix}$ for $j = 1, 2, 3, 4$;
$\begin{pmatrix} x_9^j & x_{13}^j & x_{10}^j & x_{14}^j \\ x_3^j & x_{16}^j & x_{12}^j & x_7^j \end{pmatrix}, \begin{pmatrix} x_5^j & x_9^j & x_6^j & x_{10}^j \\ x_7^j & x_{12}^j & x_8^j & x_{11}^j \end{pmatrix}$ for $j = 1, 2, 5$;

$\begin{pmatrix} x_5^j & x_6^j & x_{11}^j & x_{12}^j \\ x_5^5 & x_6^5 & x_{11}^5 & x_{12}^5 \end{pmatrix}, \begin{pmatrix} x_7^j & x_8^j & x_9^j & x_{10}^j \\ x_7^5 & x_8^5 & x_9^5 & x_{10}^5 \end{pmatrix}$ for $j = 1, 2, 3, 4$;
$\begin{pmatrix} x_{12}^j & x_{15}^j & x_{15}^k & x_{12}^k \\ x_2^j & x_8^j & x_{14}^k & x_9^k \end{pmatrix}, \begin{pmatrix} x_{11}^j & x_{16}^j & x_{16}^k & x_{11}^k \\ x_4^j & x_6^j & x_{13}^k & x_{10}^k \end{pmatrix}$ for $(j, k) = (1, 3), (2, 4)$;
$\begin{pmatrix} x_7^j & x_{11}^j & x_8^j & x_{12}^j \\ x_9^j & x_{15}^j & x_{10}^j & x_6^j \end{pmatrix}, \begin{pmatrix} x_1^j & x_7^j & x_2^j & x_8^j \\ x_4^j & x_{16}^j & x_9^j & x_{13}^j \end{pmatrix}$ for $j = 1, 2, 3, 4, 5$;
$\begin{pmatrix} x_1^j & x_5^j & x_2^j & x_6^j \\ x_3^j & x_8^j & x_{12}^j & x_{16}^j \end{pmatrix}$ for $j = 3, 4, 5$; $\begin{pmatrix} x_3^j & x_7^j & x_4^j & x_8^j \\ x_{13}^j & x_7^k & x_{14}^j & x_{16}^j \end{pmatrix}$ for $(j, k) = (3, 1), (4, 2)$;
$\begin{pmatrix} x_1^j & x_{13}^j & x_2^j & x_{14}^j \\ x_{k_1}^{k_2} & x_6^j & x_{10}^j & x_5^j \end{pmatrix}$ for $j = k_2 = 3, 4, 5, k_1 = 11 \& (j, k_1, k_2) = (1, 1, 3), (2, 1, 4)$;
$\begin{pmatrix} x_3^j & x_{15}^j & x_4^j & x_{16}^j \\ x_{k_1}^{k_2} & x_1^j & x_{10}^j & x_2^j \end{pmatrix}$ for $j = k_2 = 1, 2, 5, k_1 = 11 \& (j, k_1, k_2) = (3, 3, 1), (4, 3, 2)$;
$\begin{pmatrix} x_3^j & x_5^j & x_4^j & x_6^j \\ x_2^j & x_{11}^j & x_{k_1}^{k_2} & x_{14}^j \end{pmatrix}$ for $j = k_2 = 3, 4, 5, k_1 = 9 \& (j, k_1, k_2) = (1, 4, 3), (2, 4, 4)$;
$\begin{pmatrix} x_9^j & x_{15}^j & x_{10}^j & x_{16}^j \\ x_1^j & x_k^j & x_3^j & x_{14}^j \end{pmatrix}$ for $(j, k) = (1, 5), (2, 5), (3, 7), (4, 7)$;
$\begin{pmatrix} x_{11}^j & x_{13}^j & x_{12}^j & x_{14}^j \\ x_k^j & x_{15}^j & x_4^j & x_8^j \end{pmatrix}$ for $(j, k) = (1, 1), (2, 1), (3, 3), (4, 3)$;
$\begin{pmatrix} x_3^j & x_7^j & x_4^j & x_8^j \\ x_{13}^j & x_{15}^j & x_{14}^j & x_8^k \end{pmatrix}$ for $(j, k) = (1, 3), (2, 4)$;
$\begin{pmatrix} x_i^1 & x_i^2 & x_i^3 & x_i^4 \\ x_k^1 & x_k^2 & x_k^3 & x_k^4 \end{pmatrix}$ for $(i, k) = (1, 12), (2, 4), (3, 12), (6, 7)(7, 13), (9, 11), (10, 1),$
$(11, 2), (12, 16), (13, 5), (15, 6), (16, 5)$;
$\begin{pmatrix} x_i^1 & x_i^2 & x_i^3 & x_i^4 \\ x_{k_1}^1 & x_{k_1}^2 & x_{k_2}^3 & x_{k_2}^4 \end{pmatrix}$ for $(i, k_1, k_2) = (4, 9, 11), (5, 8, 15), (8, 16, 15), (14, 15, 7)$;
$\begin{pmatrix} x_1^5 & x_2^5 & x_{15}^5 & x_{16}^5 \\ x_{12}^5 & x_4^5 & x_6^5 & x_5^5 \end{pmatrix}, \begin{pmatrix} x_3^5 & x_4^5 & x_{13}^5 & x_{14}^5 \\ x_{12}^5 & x_{11}^5 & x_5^5 & x_{15}^5 \end{pmatrix}, \begin{pmatrix} x_5^5 & x_6^5 & x_{11}^5 & x_{12}^5 \\ x_{15}^5 & x_7^5 & x_2^5 & x_{16}^5 \end{pmatrix}, \begin{pmatrix} x_7^5 & x_8^5 & x_9^5 & x_{10}^5 \\ x_{13}^5 & x_{15}^5 & x_{11}^5 & x_1^5 \end{pmatrix},$
$\begin{pmatrix} x_5^3 & x_9^3 & x_6^3 & x_{10}^3 \\ x_7^3 & x_9^1 & x_8^3 & x_{10}^1 \end{pmatrix}, \begin{pmatrix} x_5^4 & x_9^4 & x_6^4 & x_{10}^4 \\ x_7^4 & x_9^2 & x_8^4 & x_{10}^2 \end{pmatrix}, \begin{pmatrix} x_9^5 & x_{15}^5 & x_{10}^5 & x_{16}^5 \\ x_1^5 & x_{12}^5 & x_3^5 & x_{14}^5 \end{pmatrix}, \begin{pmatrix} x_{11}^5 & x_{13}^5 & x_{12}^5 & x_{14}^5 \\ x_{16}^5 & x_{15}^5 & x_4^5 & x_8^5 \end{pmatrix},$
$\begin{pmatrix} x_3^5 & x_7^5 & x_4^5 & x_8^5 \\ x_{13}^5 & x_{15}^5 & x_{14}^5 & x_{16}^5 \end{pmatrix}, \begin{pmatrix} x_1^1 & x_5^1 & x_2^1 & x_6^1 \\ x_3^1 & x_5^3 & x_2^3 & x_6^3 \end{pmatrix}, \begin{pmatrix} x_1^2 & x_5^2 & x_2^2 & x_6^2 \\ x_3^2 & x_5^4 & x_2^4 & x_6^4 \end{pmatrix}, \begin{pmatrix} x_9^3 & x_{13}^3 & x_{10}^3 & x_{14}^3 \\ x_3^3 & x_{13}^1 & x_{12}^3 & x_{14}^1 \end{pmatrix},$
$\begin{pmatrix} x_9^4 & x_{13}^4 & x_{10}^4 & x_{14}^4 \\ x_3^4 & x_{13}^2 & x_{12}^4 & x_{14}^2 \end{pmatrix}.$

3.3.5. An L_8- decomposition of $K_{16} \square K_{13}$

$\begin{pmatrix} x_1^j & x_2^j & x_{15}^j & x_{16}^j \\ x_{12}^j & x_4^j & x_6^j & x_5^j \end{pmatrix}, \begin{pmatrix} x_3^j & x_5^j & x_4^j & x_6^j \\ x_2^j & x_{11}^j & x_9^j & x_{14}^j \end{pmatrix}$, for $j = 2, \ldots, 8, 10, \ldots, 13$;
$\begin{pmatrix} x_5^j & x_6^j & x_{11}^j & x_{12}^j \\ x_{15}^j & x_7^j & x_2^j & x_{16}^j \end{pmatrix}, \begin{pmatrix} x_7^j & x_8^j & x_9^j & x_{10}^j \\ x_{13}^j & x_{15}^j & x_{11}^j & x_1^j \end{pmatrix}$ for $j = 1, \ldots, 13$;
$\begin{pmatrix} x_1^j & x_7^j & x_2^j & x_8^j \\ x_4^j & x_{16}^j & x_9^j & x_{13}^j \end{pmatrix}, \begin{pmatrix} x_5^j & x_9^j & x_6^j & x_{10}^j \\ x_7^j & x_{12}^j & x_8^j & x_{11}^j \end{pmatrix}$ for $j = 1, \ldots, 13$;
$\begin{pmatrix} x_1^j & x_{13}^j & x_2^j & x_{14}^j \\ x_{11}^j & x_6^j & x_{10}^j & x_5^j \end{pmatrix}, \begin{pmatrix} x_3^j & x_{15}^j & x_4^j & x_{16}^j \\ x_{11}^j & x_1^j & x_{10}^j & x_2^j \end{pmatrix}$ for $j = 2, \ldots, 6, 8, 10, 12, 13$;
$\begin{pmatrix} x_9^j & x_{15}^j & x_{10}^j & x_{16}^j \\ x_1^j & x_{12}^j & x_3^j & x_{14}^j \end{pmatrix}, \begin{pmatrix} x_{11}^j & x_{13}^j & x_{12}^j & x_{14}^j \\ x_{16}^j & x_{15}^j & x_4^j & x_8^j \end{pmatrix}$ for $j = 2, \ldots 6, 8, 10, 12, 13$;
$\begin{pmatrix} x_7^j & x_{11}^j & x_8^j & x_{12}^j \\ x_9^j & x_{15}^j & x_{10}^j & x_6^j \end{pmatrix}, \begin{pmatrix} x_1^j & x_5^j & x_2^j & x_6^j \\ x_3^j & x_8^j & x_{12}^j & x_{16}^j \end{pmatrix}$ for $j = 1, \ldots, 6, 8, 9, 10, 12, 13$;
, $\begin{pmatrix} x_i^1 & x_i^7 & x_i^2 & x_i^8 \\ x_i^5 & x_i^4 & x_i^6 & x_i^3 \end{pmatrix}, \begin{pmatrix} x_i^4 & x_i^5 & x_i^7 & x_i^6 \\ x_i^2 & x_i^3 & x_i^{10} & x_i^1 \end{pmatrix}$ for $i = 1, 2, ..., 16$;
$\begin{pmatrix} x_i^1 & x_i^2 & x_i^{10} & x_i^9 \\ x_i^4 & x_i^{12} & x_i^6 & x_i^5 \end{pmatrix}, \begin{pmatrix} x_i^5 & x_i^6 & x_i^{13} & x_i^{10} \\ x_i^2 & x_i^3 & x_i^{11} & x_i^{12} \end{pmatrix}$, for $i = 1, 2, ..., 16$;
$\begin{pmatrix} x_7^j & x_{11}^j & x_8^j & x_{12}^j \\ x_{14}^j & x_{15}^j & x_{10}^j & x_6^j \end{pmatrix}, \begin{pmatrix} x_1^j & x_{13}^j & x_2^j & x_{14}^j \\ x_{11}^j & x_6^j & x_{16}^j & x_5^j \end{pmatrix}$, for $j = 7, 11$;

$\begin{pmatrix} x_i^3 & x_i^4 & x_i^8 & x_i^7 \\ x_i^{13} & x_i^{11} & x_i^5 & x_i^9 \end{pmatrix}$, $\begin{pmatrix} x_i^2 & x_i^3 & x_i^{12} & x_i^{11} \\ x_i^{13} & x_i^1 & x_i^8 & x_i^{10} \end{pmatrix}$, for $i = 1, 2, 5, 6, 7, 8, 11, \dots, 16$;
$\begin{pmatrix} x_3^j & x_4^j & x_{13}^j & x_{14}^j \\ x_3^{k_1} & x_4^{k_1} & x_{13}^{k_2} & x_{15}^j \end{pmatrix}$, $\begin{pmatrix} x_9^j & x_{13}^j & x_{10}^j & x_{14}^j \\ x_9^{k_1} & x_{16}^j & x_{10}^{k_1} & x_{14}^{k_2} \end{pmatrix}$ for $(j, k_1, k_2) = (1, 10, 12), (9, 7, 13)$;
$\begin{pmatrix} x_3^j & x_{15}^j & x_4^j & x_{16}^j \\ x_{11}^j & x_{13}^j & x_{10}^j & x_{16}^k \end{pmatrix}$, $\begin{pmatrix} x_{11}^j & x_{13}^j & x_{12}^j & x_{14}^j \\ x_{16}^j & x_{13}^k & x_{15}^j & x_8^j \end{pmatrix}$ for $(j, k) = (7, 13), (11, 8)$;
$\begin{pmatrix} x_i^3 & x_i^4 & x_i^8 & x_i^7 \\ x_i^{13} & x_i^{11} & x_i^5 & x_k^7 \end{pmatrix}$, $\begin{pmatrix} x_i^2 & x_i^3 & x_i^{12} & x_i^{11} \\ x_i^{13} & x_i^1 & x_i^8 & x_k^{11} \end{pmatrix}$ for $(i, k) = (3, 1), (4, 12), (9, 7), (10, 2)$;
$\begin{pmatrix} x_3^j & x_5^j & x_4^j & x_6^j \\ x_2^j & x_{13}^j & x_9^j & x_{14}^j \end{pmatrix}$ for $j = 1, 9$; $\begin{pmatrix} x_1^j & x_2^j & x_{15}^j & x_{16}^j \\ x_{12}^j & x_4^j & x_6^j & x_{16}^k \end{pmatrix}$ for $(j, k) = (1, 12), (9, 11)$;
$\begin{pmatrix} x_3^j & x_7^j & x_4^j & x_8^j \\ x_{13}^j & x_{15}^j & x_{14}^j & x_{16}^j \end{pmatrix}$ for $j = 1, \dots, 13$; $\begin{pmatrix} x_{11}^j & x_{13}^j & x_{12}^j & x_{14}^j \\ x_{16}^j & x_{15}^j & x_4^j & x_{14}^k \end{pmatrix}$ for $(j, k) = (1, 11), (9, 11)$;
$\begin{pmatrix} x_1^j & x_5^j & x_2^j & x_6^j \\ x_{15}^j & x_8^j & x_{12}^j & x_{16}^j \end{pmatrix}$ for $j = 7, 11$; $\begin{pmatrix} x_9^j & x_{13}^j & x_{10}^j & x_{14}^j \\ x_3^j & x_{16}^j & x_{12}^j & x_7^j \end{pmatrix}$ for $j = 2, \dots, 6, 8, 12, 13$;
$\begin{pmatrix} x_3^j & x_4^j & x_{13}^j & x_{14}^j \\ x_{12}^j & x_{11}^j & x_5^j & x_{15}^j \end{pmatrix}$ for $j = 2, \dots, 8, 11, 12, 13$;
$\begin{pmatrix} x_9^j & x_{13}^j & x_{10}^j & x_{14}^j \\ x_3^j & x_{16}^j & x_{12}^j & x_{14}^k \end{pmatrix}$ for $(j, k) = (7, 13), (11, 8)$;
$\begin{pmatrix} x_9^j & x_{15}^j & x_{10}^j & x_{16}^j \\ x_1^j & x_{15}^k & x_3^j & x_{14}^j \end{pmatrix}$ for $(j, k) = (1, 12), (7, 13), (9, 13), (11, 8)$;
$\begin{pmatrix} x_i^3 & x_i^9 & x_i^4 & x_i^{10} \\ x_i^{11} & x_i^2 & x_i^k & x_i^1 \end{pmatrix}$ for $i = 1, 2, 5, 6, 7, 8, 11, 12,\ k = 12$ & $i = 13, 14, 15, 16$ $k = 13$;
$\begin{pmatrix} x_i^3 & x_i^9 & x_i^4 & x_i^{10} \\ x_i^{11} & x_i^2 & x_i^{12} & x_k^{10} \end{pmatrix}$ for $(i, k) = (3, 12), (4, 11), (9, 3), (10, 12)$;
$\begin{pmatrix} x_i^5 & x_i^{11} & x_i^6 & x_i^{12} \\ x_i^{13} & x_i^7 & x_i^8 & x_i^k \end{pmatrix}$ for $i = 1, 2, ..., 12,\ k = 1$ & $i = 13, 14, 15, 16$ $k = 4$;
$\begin{pmatrix} x_i^8 & x_i^9 & x_i^{12} & x_i^{13} \\ x_i^{10} & x_i^6 & x_i^7 & x_i^k \end{pmatrix}$ for $i = 1, 2, ..., 12,\ k = 4$ & $i = 13, 14, 15, 16$ $k = 1$;
$\begin{pmatrix} x_i^1 & x_i^{11} & x_i^9 & x_i^{13} \\ x_k^1 & x_i^8 & x_k^9 & x_i^7 \end{pmatrix}$ for $(i, k) = (1, 15), (2, 16), (3, 12), (4, 11), (5, 16), (6, 13), (7, 14),$
$(8, 14), (9, 3), (10, 12), (11, 5), (12, 15)$;
$\begin{pmatrix} x_3^{10} & x_4^{10} & x_{13}^{10} & x_{14}^{10} \\ x_3^{11} & x_4^{11} & x_5^{10} & x_{15}^{10} \end{pmatrix}$, $\begin{pmatrix} x_1^1 & x_{13}^1 & x_2^1 & x_{14}^1 \\ x_{11}^1 & x_{13}^{11} & x_{10}^1 & x_5^1 \end{pmatrix}$, $\begin{pmatrix} x_1^9 & x_{13}^9 & x_2^9 & x_{14}^9 \\ x_{11}^9 & x_{13}^{11} & x_{10}^9 & x_5^9 \end{pmatrix}$, $\begin{pmatrix} x_3^1 & x_{15}^1 & x_4^1 & x_{16}^1 \\ x_{11}^1 & x_{15}^{11} & x_{10}^1 & x_{16}^{11} \end{pmatrix}$,
$\begin{pmatrix} x_3^9 & x_{15}^9 & x_4^9 & x_{16}^9 \\ x_{11}^9 & x_{15}^{11} & x_{10}^9 & x_{16}^{13} \end{pmatrix}$, $\begin{pmatrix} x_9^{10} & x_{13}^{10} & x_{10}^{10} & x_{14}^{10} \\ x_9^{11} & x_{16}^{10} & x_{10}^{11} & x_7^{10} \end{pmatrix}$.

3.3.6. An L_8- decomposition of $K_{16} \Box K_7$

$\begin{pmatrix} x_1^j & x_2^j & x_{15}^j & x_{16}^j \\ x_1^k & x_2^k & x_{15}^k & x_{16}^k \end{pmatrix}$, $\begin{pmatrix} x_3^j & x_4^j & x_{13}^j & x_{14}^j \\ x_3^k & x_4^k & x_{13}^k & x_{14}^k \end{pmatrix}$ for $(j, k) = (1, 4), (2, 3), (3, 6), (4, 3), (5, 6),$
$(6, 2), (7, 6)$;
$\begin{pmatrix} x_1^j & x_{13}^j & x_2^j & x_{14}^j \\ x_{11}^j & x_6^j & x_{10}^j & x_5^j \end{pmatrix}$, $\begin{pmatrix} x_1^j & x_7^j & x_2^j & x_8^j \\ x_4^j & x_{16}^j & x_9^j & x_{13}^j \end{pmatrix}$ for $j = 1, 3, 6, 7$;
$\begin{pmatrix} x_1^j & x_7^j & x_2^j & x_8^j \\ x_4^j & x_{16}^j & x_{11}^j & x_{13}^j \end{pmatrix}$ for $j = 4, 5$; $\begin{pmatrix} x_7^j & x_{11}^j & x_8^j & x_{12}^j \\ x_9^j & x_{15}^j & x_{10}^j & x_6^j \end{pmatrix}$ for $j = 1, 3, 4, 6$;
$\begin{pmatrix} x_9^j & x_{15}^j & x_{10}^j & x_{16}^j \\ x_1^j & x_{12}^j & x_3^j & x_{14}^j \end{pmatrix}$ for $j = 1, 6$; $\begin{pmatrix} x_5^j & x_9^j & x_6^j & x_{10}^j \\ x_7^j & x_{12}^j & x_8^j & x_{11}^j \end{pmatrix}$ for $j = 1, 4, 6, 7$;
$\begin{pmatrix} x_{11}^j & x_{13}^j & x_{12}^j & x_{14}^j \\ x_{16}^j & x_{15}^j & x_4^j & x_8^j \end{pmatrix}$ for $j = 1, 6$; $\begin{pmatrix} x_3^j & x_7^j & x_4^j & x_8^j \\ x_{13}^j & x_{15}^j & x_{14}^j & x_{16}^j \end{pmatrix}$ for $j = 1, 2, 4, 5, 6$;
$\begin{pmatrix} x_3^j & x_7^j & x_4^j & x_8^j \\ x_{10}^j & x_{15}^j & x_{14}^j & x_{16}^j \end{pmatrix}$ for $j = 3, 7$; $\begin{pmatrix} x_1^j & x_5^j & x_2^j & x_6^j \\ x_3^j & x_8^j & x_{12}^j & x_{16}^j \end{pmatrix}$ for $j = 1, 3, 4, 5, 6, 7$;
$\begin{pmatrix} x_3^j & x_{15}^j & x_4^j & x_{16}^j \\ x_{11}^j & x_1^j & x_{10}^j & x_2^j \end{pmatrix}$ for $j = 1, 6, 7$; $\begin{pmatrix} x_1^j & x_{13}^j & x_2^j & x_{14}^j \\ x_{11}^j & x_{13}^k & x_{10}^j & x_5^j \end{pmatrix}$ for $(j, k) = (2, 5), (4, 7), (5, 4)$;
$\begin{pmatrix} x_3^j & x_5^j & x_4^j & x_6^j \\ x_2^j & x_{11}^j & x_9^j & x_{14}^j \end{pmatrix}$ for $j = 1, ..., 7$; $\begin{pmatrix} x_3^j & x_{15}^j & x_4^j & x_{16}^j \\ x_{11}^j & x_{15}^k & x_{10}^j & x_{16}^k \end{pmatrix}$ $(j, k) = (2, 5), (4, 7), (5, 4)$;

$\begin{pmatrix} x_9^j & x_{13}^j & x_{10}^j & x_{14}^j \\ x_3^j & x_{16}^j & x_{12}^j & x_7^j \end{pmatrix}$ for $j = 1, 3, 4, 5, 6, 7$;
$\begin{pmatrix} x_7^j & x_8^j & x_9^j & x_{10}^j \\ x_7^k & x_8^k & x_9^k & x_{10}^k \end{pmatrix}$ for $(j, k) = (1, 4), (5, 6), (6, 2), (7, 6)$;
$\begin{pmatrix} x_7^j & x_8^j & x_9^j & x_{10}^j \\ x_7^{k_1} & x_8^{k_2} & x_9^{k_2} & x_{10}^{k_2} \end{pmatrix}$ for $(j, k_1, k_2) = (2, 4, 3), (3, 4, 6), (4, 6, 3)$;
$\begin{pmatrix} x_9^j & x_{15}^j & x_{10}^j & x_{16}^j \\ x_1^j & x_{12}^j & x_{10}^k & x_{14}^j \end{pmatrix}$ for $(j, k) = (3, 7), (4, 7), (5, 2), (7, 2)$;
$\begin{pmatrix} x_5^j & x_6^j & x_{11}^j & x_{12}^j \\ x_5^k & x_6^k & x_{11}^k & x_{12}^k \end{pmatrix}$ for $(j, k) = (1, 4), (4, 3), (5, 6), (7, 6)$;
$\begin{pmatrix} x_i^1 & x_i^3 & x_i^5 & x_i^7 \\ x_k^1 & x_k^3 & x_k^5 & x_k^7 \end{pmatrix}$ for $(i, k) = (1, 12), (2, 4), (3, 12), (4, 11), (8, 15), (9, 11), (10, 1),$ $(12, 16), (13, 5)$;
$\begin{pmatrix} x_i^1 & x_i^2 & x_i^4 & x_i^6 \\ x_i^5 & x_k^2 & x_k^4 & x_k^6 \end{pmatrix}$ for $(i, k) = (1, 12), (2, 4), (3, 12), (4, 11), (5, 15), (8, 15), (12, 16),$ $(14, 15)$;
$\begin{pmatrix} x_i^2 & x_i^5 & x_i^4 & x_i^7 \\ x_k^2 & x_k^5 & x_k^4 & x_i^3 \end{pmatrix}$ for $(i, k) = (1, 15), (2, 16), (4, 12), (5, 16), (6, 13), (8, 14), (9, 2)$;
$\begin{pmatrix} x_5^2 & x_6^2 & x_{11}^2 & x_{12}^2 \\ x_7^2 & x_6^3 & x_{11}^3 & x_{12}^3 \end{pmatrix}, \begin{pmatrix} x_5^3 & x_6^3 & x_{11}^3 & x_{12}^3 \\ x_5^6 & x_6^6 & x_{11}^6 & x_{12}^7 \end{pmatrix}, \begin{pmatrix} x_5^6 & x_6^6 & x_{11}^6 & x_{12}^6 \\ x_5^2 & x_6^2 & x_{11}^2 & x_{12}^3 \end{pmatrix}, \begin{pmatrix} x_3^3 & x_{15}^3 & x_4^3 & x_{16}^3 \\ x_{11}^3 & x_1^3 & x_{10}^3 & x_{16}^7 \end{pmatrix},$
$\begin{pmatrix} x_7^2 & x_{11}^2 & x_8^2 & x_{12}^2 \\ x_{16}^2 & x_{11}^7 & x_6^2 & x_{12}^7 \end{pmatrix}, \begin{pmatrix} x_7^5 & x_{11}^5 & x_8^5 & x_{12}^5 \\ x_9^5 & x_{15}^5 & x_{10}^5 & x_{12}^2 \end{pmatrix}, \begin{pmatrix} x_7^7 & x_{11}^7 & x_8^7 & x_{12}^7 \\ x_9^7 & x_{11}^3 & x_{10}^7 & x_6^7 \end{pmatrix}, \begin{pmatrix} x_5^2 & x_9^2 & x_6^2 & x_{10}^2 \\ x_{13}^2 & x_{11}^2 & x_{12}^2 & x_1^2 \end{pmatrix},$
$\begin{pmatrix} x_5^3 & x_9^3 & x_6^3 & x_{10}^3 \\ x_{16}^3 & x_{12}^3 & x_8^3 & x_{11}^3 \end{pmatrix}, \begin{pmatrix} x_5^5 & x_9^5 & x_6^5 & x_{10}^5 \\ x_7^5 & x_{12}^5 & x_8^5 & x_{10}^4 \end{pmatrix}, \begin{pmatrix} x_1^2 & x_7^2 & x_2^2 & x_8^2 \\ x_4^2 & x_6^2 & x_{11}^2 & x_{10}^2 \end{pmatrix}, \begin{pmatrix} x_9^2 & x_{15}^2 & x_{10}^2 & x_{16}^2 \\ x_7^2 & x_{12}^2 & x_3^2 & x_{14}^2 \end{pmatrix},$
$\begin{pmatrix} x_{11}^2 & x_{13}^2 & x_{12}^2 & x_{14}^2 \\ x_{16}^2 & x_{15}^2 & x_{12}^6 & x_{14}^5 \end{pmatrix}, \begin{pmatrix} x_{11}^3 & x_{13}^3 & x_{12}^3 & x_{14}^3 \\ x_{16}^3 & x_7^3 & x_4^3 & x_{14}^7 \end{pmatrix}, \begin{pmatrix} x_{11}^4 & x_{13}^4 & x_{12}^4 & x_{14}^4 \\ x_{11}^5 & x_{15}^4 & x_{12}^7 & x_{14}^7 \end{pmatrix}, \begin{pmatrix} x_{11}^5 & x_{13}^5 & x_{12}^5 & x_{14}^5 \\ x_{10}^5 & x_{15}^5 & x_{12}^4 & x_{14}^4 \end{pmatrix},$
$\begin{pmatrix} x_{11}^7 & x_{13}^7 & x_{12}^7 & x_{14}^7 \\ x_{16}^7 & x_{13}^2 & x_4^7 & x_8^7 \end{pmatrix}, \begin{pmatrix} x_1^2 & x_5^2 & x_2^2 & x_6^2 \\ x_9^2 & x_8^2 & x_{12}^2 & x_{16}^2 \end{pmatrix}, \begin{pmatrix} x_9^2 & x_{13}^2 & x_{10}^2 & x_{14}^2 \\ x_3^2 & x_{16}^2 & x_{12}^2 & x_{14}^7 \end{pmatrix}, \begin{pmatrix} x_5^1 & x_5^3 & x_5^5 & x_5^7 \\ x_{15}^1 & x_5^2 & x_{15}^5 & x_{15}^7 \end{pmatrix},$
$\begin{pmatrix} x_6^1 & x_6^3 & x_6^5 & x_6^7 \\ x_7^1 & x_{15}^3 & x_{12}^5 & x_7^7 \end{pmatrix}, \begin{pmatrix} x_7^1 & x_7^3 & x_7^5 & x_7^7 \\ x_{13}^1 & x_5^3 & x_{13}^5 & x_{13}^7 \end{pmatrix}, \begin{pmatrix} x_{11}^1 & x_{11}^3 & x_{11}^5 & x_{11}^7 \\ x_2^1 & x_2^3 & x_{11}^2 & x_2^7 \end{pmatrix}, \begin{pmatrix} x_{14}^1 & x_{14}^3 & x_{14}^5 & x_{14}^7 \\ x_{15}^1 & x_8^3 & x_{15}^5 & x_{15}^7 \end{pmatrix},$
$\begin{pmatrix} x_{15}^1 & x_{15}^3 & x_{15}^5 & x_{15}^7 \\ x_6^1 & x_5^3 & x_6^5 & x_6^7 \end{pmatrix}, \begin{pmatrix} x_{16}^1 & x_{16}^3 & x_{16}^5 & x_{16}^7 \\ x_5^1 & x_2^3 & x_{11}^5 & x_5^7 \end{pmatrix}, \begin{pmatrix} x_6^1 & x_6^2 & x_6^4 & x_6^6 \\ x_6^5 & x_{15}^2 & x_7^4 & x_7^6 \end{pmatrix}, \begin{pmatrix} x_9^1 & x_9^2 & x_9^4 & x_9^6 \\ x_9^5 & x_{12}^2 & x_{11}^4 & x_{11}^6 \end{pmatrix},$
$\begin{pmatrix} x_{10}^1 & x_{10}^2 & x_{10}^4 & x_{10}^6 \\ x_{10}^5 & x_{11}^2 & x_1^4 & x_1^6 \end{pmatrix}, \begin{pmatrix} x_{11}^1 & x_{11}^2 & x_{11}^4 & x_{11}^6 \\ x_{11}^5 & x_{15}^2 & x_{11}^7 & x_2^6 \end{pmatrix}, \begin{pmatrix} x_{13}^1 & x_{13}^2 & x_{13}^4 & x_{13}^6 \\ x_{13}^5 & x_8^2 & x_5^4 & x_5^6 \end{pmatrix}, \begin{pmatrix} x_{15}^1 & x_{15}^2 & x_{15}^4 & x_{15}^6 \\ x_{15}^5 & x_{15}^7 & x_6^4 & x_6^6 \end{pmatrix},$
$\begin{pmatrix} x_{16}^1 & x_{16}^2 & x_{16}^4 & x_{16}^6 \\ x_{16}^5 & x_{16}^7 & x_{11}^4 & x_5^6 \end{pmatrix}, \begin{pmatrix} x_3^2 & x_3^5 & x_3^4 & x_3^7 \\ x_1^2 & x_{10}^5 & x_{10}^4 & x_3^3 \end{pmatrix}, \begin{pmatrix} x_7^2 & x_7^5 & x_7^4 & x_7^7 \\ x_{14}^2 & x_6^5 & x_{13}^4 & x_7^3 \end{pmatrix}, \begin{pmatrix} x_7^1 & x_7^2 & x_7^3 & x_7^6 \\ x_7^5 & x_{13}^2 & x_6^3 & x_{13}^6 \end{pmatrix},$
$\begin{pmatrix} x_{13}^3 & x_{13}^7 & x_{15}^7 & x_{15}^3 \\ x_3^3 & x_3^7 & x_{11}^7 & x_{14}^3 \end{pmatrix}.$

3.4. L_8-decomposition required for Lemma 2.7

3.4.1. An L_8- decomposition of $K_{19} \backslash K_3$

Let $V(K_{19}) = \{x_1, x_2, ..., x_{19}\}$ and the K_3 be $(x_{13}x_{15}x_{17})$.
$\begin{pmatrix} x_1 & x_2 & x_{18} & x_{19} \\ x_9 & x_{12} & x_5 & x_6 \end{pmatrix}, \begin{pmatrix} x_3 & x_4 & x_{16} & x_{17} \\ x_9 & x_1 & x_{18} & x_{19} \end{pmatrix}, \begin{pmatrix} x_5 & x_6 & x_{14} & x_{15} \\ x_3 & x_4 & x_{16} & x_1 \end{pmatrix}, \begin{pmatrix} x_7 & x_8 & x_{12} & x_{13} \\ x_9 & x_2 & x_{16} & x_{14} \end{pmatrix},$
$\begin{pmatrix} x_9 & x_{10} & x_{11} & x_{12} \\ x_4 & x_{16} & x_7 & x_{17} \end{pmatrix}, \begin{pmatrix} x_1 & x_{16} & x_2 & x_{17} \\ x_{12} & x_3 & x_9 & x_4 \end{pmatrix}, \begin{pmatrix} x_3 & x_{18} & x_4 & x_{19} \\ x_{10} & x_1 & x_2 & x_{12} \end{pmatrix}, \begin{pmatrix} x_5 & x_{12} & x_6 & x_{13} \\ x_{11} & x_{18} & x_3 & x_{16} \end{pmatrix},$
$\begin{pmatrix} x_7 & x_{14} & x_8 & x_{15} \\ x_2 & x_{17} & x_1 & x_3 \end{pmatrix}, \begin{pmatrix} x_1 & x_{10} & x_2 & x_{11} \\ x_{14} & x_{13} & x_{19} & x_3 \end{pmatrix}, \begin{pmatrix} x_3 & x_{12} & x_4 & x_{13} \\ x_1 & x_7 & x_{11} & x_2 \end{pmatrix}, \begin{pmatrix} x_1 & x_5 & x_2 & x_6 \\ x_{13} & x_{14} & x_{15} & x_{11} \end{pmatrix},$
$\begin{pmatrix} x_7 & x_{16} & x_8 & x_{17} \\ x_6 & x_{19} & x_5 & x_{11} \end{pmatrix}, \begin{pmatrix} x_9 & x_{18} & x_{10} & x_{19} \\ x_{11} & x_6 & x_4 & x_5 \end{pmatrix}, \begin{pmatrix} x_{11} & x_{14} & x_{12} & x_{15} \\ x_{13} & x_2 & x_{10} & x_4 \end{pmatrix}, \begin{pmatrix} x_3 & x_7 & x_4 & x_8 \\ x_2 & x_1 & x_{14} & x_9 \end{pmatrix},$
$\begin{pmatrix} x_5 & x_{16} & x_6 & x_{17} \\ x_7 & x_9 & x_8 & x_{18} \end{pmatrix}, \begin{pmatrix} x_9 & x_{14} & x_{10} & x_{15} \\ x_{13} & x_3 & x_{17} & x_{16} \end{pmatrix}, \begin{pmatrix} x_{11} & x_{18} & x_{13} & x_{19} \\ x_{16} & x_{14} & x_8 & x_{15} \end{pmatrix}, \begin{pmatrix} x_5 & x_9 & x_6 & x_{10} \\ x_4 & x_{17} & x_{15} & x_8 \end{pmatrix},$

$$\begin{pmatrix} x_7 & x_{18} & x_8 & x_{19} \\ x_{10} & x_{15} & x_{11} & x_{14} \end{pmatrix}.$$

3.4.2. An L_8- decomposition of $K_{15}\Box K_3$

$$\begin{pmatrix} x_7^j & x_8^j & x_9^j & x_{10}^j \\ x_{15}^j & x_{14}^j & x_1^j & x_5^j \end{pmatrix} \text{ for } j=2,3; \begin{pmatrix} x_5^j & x_{13}^j & x_6^j & x_{15}^j \\ x_{14}^j & x_4^j & x_7^j & x_2^j \end{pmatrix} \text{ for } j=1,2,3;$$
$$\begin{pmatrix} x_3^j & x_{11}^j & x_9^j & x_{12}^j \\ x_8^j & x_6^j & x_{15}^j & x_{10}^j \end{pmatrix} \text{ for } j=1,3; \begin{pmatrix} x_1^j & x_{12}^j & x_2^j & x_{13}^j \\ x_4^j & x_k^j & x_7^j & x_{14}^j \end{pmatrix} \text{ for } (j,k)=(1,15),(2,5),(3,15);$$
$$\begin{pmatrix} x_{10}^j & x_{13}^j & x_{11}^j & x_{14}^j \\ x_{15}^j & x_8^j & x_k^j & x_1^j \end{pmatrix} \text{ for } (j,k)=(1,4),(2,12),(3,4);$$
$$\begin{pmatrix} x_3^j & x_7^j & x_4^j & x_9^j \\ x_{10}^j & x_{13}^j & x_k^j & x_{14}^j \end{pmatrix} \text{ for } (j,k)=(1,5),(2,8),(3,8);$$
$$\begin{pmatrix} x_1^1 & x_2^1 & x_{14}^1 & x_{15}^1 \\ x_7^1 & x_8^1 & x_{12}^1 & x_{15}^3 \end{pmatrix}, \begin{pmatrix} x_1^2 & x_2^2 & x_{14}^2 & x_{15}^2 \\ x_1^3 & x_8^2 & x_{14}^3 & x_{15}^3 \end{pmatrix}, \begin{pmatrix} x_1^3 & x_2^3 & x_{14}^3 & x_{15}^3 \\ x_7^3 & x_8^3 & x_{14}^1 & x_{13}^3 \end{pmatrix}, \begin{pmatrix} x_3^1 & x_4^1 & x_{12}^1 & x_{13}^1 \\ x_3^3 & x_4^3 & x_5^1 & x_9^1 \end{pmatrix},$$
$$\begin{pmatrix} x_3^2 & x_4^2 & x_{12}^2 & x_{13}^2 \\ x_3^3 & x_4^3 & x_{14}^2 & x_{13}^3 \end{pmatrix}, \begin{pmatrix} x_3^3 & x_4^3 & x_{12}^3 & x_{13}^3 \\ x_1^3 & x_2^3 & x_5^3 & x_9^3 \end{pmatrix}, \begin{pmatrix} x_5^1 & x_6^1 & x_{10}^1 & x_{11}^1 \\ x_3^1 & x_6^2 & x_2^1 & x_{15}^1 \end{pmatrix}, \begin{pmatrix} x_5^2 & x_6^2 & x_{10}^2 & x_{11}^2 \\ x_3^2 & x_4^2 & x_{10}^3 & x_{11}^3 \end{pmatrix},$$
$$\begin{pmatrix} x_5^3 & x_6^3 & x_{10}^3 & x_{11}^3 \\ x_3^3 & x_4^3 & x_2^3 & x_{15}^3 \end{pmatrix}, \begin{pmatrix} x_7^1 & x_8^1 & x_9^1 & x_{10}^1 \\ x_7^3 & x_{14}^1 & x_1^1 & x_5^1 \end{pmatrix}, \begin{pmatrix} x_3^1 & x_{14}^1 & x_4^1 & x_{15}^1 \\ x_2^1 & x_{14}^2 & x_{10}^1 & x_8^1 \end{pmatrix}, \begin{pmatrix} x_3^2 & x_{14}^2 & x_4^2 & x_{15}^2 \\ x_2^2 & x_7^2 & x_{10}^2 & x_{12}^2 \end{pmatrix},$$
$$\begin{pmatrix} x_3^3 & x_{14}^3 & x_4^3 & x_{15}^3 \\ x_2^3 & x_7^3 & x_{10}^3 & x_8^3 \end{pmatrix}, \begin{pmatrix} x_5^1 & x_8^1 & x_6^1 & x_9^1 \\ x_5^2 & x_1^1 & x_6^3 & x_2^1 \end{pmatrix}, \begin{pmatrix} x_5^2 & x_8^2 & x_6^2 & x_9^2 \\ x_4^2 & x_1^2 & x_6^3 & x_9^3 \end{pmatrix}, \begin{pmatrix} x_5^3 & x_8^3 & x_6^3 & x_9^3 \\ x_4^3 & x_1^3 & x_3^3 & x_2^3 \end{pmatrix},$$
$$\begin{pmatrix} x_7^1 & x_{11}^1 & x_8^1 & x_{12}^1 \\ x_{14}^1 & x_1^1 & x_8^3 & x_6^1 \end{pmatrix}, \begin{pmatrix} x_7^2 & x_{11}^2 & x_8^2 & x_{12}^2 \\ x_7^3 & x_1^2 & x_8^3 & x_6^2 \end{pmatrix}, \begin{pmatrix} x_7^3 & x_{11}^3 & x_8^3 & x_{12}^3 \\ x_9^3 & x_1^3 & x_{10}^3 & x_6^3 \end{pmatrix}, \begin{pmatrix} x_1^1 & x_5^1 & x_2^1 & x_6^1 \\ x_{10}^1 & x_5^3 & x_{11}^1 & x_{14}^1 \end{pmatrix},$$
$$\begin{pmatrix} x_1^2 & x_5^2 & x_2^2 & x_6^2 \\ x_{10}^2 & x_5^3 & x_{11}^2 & x_{14}^2 \end{pmatrix}, \begin{pmatrix} x_1^3 & x_5^3 & x_2^3 & x_6^3 \\ x_{10}^3 & x_7^3 & x_2^2 & x_{14}^3 \end{pmatrix}, \begin{pmatrix} x_3^2 & x_{11}^2 & x_9^2 & x_{12}^2 \\ x_8^2 & x_6^2 & x_{15}^2 & x_{12}^3 \end{pmatrix},$$
$$\begin{pmatrix} x_{11}^1 & x_{12}^1 & x_{12}^3 & x_{11}^3 \\ x_{11}^2 & x_{12}^2 & x_{14}^3 & x_2^3 \end{pmatrix}, \begin{pmatrix} x_1^1 & x_3^1 & x_3^2 & x_1^2 \\ x_1^3 & x_6^1 & x_6^2 & x_7^2 \end{pmatrix}, \begin{pmatrix} x_2^1 & x_4^1 & x_4^2 & x_2^2 \\ x_2^3 & x_6^1 & x_{11}^2 & x_{10}^2 \end{pmatrix},$$
$$\begin{pmatrix} x_7^1 & x_9^1 & x_9^2 & x_7^2 \\ x_5^1 & x_9^3 & x_2^2 & x_5^2 \end{pmatrix}, \begin{pmatrix} x_8^1 & x_{10}^1 & x_{10}^2 & x_8^2 \\ x_4^1 & x_{10}^3 & x_{12}^2 & x_{15}^2 \end{pmatrix}, \begin{pmatrix} x_{13}^1 & x_{15}^1 & x_{15}^2 & x_{13}^2 \\ x_{13}^3 & x_7^1 & x_{11}^2 & x_9^2 \end{pmatrix}.$$

3.5. L_8-decomposition required for Lemma 2.8

3.5.1. An L_8- decomposition of $K_{13}\Box K_5$

$$\begin{pmatrix} x_5^j & x_9^j & x_6^j & x_{10}^j \\ x_3^j & x_9^3 & x_4^j & x_{10}^3 \end{pmatrix}, \begin{pmatrix} x_3^j & x_{12}^j & x_4^j & x_{13}^j \\ x_{11}^j & x_2^j & x_9^j & x_7^j \end{pmatrix} \text{ for } j=2,4;$$
$$\begin{pmatrix} x_7^j & x_8^j & x_{10}^j & x_{11}^j \\ x_9^j & x_2^j & x_1^j & x_{13}^j \end{pmatrix} \text{ for } j=1,5; \begin{pmatrix} x_1^j & x_{13}^j & x_8^j & x_{12}^j \\ x_7^j & x_{13}^3 & x_9^j & x_{12}^3 \end{pmatrix} \text{ for } j=2,5;$$
$$\begin{pmatrix} x_9^j & x_{12}^j & x_{10}^j & x_{13}^j \\ x_8^j & x_6^j & x_4^j & x_5^j \end{pmatrix} \text{ for } j=3,4; \begin{pmatrix} x_i^1 & x_i^2 & x_i^4 & x_i^3 \\ x_k^1 & x_k^2 & x_k^4 & x_k^3 \end{pmatrix} \text{ for } (i,k)=(5,7),(8,11);$$
$$\begin{pmatrix} x_1^j & x_5^j & x_2^j & x_6^j \\ x_{k_1}^j & x_5^{k_2} & x_2^{k_2} & x_6^{k_2} \end{pmatrix} \text{ for } (j,k_1,k_2)=(1,9,4),(2,8,5),(3,9,2);$$
$$\begin{pmatrix} x_i^1 & x_i^2 & x_i^5 & x_i^4 \\ x_{k_1}^1 & x_{k_2}^2 & x_{k_1}^5 & x_{k_2}^4 \end{pmatrix} \text{ for } (i,k_1,k_2)=(9,10,11),(10,2,1),(13,6,6);$$
$$\begin{pmatrix} x_1^4 & x_5^4 & x_2^4 & x_6^4 \\ x_1^5 & x_5^5 & x_{11}^4 & x_6^5 \end{pmatrix}, \begin{pmatrix} x_1^5 & x_5^5 & x_2^5 & x_6^5 \\ x_1^3 & x_5^3 & x_2^3 & x_6^3 \end{pmatrix}, \begin{pmatrix} x_3^1 & x_7^1 & x_4^1 & x_8^1 \\ x_3^3 & x_7^5 & x_4^3 & x_8^4 \end{pmatrix}, \begin{pmatrix} x_3^2 & x_7^2 & x_4^2 & x_8^2 \\ x_1^2 & x_7^4 & x_4^3 & x_8^5 \end{pmatrix},$$
$$\begin{pmatrix} x_3^3 & x_7^3 & x_4^3 & x_8^3 \\ x_9^3 & x_1^3 & x_2^3 & x_8^2 \end{pmatrix}, \begin{pmatrix} x_3^4 & x_7^4 & x_4^4 & x_8^4 \\ x_3^2 & x_7^3 & x_4^3 & x_8^5 \end{pmatrix}, \begin{pmatrix} x_3^5 & x_7^5 & x_4^5 & x_8^5 \\ x_3^3 & x_5^5 & x_{11}^5 & x_8^3 \end{pmatrix}, \begin{pmatrix} x_5^1 & x_9^1 & x_6^1 & x_{10}^1 \\ x_5^5 & x_9^3 & x_{12}^1 & x_{10}^3 \end{pmatrix},$$
$$\begin{pmatrix} x_5^3 & x_9^3 & x_6^3 & x_{10}^3 \\ x_3^3 & x_7^3 & x_4^3 & x_1^3 \end{pmatrix}, \begin{pmatrix} x_5^5 & x_9^5 & x_6^5 & x_{10}^5 \\ x_{13}^5 & x_9^3 & x_6^1 & x_{10}^3 \end{pmatrix}, \begin{pmatrix} x_1^1 & x_4^1 & x_5^1 & x_{11}^1 \\ x_2^1 & x_6^1 & x_8^1 & x_{11}^3 \end{pmatrix}, \begin{pmatrix} x_1^2 & x_4^2 & x_5^2 & x_{11}^2 \\ x_2^2 & x_4^4 & x_8^2 & x_{11}^5 \end{pmatrix},$$
$$\begin{pmatrix} x_1^3 & x_4^3 & x_5^3 & x_{11}^3 \\ x_1^4 & x_{13}^3 & x_8^3 & x_6^3 \end{pmatrix}, \begin{pmatrix} x_1^4 & x_4^4 & x_5^4 & x_{11}^4 \\ x_2^4 & x_3^4 & x_8^4 & x_{11}^1 \end{pmatrix}, \begin{pmatrix} x_1^5 & x_4^5 & x_5^5 & x_{11}^5 \\ x_2^5 & x_6^5 & x_8^5 & x_3^5 \end{pmatrix}, \begin{pmatrix} x_2^1 & x_3^1 & x_6^1 & x_7^1 \\ x_{11}^1 & x_4^1 & x_5^1 & x_{13}^1 \end{pmatrix},$$
$$\begin{pmatrix} x_2^2 & x_3^2 & x_6^2 & x_7^2 \\ x_{11}^2 & x_4^2 & x_5^2 & x_{10}^2 \end{pmatrix}, \begin{pmatrix} x_2^3 & x_3^3 & x_6^3 & x_7^3 \\ x_{11}^3 & x_3^2 & x_5^3 & x_{10}^3 \end{pmatrix}, \begin{pmatrix} x_2^4 & x_3^4 & x_6^4 & x_7^4 \\ x_9^4 & x_1^4 & x_5^4 & x_{10}^4 \end{pmatrix}, \begin{pmatrix} x_2^5 & x_3^5 & x_6^5 & x_7^5 \\ x_{11}^5 & x_4^5 & x_5^5 & x_7^3 \end{pmatrix},$$

$$\begin{pmatrix} x_7^2 & x_8^2 & x_{10}^2 & x_{11}^2 \\ x_9^2 & x_2^2 & x_3^2 & x_4^2 \end{pmatrix}, \begin{pmatrix} x_7^3 & x_8^3 & x_{10}^3 & x_{11}^3 \\ x_7^2 & x_1^3 & x_3^3 & x_{11}^4 \end{pmatrix}, \begin{pmatrix} x_7^4 & x_8^4 & x_{10}^4 & x_{11}^4 \\ x_9^4 & x_1^4 & x_3^4 & x_4^4 \end{pmatrix}, \begin{pmatrix} x_1^1 & x_{13}^1 & x_8^1 & x_{12}^1 \\ x_7^1 & x_{13}^3 & x_9^1 & x_5^1 \end{pmatrix},$$
$$\begin{pmatrix} x_1^3 & x_{13}^3 & x_8^3 & x_{12}^3 \\ x_3^3 & x_6^3 & x_2^3 & x_7^3 \end{pmatrix}, \begin{pmatrix} x_1^4 & x_{13}^4 & x_8^4 & x_{12}^4 \\ x_7^4 & x_{13}^3 & x_2^4 & x_{12}^3 \end{pmatrix}, \begin{pmatrix} x_3^1 & x_{12}^1 & x_4^1 & x_{13}^1 \\ x_{11}^1 & x_2^1 & x_4^5 & x_{13}^5 \end{pmatrix}, \begin{pmatrix} x_3^3 & x_4^3 & x_{12}^3 & x_{13}^3 \\ x_{11}^3 & x_9^3 & x_5^3 & x_7^3 \end{pmatrix},$$
$$\begin{pmatrix} x_3^5 & x_{12}^5 & x_4^5 & x_{13}^5 \\ x_1^5 & x_2^5 & x_4^3 & x_7^5 \end{pmatrix}, \begin{pmatrix} x_9^1 & x_{12}^1 & x_{10}^1 & x_{13}^1 \\ x_{11}^1 & x_{12}^3 & x_4^1 & x_5^1 \end{pmatrix}, \begin{pmatrix} x_9^2 & x_{12}^2 & x_{10}^2 & x_{13}^2 \\ x_2^2 & x_6^2 & x_4^2 & x_5^2 \end{pmatrix}, \begin{pmatrix} x_9^5 & x_{12}^5 & x_{10}^5 & x_{13}^5 \\ x_{11}^5 & x_6^5 & x_4^5 & x_2^5 \end{pmatrix},$$
$$\begin{pmatrix} x_1^1 & x_1^3 & x_1^2 & x_1^4 \\ x_3^1 & x_2^3 & x_1^5 & x_9^4 \end{pmatrix}, \begin{pmatrix} x_2^1 & x_2^2 & x_2^4 & x_2^3 \\ x_4^1 & x_{13}^2 & x_{13}^4 & x_{10}^3 \end{pmatrix}, \begin{pmatrix} x_3^1 & x_3^2 & x_3^5 & x_3^4 \\ x_9^1 & x_9^2 & x_9^5 & x_3^3 \end{pmatrix}, \begin{pmatrix} x_4^1 & x_4^2 & x_4^5 & x_4^4 \\ x_{11}^1 & x_2^2 & x_2^5 & x_2^4 \end{pmatrix},$$
$$\begin{pmatrix} x_6^1 & x_6^2 & x_6^4 & x_6^3 \\ x_{11}^1 & x_8^2 & x_8^4 & x_8^3 \end{pmatrix}, \begin{pmatrix} x_7^1 & x_7^2 & x_7^5 & x_7^4 \\ x_7^3 & x_{12}^2 & x_{12}^5 & x_{12}^4 \end{pmatrix}, \begin{pmatrix} x_{11}^2 & x_{11}^3 & x_{11}^5 & x_{11}^4 \\ x_6^2 & x_{13}^3 & x_6^5 & x_6^4 \end{pmatrix}, \begin{pmatrix} x_{12}^1 & x_{12}^2 & x_{12}^5 & x_{12}^4 \\ x_7^1 & x_{11}^2 & x_5^5 & x_{11}^4 \end{pmatrix},$$
$$\begin{pmatrix} x_2^1 & x_2^5 & x_9^5 & x_9^1 \\ x_{13}^1 & x_2^4 & x_4^5 & x_4^1 \end{pmatrix}, \begin{pmatrix} x_9^2 & x_9^4 & x_{10}^4 & x_{10}^2 \\ x_1^2 & x_3^4 & x_2^4 & x_2^2 \end{pmatrix}, \begin{pmatrix} x_{11}^1 & x_{11}^5 & x_{12}^5 & x_{12}^1 \\ x_{11}^2 & x_8^5 & x_{13}^5 & x_{13}^1 \end{pmatrix}, \begin{pmatrix} x_{12}^2 & x_{12}^4 & x_{13}^4 & x_{13}^2 \\ x_5^2 & x_5^4 & x_{11}^4 & x_{11}^2 \end{pmatrix},$$
$$\begin{pmatrix} x_2^3 & x_9^3 & x_{11}^3 & x_{12}^3 \\ x_{13}^3 & x_{10}^3 & x_4^3 & x_3^3 \end{pmatrix}, \begin{pmatrix} x_3^1 & x_3^5 & x_{10}^5 & x_{10}^1 \\ x_5^1 & x_5^5 & x_7^5 & x_7^1 \end{pmatrix}, \begin{pmatrix} x_1^1 & x_1^5 & x_8^5 & x_8^1 \\ x_1^2 & x_9^5 & x_6^5 & x_6^1 \end{pmatrix}.$$

3.6. L_8-decomposition required for Lemma 2.9

3.6.1. An L_8- decomposition of $K_{11}\Box K_7$

$$\begin{pmatrix} x_1^j & x_5^j & x_2^j & x_6^j \\ x_9^j & x_7^j & x_4^j & x_8^j \end{pmatrix} \text{ for } j = 2, 5; \begin{pmatrix} x_1^j & x_4^j & x_5^j & x_{11}^j \\ x_3^j & x_9^j & x_8^j & x_{11}^k \end{pmatrix} \text{ for } (j, k) = (2, 3), (3, 4);$$
$$\begin{pmatrix} x_5^j & x_9^j & x_6^j & x_{10}^j \\ x_3^j & x_{11}^j & x_4^j & x_1^j \end{pmatrix} \text{ for } j = 5, 6; \begin{pmatrix} x_7^j & x_8^j & x_{10}^j & x_{11}^j \\ x_9^j & x_1^j & x_2^j & x_4^j \end{pmatrix} \text{ for } j = 1, 2, 5;$$
$$\begin{pmatrix} x_2^j & x_3^j & x_6^j & x_7^j \\ x_1^j & x_{11}^j & x_5^j & x_7^k \end{pmatrix} \text{ for } (j, k) = (5, 2), (6, 2);$$
$$\begin{pmatrix} x_7^j & x_8^j & x_{10}^j & x_{11}^j \\ x_9^j & x_1^j & x_{10}^k & x_4^j \end{pmatrix} \text{ for } (j, k) = (3, 6), (6, 7);$$
$$\begin{pmatrix} x_i^1 & x_i^3 & x_i^5 & x_i^7 \\ x_i^4 & x_i^2 & x_i^6 & x_k^7 \end{pmatrix} \text{ for } (i, k) = (5, 9), (6, 8), (7, 1), (8, 9);$$
$$\begin{pmatrix} x_i^1 & x_i^3 & x_i^5 & x_i^7 \\ x_{k_1}^1 & x_i^2 & x_i^6 & x_{k_2}^7 \end{pmatrix} \text{ for } (i, k_1, k_2) = (1, 9, 9), (4, 3, 3), (9, 8, 2);$$
$$\begin{pmatrix} x_i^1 & x_i^2 & x_i^4 & x_i^6 \\ x_i^5 & x_k^2 & x_i^3 & x_i^7 \end{pmatrix} \text{ for } (i, k) = (4, 3), (8, 9), (9, 2);$$
$$\begin{pmatrix} x_i^2 & x_i^5 & x_i^4 & x_i^7 \\ x_i^6 & x_k^5 & x_k^4 & x_i^3 \end{pmatrix} \text{ for } (i, k) = (3, 10), (8, 9);$$
$$\begin{pmatrix} x_1^1 & x_5^1 & x_2^1 & x_6^1 \\ x_{10}^1 & x_3^1 & x_9^1 & x_8^1 \end{pmatrix}, \begin{pmatrix} x_1^3 & x_5^3 & x_2^3 & x_6^3 \\ x_9^3 & x_7^3 & x_{10}^3 & x_{11}^3 \end{pmatrix}, \begin{pmatrix} x_1^4 & x_5^4 & x_2^4 & x_6^4 \\ x_1^1 & x_{10}^4 & x_2^1 & x_8^4 \end{pmatrix}, \begin{pmatrix} x_1^6 & x_5^6 & x_2^6 & x_6^6 \\ x_1^3 & x_5^3 & x_2^2 & x_6^3 \end{pmatrix},$$
$$\begin{pmatrix} x_1^7 & x_5^7 & x_2^7 & x_6^7 \\ x_1^6 & x_5^6 & x_2^6 & x_{11}^7 \end{pmatrix}, \begin{pmatrix} x_3^1 & x_7^1 & x_4^1 & x_8^1 \\ x_{11}^1 & x_1^1 & x_{10}^1 & x_2^1 \end{pmatrix}, \begin{pmatrix} x_3^2 & x_7^2 & x_4^2 & x_8^2 \\ x_9^2 & x_1^2 & x_{10}^2 & x_2^2 \end{pmatrix}, \begin{pmatrix} x_3^3 & x_7^3 & x_4^3 & x_8^3 \\ x_5^3 & x_1^3 & x_{10}^3 & x_2^3 \end{pmatrix},$$
$$\begin{pmatrix} x_3^4 & x_7^4 & x_4^4 & x_8^4 \\ x_3^3 & x_1^4 & x_9^4 & x_2^4 \end{pmatrix}, \begin{pmatrix} x_3^5 & x_7^5 & x_4^5 & x_8^5 \\ x_9^5 & x_1^5 & x_{10}^5 & x_5^5 \end{pmatrix}, \begin{pmatrix} x_3^6 & x_7^6 & x_4^6 & x_8^6 \\ x_9^6 & x_7^3 & x_{10}^6 & x_5^6 \end{pmatrix}, \begin{pmatrix} x_3^7 & x_7^7 & x_4^7 & x_8^7 \\ x_5^7 & x_7^6 & x_6^7 & x_2^7 \end{pmatrix},$$
$$\begin{pmatrix} x_5^1 & x_9^1 & x_6^1 & x_{10}^1 \\ x_8^1 & x_{11}^1 & x_4^1 & x_{10}^4 \end{pmatrix}, \begin{pmatrix} x_5^2 & x_9^2 & x_6^2 & x_{10}^2 \\ x_3^2 & x_{11}^2 & x_4^2 & x_{10}^3 \end{pmatrix}, \begin{pmatrix} x_5^3 & x_9^3 & x_6^3 & x_{10}^3 \\ x_5^7 & x_9^6 & x_4^3 & x_1^3 \end{pmatrix}, \begin{pmatrix} x_7^7 & x_8^7 & x_{10}^7 & x_{11}^7 \\ x_9^7 & x_1^7 & x_2^7 & x_3^7 \end{pmatrix},$$
$$\begin{pmatrix} x_1^1 & x_4^1 & x_5^1 & x_{11}^1 \\ x_3^1 & x_9^1 & x_7^1 & x_8^1 \end{pmatrix}, \begin{pmatrix} x_1^4 & x_4^4 & x_5^4 & x_{11}^4 \\ x_3^4 & x_{10}^4 & x_8^4 & x_9^4 \end{pmatrix}, \begin{pmatrix} x_1^5 & x_4^5 & x_5^5 & x_{11}^5 \\ x_3^5 & x_9^5 & x_5^2 & x_6^5 \end{pmatrix}, \begin{pmatrix} x_1^6 & x_4^6 & x_5^6 & x_{11}^6 \\ x_3^1 & x_9^6 & x_5^2 & x_{11}^2 \end{pmatrix},$$
$$\begin{pmatrix} x_1^7 & x_4^7 & x_5^7 & x_{11}^7 \\ x_3^7 & x_2^7 & x_8^7 & x_{11}^6 \end{pmatrix}, \begin{pmatrix} x_2^1 & x_3^1 & x_6^1 & x_7^1 \\ x_{11}^1 & x_9^1 & x_5^1 & x_{10}^1 \end{pmatrix}, \begin{pmatrix} x_2^2 & x_3^2 & x_6^2 & x_7^2 \\ x_1^2 & x_3^3 & x_5^2 & x_{10}^2 \end{pmatrix}, \begin{pmatrix} x_2^3 & x_3^3 & x_6^3 & x_7^3 \\ x_1^3 & x_9^3 & x_5^3 & x_{10}^3 \end{pmatrix},$$
$$\begin{pmatrix} x_2^4 & x_3^4 & x_6^4 & x_7^4 \\ x_1^4 & x_{11}^4 & x_9^4 & x_{10}^4 \end{pmatrix}, \begin{pmatrix} x_2^7 & x_3^7 & x_6^7 & x_7^7 \\ x_2^3 & x_9^7 & x_5^7 & x_{10}^7 \end{pmatrix}, \begin{pmatrix} x_2^1 & x_2^3 & x_2^5 & x_2^7 \\ x_4^1 & x_2^4 & x_8^5 & x_{11}^7 \end{pmatrix}, \begin{pmatrix} x_3^1 & x_3^3 & x_3^5 & x_3^7 \\ x_{10}^1 & x_4^3 & x_3^6 & x_{10}^7 \end{pmatrix},$$
$$\begin{pmatrix} x_{10}^1 & x_{10}^3 & x_{10}^5 & x_{10}^7 \\ x_9^1 & x_3^3 & x_7^5 & x_9^7 \end{pmatrix}, \begin{pmatrix} x_{11}^1 & x_{11}^3 & x_{11}^5 & x_{11}^7 \\ x_{11}^4 & x_9^3 & x_{11}^6 & x_8^7 \end{pmatrix}, \begin{pmatrix} x_1^1 & x_1^2 & x_1^4 & x_1^6 \\ x_2^1 & x_{10}^2 & x_1^3 & x_9^6 \end{pmatrix}, \begin{pmatrix} x_2^1 & x_2^2 & x_2^4 & x_2^6 \\ x_2^5 & x_{11}^2 & x_9^4 & x_8^6 \end{pmatrix},$$
$$\begin{pmatrix} x_3^1 & x_3^2 & x_3^4 & x_3^6 \\ x_3^5 & x_{10}^2 & x_9^4 & x_3^7 \end{pmatrix}, \begin{pmatrix} x_5^1 & x_5^2 & x_5^4 & x_5^6 \\ x_5^5 & x_5^7 & x_5^3 & x_7^6 \end{pmatrix}, \begin{pmatrix} x_6^1 & x_6^2 & x_6^4 & x_6^6 \\ x_{11}^1 & x_{11}^2 & x_6^3 & x_8^6 \end{pmatrix}, \begin{pmatrix} x_7^1 & x_7^2 & x_7^4 & x_7^6 \\ x_7^5 & x_7^7 & x_7^3 & x_1^6 \end{pmatrix},$$
$$\begin{pmatrix} x_{10}^1 & x_{10}^2 & x_{10}^4 & x_{10}^6 \\ x_{10}^5 & x_9^2 & x_{10}^3 & x_7^6 \end{pmatrix}, \begin{pmatrix} x_{11}^1 & x_{11}^2 & x_{11}^4 & x_{11}^6 \\ x_{11}^5 & x_8^2 & x_{10}^4 & x_6^6 \end{pmatrix}, \begin{pmatrix} x_1^2 & x_1^5 & x_1^4 & x_1^7 \\ x_1^6 & x_1^1 & x_9^4 & x_1^3 \end{pmatrix}, \begin{pmatrix} x_2^2 & x_2^5 & x_2^4 & x_2^7 \\ x_2^3 & x_{11}^5 & x_{11}^4 & x_1^7 \end{pmatrix},$$

$$\begin{pmatrix} x_4^2 & x_4^5 & x_4^4 & x_4^7 \\ x_4^6 & x_3^5 & x_4^1 & x_4^3 \end{pmatrix}, \begin{pmatrix} x_2^6 & x_4^6 & x_3^6 & x_{10}^6 \\ x_2^5 & x_4^3 & x_3^3 & x_{10}^5 \end{pmatrix}, \begin{pmatrix} x_6^2 & x_6^5 & x_6^4 & x_6^7 \\ x_6^6 & x_6^1 & x_{11}^4 & x_6^3 \end{pmatrix}, \begin{pmatrix} x_9^2 & x_9^5 & x_9^4 & x_9^7 \\ x_9^6 & x_2^5 & x_9^1 & x_{11}^7 \end{pmatrix},$$
$$\begin{pmatrix} x_{10}^2 & x_{10}^5 & x_{10}^4 & x_{10}^7 \\ x_{10}^6 & x_9^5 & x_1^4 & x_{10}^3 \end{pmatrix}, \begin{pmatrix} x_{11}^2 & x_{11}^5 & x_{11}^4 & x_{11}^7 \\ x_3^2 & x_8^5 & x_8^4 & x_{11}^3 \end{pmatrix}, \begin{pmatrix} x_2^3 & x_9^3 & x_8^3 & x_{11}^3 \\ x_4^3 & x_{10}^3 & x_6^3 & x_3^3 \end{pmatrix}, \begin{pmatrix} x_2^6 & x_9^6 & x_8^6 & x_{11}^6 \\ x_2^3 & x_{10}^6 & x_8^3 & x_{11}^3 \end{pmatrix}$$
$$\begin{pmatrix} x_3^4 & x_4^4 & x_6^4 & x_5^4 \\ x_3^1 & x_4^1 & x_{10}^4 & x_9^4 \end{pmatrix}, \begin{pmatrix} x_7^4 & x_8^4 & x_{10}^4 & x_9^4 \\ x_{11}^4 & x_1^4 & x_2^4 & x_4^4 \end{pmatrix}, \begin{pmatrix} x_4^7 & x_9^7 & x_6^7 & x_{10}^7 \\ x_{11}^7 & x_9^3 & x_6^6 & x_1^7 \end{pmatrix}, \begin{pmatrix} x_5^4 & x_7^4 & x_7^7 & x_5^7 \\ x_5^5 & x_7^5 & x_7^3 & x_{10}^7 \end{pmatrix}.$$

3.7. L_8-decomposition required for Lemma 2.10

3.7.1. An L_8- decomposition of $K_9 \square K_9$

$\begin{pmatrix} x_1^j & x_2^j & x_3^j & x_4^j \\ x_7^j & x_5^j & x_6^j & x_8^j \end{pmatrix}$ for $j = 4, 6, 7, 8$; $\begin{pmatrix} x_5^j & x_6^j & x_7^j & x_8^j \\ x_4^j & x_1^j & x_2^j & x_3^j \end{pmatrix}$ for $j = 1, 2, 4, 5, 6, 7, 8$;
$\begin{pmatrix} x_i^1 & x_i^2 & x_i^3 & x_i^4 \\ x_i^7 & x_i^5 & x_i^6 & x_i^8 \end{pmatrix}$ for $i = 2, 4, 6$; $\begin{pmatrix} x_1^j & x_8^j & x_2^j & x_9^j \\ x_5^j & x_6^j & x_4^j & x_7^j \end{pmatrix}$ for $j = 1, 2, 3, 4, 7, 8, 9$;
$\begin{pmatrix} x_i^1 & x_i^2 & x_i^3 & x_i^4 \\ x_i^7 & x_i^6 & x_k^3 & x_i^8 \end{pmatrix}$ for $(i, k) = (1, 7), (7, 2)$; $\begin{pmatrix} x_i^1 & x_i^2 & x_i^3 & x_i^4 \\ x_i^7 & x_k^2 & x_i^6 & x_i^8 \end{pmatrix}$ for $(i, k) = (3, 5), (8, 4)$;
$\begin{pmatrix} x_i^1 & x_i^2 & x_i^3 & x_i^4 \\ x_k^1 & x_i^5 & x_i^6 & x_k^4 \end{pmatrix}$ for $(i, k) = (5, 3), (9, 6)$; $\begin{pmatrix} x_i^5 & x_i^6 & x_i^7 & x_i^8 \\ x_i^4 & x_i^1 & x_i^2 & x_i^3 \end{pmatrix}$ for $i = 1, 3, 4, 6, 7, 8$;
$\begin{pmatrix} x_i^1 & x_i^8 & x_i^2 & x_i^9 \\ x_i^5 & x_i^6 & x_i^4 & x_i^7 \end{pmatrix}$ for $i = 1, 3, \ldots, 9$; $\begin{pmatrix} x_i^3 & x_i^7 & x_i^4 & x_i^9 \\ x_i^1 & x_i^5 & x_i^6 & x_i^8 \end{pmatrix}$ for $i = 1, 3, 4, 5, 7, 8, 9$;
$\begin{pmatrix} x_3^j & x_7^j & x_4^j & x_9^j \\ x_1^j & x_5^j & x_6^j & x_8^j \end{pmatrix}$ for $j = 1, 2, 3, 4, 6, 7, 8, 9$;

$$\begin{pmatrix} x_1^1 & x_2^1 & x_3^1 & x_4^1 \\ x_7^1 & x_2^5 & x_6^1 & x_8^1 \end{pmatrix}, \begin{pmatrix} x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_7^2 & x_5^2 & x_6^2 & x_4^6 \end{pmatrix}, \begin{pmatrix} x_1^3 & x_2^3 & x_3^3 & x_4^3 \\ x_1^6 & x_2^8 & x_6^3 & x_8^3 \end{pmatrix}, \begin{pmatrix} x_1^5 & x_2^5 & x_3^5 & x_4^5 \\ x_1^3 & x_2^7 & x_6^5 & x_4^3 \end{pmatrix},$$
$$\begin{pmatrix} x_1^9 & x_2^9 & x_3^9 & x_4^9 \\ x_1^6 & x_5^9 & x_6^9 & x_4^6 \end{pmatrix}, \begin{pmatrix} x_5^3 & x_6^3 & x_7^3 & x_8^3 \\ x_4^3 & x_1^3 & x_7^6 & x_8^5 \end{pmatrix}, \begin{pmatrix} x_5^9 & x_6^9 & x_7^9 & x_8^9 \\ x_5^5 & x_9^9 & x_2^9 & x_3^9 \end{pmatrix}, \begin{pmatrix} x_1^5 & x_8^5 & x_2^5 & x_9^5 \\ x_5^5 & x_6^5 & x_2^3 & x_7^5 \end{pmatrix},$$
$$\begin{pmatrix} x_1^6 & x_8^6 & x_2^6 & x_9^6 \\ x_5^6 & x_8^2 & x_4^6 & x_7^6 \end{pmatrix}, \begin{pmatrix} x_3^5 & x_7^5 & x_4^5 & x_9^5 \\ x_1^5 & x_7^2 & x_6^5 & x_8^5 \end{pmatrix}, \begin{pmatrix} x_2^5 & x_2^6 & x_2^7 & x_2^8 \\ x_2^4 & x_2^1 & x_2^2 & x_6^8 \end{pmatrix}, \begin{pmatrix} x_5^5 & x_5^6 & x_5^7 & x_5^8 \\ x_7^5 & x_5^1 & x_5^2 & x_5^3 \end{pmatrix},$$
$$\begin{pmatrix} x_9^5 & x_9^6 & x_9^7 & x_9^8 \\ x_5^5 & x_9^9 & x_9^2 & x_9^3 \end{pmatrix}, \begin{pmatrix} x_2^1 & x_2^8 & x_2^2 & x_2^9 \\ x_6^1 & x_2^6 & x_2^4 & x_2^7 \end{pmatrix}, \begin{pmatrix} x_2^3 & x_2^7 & x_2^4 & x_2^9 \\ x_2^1 & x_6^7 & x_2^6 & x_2^8 \end{pmatrix}, \begin{pmatrix} x_6^3 & x_6^7 & x_6^4 & x_6^9 \\ x_6^1 & x_6^5 & x_2^4 & x_6^8 \end{pmatrix},$$
$$\begin{pmatrix} x_6^2 & x_6^6 & x_9^6 & x_9^2 \\ x_2^2 & x_8^6 & x_5^6 & x_5^2 \end{pmatrix}, \begin{pmatrix} x_6^3 & x_6^5 & x_9^5 & x_9^3 \\ x_2^3 & x_2^5 & x_9^9 & x_5^3 \end{pmatrix}, \begin{pmatrix} x_5^4 & x_5^8 & x_9^8 & x_9^4 \\ x_5^5 & x_3^8 & x_6^8 & x_9^5 \end{pmatrix}, \begin{pmatrix} x_5^1 & x_5^7 & x_9^7 & x_9^1 \\ x_2^1 & x_3^7 & x_6^7 & x_9^6 \end{pmatrix},$$
$$\begin{pmatrix} x_3^6 & x_3^9 & x_5^9 & x_5^6 \\ x_3^2 & x_3^5 & x_9^9 & x_5^2 \end{pmatrix}, \begin{pmatrix} x_3^3 & x_3^5 & x_5^5 & x_5^3 \\ x_8^3 & x_3^2 & x_2^5 & x_2^3 \end{pmatrix}, \begin{pmatrix} x_2^6 & x_2^9 & x_6^9 & x_6^6 \\ x_2^2 & x_2^5 & x_6^5 & x_6^4 \end{pmatrix}, \begin{pmatrix} x_1^5 & x_1^9 & x_7^9 & x_7^5 \\ x_1^2 & x_6^9 & x_7^6 & x_7^3 \end{pmatrix},$$
$$\begin{pmatrix} x_4^5 & x_4^9 & x_8^9 & x_8^5 \\ x_2^5 & x_5^9 & x_8^6 & x_8^2 \end{pmatrix}.$$