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An extended radius of convergence comparison between two sixth order methods under general continuity for solving equations

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Abstract

In this paper we compare the radius of convergence of two sixth convergence order methods for solving nonlinear equation. We present the local convergence analysis not given before, which is based on the first Fréchet derivative that only appears on the method. Numerical examples where the theoretical results are tested complete the paper.

Keywords: Newton/Chebyshev method Banach Space Local/semi-local Convergence.
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1. Introduction

In this paper we compare the radii of convergence of two sixth convergence order methods for solving nonlinear equation

$$F(x) = 0, \tag{1.1}$$

where $F : \Omega \subset X \longrightarrow Y$ is continuously Fréchet differentiable, X, Y are Banach spaces, and Ω is a nonempty convex set.

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The methods under consideration in this paper are defined for each $n = 0, 1, 2, \ldots$ by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ z_n &= y_n - (2I - F'(x_n)^{-1}F'(y_n))F'(x_n)^{-1}F(y_n) \\ x_{n+1} &= z_n - (2I - F'(x_n)^{-1}F'(y_n))F'(x_n)^{-1}F(z_n) \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ z_n &= x_n - \frac{1}{2}(F'(x_n)^{-1} + F'(y_n)^{-1})F(x_n) \\ x_{n+1} &= z_n - \frac{1}{2}(F'(x_n)^{-1}F'(y_n)^{-1})F'(x_n)F'(y_n)^{-1}F(z_n). \end{aligned} \tag{1.3}$$

The convergence order of iterative methods, in general, was obtained using Taylor expansions and conditions on high order derivatives not appearing on the method. These conditions limit the applicability of the methods [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]. For example: Let $X = Y = \mathbb{R}$, $\Omega = [-\frac{1}{2}, \frac{3}{2}]$. Define f on Ω by

$$f(t) = \begin{cases} t^3 \log t^2 + t^5 - t^4 & if\ t \neq 0 \\ 0 & if\ t = 0. \end{cases}$$

Then, we have $t_* = 1$,

$$f'''(t) = 6 \log t^2 + 60t^2 - 24t + 22.$$

Obviously $f'''(t)$ is not bounded on Ω. So, the convergence of these methods is not guaranteed by the analysis in these papers. The convergence order of methods (1.2) and (1.3) was given using assumptions on the derivatives of order up to seven in [20] and [21], respectively. But these derivatives are not used in method (1.2) and method (1.3). The first derivative has only been used in our convergene hypotheses. Notice that this is the only derivative appearing on the method. We also provide a computable radius of convergence also not given in [20, 21]. This way we locate a set of initial points for the convergence of the method. The numerical examples are chosen to show how the radii theoretically predicted are computed. In particular, the last example shows that earlier results cannot be used to show convergence of the method. Our results significantly extends the applicability of these methods and provide a new way of looking at iterative methods. The article contains local convergence analysis in Section 2 and the numerical examples in Section 3.

2. Convergence

It is convenient to develop some real functions crucial to the local convergence of methods (1.2) and (1.3), respectively. Set $E = [0, \infty)$.

Assume that there exists function $\varphi_0 : E \longrightarrow E$ such that equation

$$\varphi_0(t) - 1 = 0 \tag{2.4}$$

has a least solution $R_0 \in E - \{0\}$. Set $E_0 = [0, R_0)$.

Assume that equation

$$\psi_1(t) - 1 = 0,$$

has a least solution $r_1 \in (0, R_0)$, where $\varphi : E_0 \longrightarrow E$ is some continuous and nondecreasing functions and

$$\psi_1(t) = \frac{\int_0^1 \varphi((1-\theta)t)d\theta}{1 - \varphi_0(t)}.$$

Assume that equation

$$\varphi_0(\psi_1(t)t) - 1 = 0 \tag{2.5}$$

has a least solution $R_1 \in E - \{0\}$. Set $R_2 = \min\{R_0, R_1\}$ and $E_1 = [0, R_2)$.

Assume that equation

$$\psi_2(t) - 1 = 0$$

has a least solution $r_2 \in (0, R_2)$, where $\varphi_1 : [0, R_2) \longrightarrow E$ is some continuous and nondecreasing function and

$$\begin{aligned}\psi_2(t) \;=\; & \Big[\psi_1(\psi_1(t)t) \\ & + \frac{(\varphi_0(t) + \varphi_0(\psi_1(t)t)) \int_0^1 \varphi_1(\theta\psi_1(t)t)d\theta}{(1 - \varphi_0(t))(1 - \varphi_0(\psi_1(t)t))} \\ & + \frac{(\varphi_0(t) + \varphi_0(\psi_1(t)t)) \int_0^1 \varphi_1(\theta\psi_1(t)t)d\theta}{(1 - \varphi_0(t))^2}\Big]\psi_1(t).\end{aligned}$$

Assume that equation

$$\varphi_0(\psi_2(t)t) - 1 = 0$$

has a least solution $R_3 \in E - \{0\}$. Set $R = \min\{R_2, R_3\}$ and $E_2 = [0, R)$.

Assume that equation

$$\varphi_3(t) - 1 = 0 \tag{2.6}$$

has a least solution $r_3 \in (0, R)$, where

$$\begin{aligned}\psi_3(t) \;=\; & \Big[\psi_1(\psi_2(t)t) \\ & + \frac{(\varphi_0(t) + \varphi_0(\psi_2(t)t)) \int_0^1 \varphi_1(\theta\psi_2(t)t)d\theta}{(1 - \varphi_0(t))(1 - \varphi_0(\psi_2(t)t))} \\ & + \frac{(\varphi_0(t) + \varphi_0(\psi_1(t)t)) \int_0^1 \varphi_1(\theta\psi_2(t)t)d\theta}{(1 - \varphi_0(t))^2}\Big]\psi_2(t).\end{aligned}$$

Next, we shall show that

$$r = \min\{r_k\},\ k = 1, 2, 3 \tag{2.7}$$

is a radius of convergence for method (1.2). It follows from (2.7) that for each $t \in [0, r)$ the following estimates hold

$$0 \le \varphi_0(t) < 1, \tag{2.8}$$

$$0 \le \varphi_0(\psi_1(t)t) < 1, \tag{2.9}$$

$$0 \le \varphi_0(\psi_2(t)t) < 1, \tag{2.10}$$

and

$$0 \le \psi_k(t) < 1. \tag{2.11}$$

We use notations $B(x, R), \bar{B}(x, R)$ to denote the open and closed balls, respectively in X with center $x \in X$ and of radius $R > 0$.

The hypotheses (H) are used: Assume:

(h1) $F : \Omega \subset X \longrightarrow Y$ is Fréchet continuously differentiable; there exists $x_* \in \Omega$ such that $F(x_*) = 0$.

(h2) There exists a continuous and nondecreasing function $\varphi_0 : E \longrightarrow E$ such that for each $x \in \Omega$

$$\|F'(x_*)^{-1}(F'(x) - F'(x_*))\| \le \varphi_0(\|x - x_*\|).$$

Set $\Omega_0 = \Omega \cap U(x_*, R_0)$.

(h3) There continuous and nondecreasing exist functions $\varphi : E_0 \longrightarrow E, \varphi_1 : E_0 \longrightarrow E$ such that for each $x, y \in \Omega_0$

$$\|F'(x_*)^{-1}(F'(y) - F'(x))\| \leq \varphi(\|y - x\|).$$

$$\|F'(x_*)^{-1}F'(x)\| \leq \varphi_1(\|x - x_*\|).$$

(h4) $\bar{B}(x_*, \rho) \subset \Omega$ for some $\rho > 0$ to be determined.

(h5) There exists $r_* \geq r$ such that $\int_0^1 \varphi_0(\theta r_*) d\theta < 1$. Set $\Omega_1 = \Omega \cap \bar{B}(x_*, r_*)$.

Then, the local convergence of method (1.2) follows given hypotheses (H) with the developed notation.

THEOREM 2.1. *Under the hypotheses (H) choose $x_0 \in B(x_*, r) - \{x_*\}$. Then, sequence $\{x_n\}$ generated by method (1.2) is well defined in $B(x_*, r)$, remains in $B(x_*, r)$ and converges to x_* for all $n = 0, 1, 2, \ldots$, so that*

$$\|y_n - x_*\| \leq \psi_1(\|x_n - x_*\|)\|x_n - x_*\| \leq \|x_n - x_*\| < r, \tag{2.12}$$

$$\|z_n - x_*\| \leq \psi_2(\|x_n - x_*\|)\|x_n - x_*\| \leq \|x_n - x_*\| \tag{2.13}$$

and

$$\|x_{n+1} - x_*\| \leq \psi_3(\|x_n - x_*\|)\|x_n - x_*\| \leq \|x_n - x_*\|, \tag{2.14}$$

where the functions $\psi_k, k = 1, 2, 3$ are given previously and radius r is given by (2.7). Moreover, x_* the only solution of equation $F(x) = 0$ in the set Ω_1 given in (h5).

Proof. Estimates (2.12)-(2.14) are shown using mathematical induction on i. Let $u \in B(x_*, r) - \{x_*\}$ be arbitrary. Then, by (2.7), (2.8), (h1) and (h2), we get in turn that

$$\|F'(x_*)^{-1}(F'(u) - F'(x_*))\| \leq \varphi_0(\|u - x_*\|) \leq \varphi_0(r) < 1 \tag{2.15}$$

implying together with a lemma on invertible operators by Banach [8] that $F'(u)^{-1} \in L(Y, X)$, with

$$\|F'(u)^{-1}F'(x_*)\| \leq \frac{1}{1 - \varphi_0(\|u - x_*\|)}. \tag{2.16}$$

The iterate y_0 is also well defined by the first three substep of method (1.2), via which we can write

$$\begin{aligned} y_0 - x_* &= x_0 - x_* - F'(x_0)^{-1}F(x_0) \\ &= [F'(x_0)^{-1}F'(x_*)] \\ &\quad \times [\int_0^1 F'(x_*)^{-1}(F'(x_* + \theta(x_0 - x_*)) - F'(x_0))d\theta(x_0 - x_*)] \end{aligned} \tag{2.17}$$

using (2.7), (2.15) (for $i = 1$), (h3), (2.16) (for $u = x_0$) and (2.17), we get in turn that

$$\begin{aligned} \|y_0 - x_*\| &\leq \frac{\int_0^1 \varphi((1 - \theta)\|x_0 - x_*\|)d\theta\|x_0 - x_*\|}{1 - \varphi_0(\|x_0 - x_*\|)} \\ &\leq \psi_1(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\| < r, \end{aligned} \tag{2.18}$$

proving $y_0 \in B(x_*, r) - \{x^*\}$, and estimate (2.12) for $n = 0$. We also have that (2.16) holds for $u = y_0$ and iterte z_0 is well defined by the second substep of method (1.2) from which we can also write.

$$\begin{aligned} z_0 - x^* &= y_0 - x^* - F'(y_0)^{-1}F(y_0) \\ &\quad +(F'(y_0)^{-1} - (2I - F'(x_n)^{-1}F(y_0))F'(x_0)^{-1}F(y_0) \\ &= y_0 - x^* - F'(y_0)^{-1}F(y_0) \\ &\quad +F'(x_0)^{-1}((F'(y_0) - F'(x_*)) \\ &\quad +(F'(x_*) - F'(x_0)))F'(x_0)^{-1}F(y_0). \end{aligned} \tag{2.19}$$

In view of (2.7), (2.15)(for $i = 2$), (2.18) and (2.19), we have in turn that

$$\begin{aligned} \|z_0 - x_*\| &\leq [\psi_1(\|y_0 - x_*\|)) \\ &\quad + \frac{(\varphi_0(\|x_0 - x_*\|) + \varphi_0(\|y_0 - x_*\|)) \int_0^1 \varphi_1(\theta\|y_0 - x_*\|)d\theta}{(1 - \varphi_0(\|x_0 - x_*\|))(1 - \varphi_0(\|y_0 - x_*\|))} \\ &\quad + \frac{(\varphi_0(\|x_0 - x_*\|) + \varphi_0(\|y_0 - x_*\|)) \int_0^1 \varphi_1(\theta\|y_0 - x_*\|)d\theta}{(1 - \varphi_0(\|x_0 - x_*\|))^2}]\|y_0 - x_*\| \\ &\leq \psi_2(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\| \end{aligned} \tag{2.20}$$

proving $z_0 \in B(x_*, r)$ and (2.13) holds for $n = 0$. Moreover, $F'(z_0)^{-1} \in L(Y, X)$ by (2.16) for $n = 0$. Iterate x_1 is also well defined by the third substep of method (1.2) from which we can also write as in (2.19) for $y_n = z_n$:

$$\begin{aligned} x_1 - x_* &= (z_0 - x_* - F'(z_0)^{-1}F(z_0) \\ &\quad +(F'(z_0)^{-1} - F'(x_0)^{-1})F(z_0) + F'(x_0)^{-1}((F'(y_0) - F'(x_0)) \\ &\quad F'(x_0)^{-1}F(z_0). \end{aligned} \tag{2.21}$$

Moreover, by (2.7), (2.15) (for $k = 3$) as in (2.20), we have in turn that

$$\begin{aligned} \|x_1 - x_*\| &\leq \left[\psi_1(\|z_0 - x_*\|) + \frac{(\varphi_0(\|x_0 - x_*\|) + \varphi_0(\|z_0 - x_*\|)) \int_0^1 \varphi_1(\theta\|z_0 - x_*\|)d\theta}{(1 - \varphi_0(\|x_0 - x_*\|))(1 - \varphi_0(\|z_0 - x_*\|))}\right. \\ &\quad \left. + \frac{(\varphi_0(\|x_0 - x_*\|) + \varphi_0(\|y_0 - x_*\|)) \int_0^1 \varphi_1(\theta\|z_0 - x_*\|)d\theta}{(1 - \varphi_0(\|x_0 - x_*\|))^2}\right] \|z_0 - x_*\| \\ &\leq \psi_3(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\|, \end{aligned} \tag{2.22}$$

proving (2.14) for $n = 0$ and $x_1 \in B(x_*, r)$. Simply switch x_0, y_0, z_0, x_1 by x_i, y_i, z_i, x_{i+1} in the preceding calculations to complete the induction for items (2.12)-(2.14). Hence, by the estimate

$$\|x_{i+1} - x_*\| \leq a\|x_i - x_*\| < r, \tag{2.23}$$

where $a = \psi_3(\|x_0 - x_*\|) \in [0, 1)$, we conclude $\lim_{i\longrightarrow\infty} x_i = x_*$ and $x_{i+1} \in B(x_*, r)$. Furthermore, let $T = \int_0^1 F'(x_* + \theta(x_{**} - x_*))d\theta$ for some $x_{**} \in \Omega_1$ with $F(x_{**}) = 0$. It then follows by (h5) that

$$\|F'(x_*)^{-1}(T - F'(x_*))\| \leq \int_0^1 \varphi_0(\theta\|x_* - x_{**})\|d\theta \leq \int_0^1 \varphi_0(\theta\tilde{r})d\theta < 1,$$

so $T^{-1} \in L(Y, X)$. Consequently, from $0 = F(x_{**}) - F(x_*) = T(x_{**} - x_*)$, we obtain $x_{**} = x_*$.

□

REMARK 2.2. *1. In view of (2.9) and the estimate*

$$\begin{aligned} \|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\| \\ &\leq 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq 1 + L_0\|x - x^*\| \end{aligned}$$

condition (2.12) can be dropped and φ_1 *can be replaced by*

$$\varphi_1(t) = 1 + \varphi_0(t)$$

or

$$\varphi_1(t) = 1 + \varphi_0(R_0),$$

since $t \in [0, R_0)$.

2. *The results obtained here can be used for operators F satisfying autonomous differential equations [2] of the form*
$$F'(x) = P(F(x))$$
where P is a continuous operator. Then, since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing x^*. For example, let $F(x) = e^x - 1$. Then, we can choose: $P(x) = x + 1$.

3. *Let $\varphi_0(t) = L_0 t$, and $\varphi(t) = Lt$. In [2, 3] we showed that $r_A = \frac{2}{2L_0+L}$ is the convergence radius of Newton's method:*
$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \text{ for each } n = 0, 1, 2, \cdots \tag{2.24}$$
under the conditions (h1) - (h3). It follows from the definition of r in (2.7) that the convergence radius r of the method (1.2) cannot be larger than the convergence radius r_A of the second order Newton's method (2.24). As already noted in [2, 3] r_A is at least as large as the convergence radius given by Rheinboldt [16]
$$r_R = \frac{2}{3L}, \tag{2.25}$$
where L_1 is the Lipschitz constant on D. The same value for r_R was given by Traub [17]. In particular, for $L_0 < L_1$ we have that
$$r_R < r_A$$
and
$$\frac{r_R}{r_A} \to \frac{1}{3} \ \ as \ \ \frac{L_0}{L_1} \to 0.$$
That is the radius of convergence r_A is at most three times larger than Rheinboldt's.

4. *We can compute the computational order of convergence (COC) defined by*
$$\xi = \ln\left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}\right) / \ln\left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}\right)$$
or the approximate computational order of convergence
$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) / \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).$$

Next, we present the local convergence analysis of method (1.3) along the same lines.

Assume that equation
$$\bar{\psi}_2(t) - 1 = 0 \tag{2.26}$$
has a least solution $\bar{r}_2 \in (0, R_2)$, where
$$\bar{\psi}_2(t) = \psi_1(t) + \frac{(\varphi_0(t) + \varphi_0(\psi_1(t)t)) \int_0^1 \varphi_1(\theta t)d\theta}{2(1 - \varphi_0(t))(1 - \varphi_0(\psi_1(t)t))}.$$
Assume that equation
$$\bar{\psi}_3 - 1 = 0 \tag{2.27}$$
has a least solution $\bar{r}_3 \in (0, R)$, where
$$\begin{aligned}\bar{\psi}_3(t) \ &= \ [\psi_1(\bar{\psi}_2(t)t) + \frac{(\varphi_0(\bar{\psi}_2(t)t) + \varphi_0(\psi_1(t)t)) \int_0^1 \varphi_1(\theta\bar{\psi}_2(t)t)d\theta}{(1 - \varphi_0(\bar{\psi}_2(t)t))(1 - \varphi_0(\psi_1(t)t))} \\ &\quad + \frac{(\varphi_0(\bar{\psi}_1(t)t) + \varphi_0(t)) \int_0^1 \varphi_1(\theta\bar{\psi}_2(t)t)d\theta}{2(1 - \varphi_0(\bar{\psi}_1(t)t))(1 - \varphi_0(t))}]\bar{\psi}_2(t).\end{aligned}$$
Set
$$\bar{r} = \min\{r_1, \bar{r}_2, \bar{r}_3\}. \tag{2.28}$$

Then, in view of the estimates

$$\begin{aligned}
x_n - x_* &= x_n - x_* - F'(x_n)^{-1}F(x_n) + \frac{1}{2}(F'(x_n)^{-1} - F'(y_n)^{-1})F(x_n),\\
\|z_n - x_*\| &\leq [\psi_1(\|x_n - x_*\|)\\
&\quad + \frac{(\varphi_0(\|x_n - x_*\|) + \varphi_0(\|y_n - x_*\|))\int_0^1 \varphi_1(\theta\|x_n - x_*\|)d\theta}{2(1-\varphi_0(\|x_n - x_*\|))(1-\varphi_0(\|y_n - x_*\|))}]\|x_n - x_*\|\\
&\leq \bar{\psi}_2(\|x_n - x_*\|)\|x_n - x_*\| \leq \|x_n - x_*\| < \bar{r},
\end{aligned}$$

$$\begin{aligned}
x_{n+1} - x_* &= z_n - x_* - F'(z_n)^{-1}F(z_n)\\
&\quad + [(F'(z_n)^{-1} - F'(y_n)^{-1}) + \frac{1}{2}(F'(y_n)^{-1} - F'(x_n)^{-1})\\
&\quad + \frac{1}{2}F'(y_n)^{-1}(F'(y_n) - F'(x_n))F'(y_n)^{-1}F(z_n),
\end{aligned}$$

$$\begin{aligned}
\|x_{n+1} - x_*\| &\leq [\psi_1(\|z_n - x_*\|) + \frac{\varphi_0(\|z_n - x_*\|) + \varphi_0(\|y_n - x_*\|)}{(1-\varphi_0(\|z_n - x_*\|))(1-\varphi_0(\|y_n - x_*\|))}\\
&\quad + \frac{1}{2}\frac{\varphi_0(\|y_n - x_*\|) + \varphi_0(\|x_n - x_*\|)}{(1-\varphi_0(\|y_n - x_*\|))(1-\varphi_0(\|x_n - x_*\|))}\\
&\quad + \frac{1}{2}\frac{\varphi_0(\|x_n - x_*\|) + \varphi_0(\|y_n - x_*\|)}{(1-\varphi_0(\|y_n - x_*\|))^2})\int_0^1 \varphi_1(\theta\|z_n - x_*\|)d\theta]\|z_n - x_*\|\\
&\leq \bar{\psi}_3(\|x_n - x_*\|)\|x_n - x_*\| \leq \|x_n - x_*\|.
\end{aligned}$$

Hence, we arrive at the corresponding local convergence for method (1.3).

THEOREM 2.3. *Under hypotheses (H) for $\rho = \bar{r}$ choose $x_0 \in B(x_*, \bar{r}) - \{x_*\}$. Then,the conclusion of Theorem 2.1 hold for method (1.3) with $\bar{\psi}_2, \bar{\psi}_3, \bar{r}$ replacing ψ_2, ψ_3 and r, respectively.*

3. Numerical Examples

EXAMPLE 3.1. *Consider the kinematic system*

$$F_1'(x) = e^x,\ F_2'(y) = (e-1)y + 1,\ F_3'(z) = 1$$

with $F_1(0) = F_2(0) = F_3(0) = 0$. Let $F = (F_1, F_2, F_3)$. Let $X = Y = \mathbb{R}^3, D = \bar{U}(0,1), x_* = (0,0,0)^T$. Define function F on D for $w = (x,y,z)^T$ by

$$F(w) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

Then, we get

$$F'(v) = \begin{bmatrix} e^x & 0 & 0\\ 0 & (e-1)y+1 & 0\\ 0 & 0 & 1 \end{bmatrix},$$

so $\varphi_0(t) = (e-1)t, \varphi(t) = e^{\frac{1}{e-1}}t, \varphi_1(t) = e^{\frac{1}{e-1}}$. Then, the radii:

$$r_1 = 0.0382692,\ r_2 = 0.207437,\ r_3 = 0.163184,\ \bar{r}_2 = 0.207429,\ \bar{r}_3 = 0.175466.$$

EXAMPLE 3.2. *Consider* $X = Y = C[0,1]$, $D = \overline{U}(0,1)$ *and* $F : D \longrightarrow Y$ *defined by*

$$F(\psi)(x) = \varphi(x) - 5\int_0^1 x\theta\psi(\theta)^3 d\theta. \tag{3.1}$$

We have that

$$F'(\psi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\psi(\theta)^2\xi(\theta)d\theta, \text{ for each } \xi \in D.$$

Then, we get that $x_* = 0$, *so* $\varphi_0(t) = 7.5t, \varphi(t) = 15t$ *and* $\varphi_1(t) = 2$. *Then, the radii:*

$$r_1 = 0.06667,\ r_2 = 0.056309,\ r_3 = 0.0293612,\ \bar{r}_2 = 0.0374367,\ \bar{r}_3 = 0.0316959.$$

EXAMPLE 3.3. *By the academic example of the introduction, we have* $\varphi_0(t) = \varphi(t) = 96.6629073t$ *and* $\varphi_1(t) = 2$. *Then, the radii:*

$$r_1 = 0.00689682,\ r_2 = 0.0079146,\ r_3 = 0.00281258,\ \bar{r}_2 = 0.00357775,\ \bar{r}_3 = 0.00299601.$$

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