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Araştırma Makalesi - Research Article

Canal Surfaces Through a Null Quaternionic Spine Curve

Null Kuaterniyonik Omurga Eğrisi Boyunca Kanal Yüzeyler

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ABSTRACT

In this study, we give the parameterizations of the canal surfaces through a null quaternionic spine curve by using the pseudo-spheres in $\mathbb{R}_1^4$. Besides, we give formulas for the Gauss and Mean curvatures and some corollaries related to the Cartan curvatures of the null quaternionic curve.

Keywords- Canal Surfaces, Cartan Frame, Null Quaternionic Curve, Semi Real Quaternions, Spine Curve

ÖZ

Bu çalışmada, $\mathbb{R}_1^4$ deki pseudo-küreleri kullanarak, bir null kuaterniyonik omurga eğrisi boyunca kanal yüzeylerin parametrizasyonları verilmiştir. Ayrıca, Gauss ve ortalama eğrilikler hesaplanmış ve null kuaterniyonik eğrinin Cartan eğrilikleri ile ilişkili bazı sonuçlar elde edilmiştir.

Anahtar Kelimeler- Kanal Yüzeyler, Cartan Çatısı, Null Kuaterniyonik Eğri, Yarı Reel Kuaterniyonlar, Omurga Eğrisi

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I. INTRODUCTION

Hamilton first defined the quaternions in 1843 with the aim of generalizing the complex numbers. He discovered that it is the best generalization if the real axis is left unchanged when the vector axis is appended by adding two more axes [1]. After him, it has been made considerable studies on the quaternions in many fields of science, for example, quaternions are used in robotic systems, video games, computer graphics, navigation systems and in several computer programs as well as mathematics and physics. Also, they derive remarkable convenience for the rotations in the three dimensional space. Many scientific studies exist on the representations of surfaces (such as constant slope or canal surfaces) by the help of quaternions [2, 4, 5, 10]. Moreover, the semi-quaternions are studied in [6, 7]. In these works, it is stated that the semi-quaternions algebra with a degenerate scalar product has the dimension four semi-Euclidean space structure with rank 2 semi-metric.

In 1987, Bharathi and Nagaraj introduced the concept of smooth quaternionic curves which lie in the three and four dimensional Euclidean spaces. They examined the Serret-Frenet formulas which are used to interpret the geometric properties of quaternionic curves [8]. Also, the quaternonic curves are described in the semi-Euclidean space by Çöken and Tuna in [9] and remarkable studies are conducted by many researchers on the different kinds of them, [10-15] are just to name a few.

On the other hand, there exist three families of curves called as spacelike, timelike or null (lightlike) depending on their causal characters in the Minkowski spacetime. Many different situations appear in the case of null curves compared to the non-null cases. The Frenet frame on a null curve was investigated and developed by many authors in this field [16-18]. Besides, in the references [19-21], it is studied the differential geometry of null quaternionic curves in semi-euclidean spaces and given the Frenet formula for null quaternionic curves by using spatial quaternions. It is also defined the Cartan frame for a null quaternionic curve in the dimension four Minkowski space in [22].

A canal surface associated to a space curve, is obtained by sweeping a family of spheres with varying radius r(s) along a space curve and this curve is called as the spine curve. When we take the radius function as a constant, then the canal surface is called a tubular (pipe) surface. The canal surfaces are rather studied in many areas besides mathematics, such as CAGD, robotic path planning or shape reconstruction ([23-26]). Moreover, the canal surfaces are useful while visualising long and thin objects such as 3D fonts, poles, brass instruments or internal organs of the body in solid surface modeling. One of the primary questions is whether the canal surface is developable or not. One knows that, on a developable surface, the Gaussian curvature is identically zero at regular points and a developable canal surface is either a cylinder or a cone in the Euclidean space [27]. Farther et al. classified the canal surfaces in Minkowski-3 space in [28] and the tubular surfaces around a null curve is studied in [30].

In this study, first we introduce the basic properties of the semi-real quaternions, null quaternionic curves and canal surfaces in $\mathbb{R}^3_1$. Then we define the non degenerate canal surfaces trough a null quaternionic spine curve by using the pseudo-spheres in the four dimensional Minkowski space. This process consists of three cases depending on the type of the pseudo-spheres which foliate the canal surface. Each case is detailed according to the casual character of the normal vector field X(s,θ)-C(s) for describing the surface parameterization. Additionally, we calculate the Gauss and Mean curvatures and give some related corollaries.

II. PRELIMINARIES

We give some basic properties of the semi-real quaternions, null quaternionic curves and canal surfaces in the following three subsections (for further information, see [6, 7, 19-21, 28, 29]).

A. Semi-real quaternions

Let Q_H be a vector space with dimension four over the field H of characteristic grater than 2. If $e_i (1 \leq i \leq 4)$ is a basis for the vector space, then set of all the semi-real quaternions can be given by

$$Q_H = \{q | q = ae_1 + be_2 + ce_3 + d;\ a, b, c, d \in \mathbb{R}, e_1, e_2, e_3 \in\ \mathbb{R}^3_1, h(e_i, e_i) = \varepsilon(e_i), 1 \leq i \leq 3\}$$

where

$e_i \times e_i = -\varepsilon(e_i), \quad 1 \leq i \leq 3$
$e_i \times e_j = \varepsilon(e_i)\varepsilon(e_j), \quad e_k \in \mathbb{R}_1^3$

Here (ijk) is an even permutation of (123). The multiplication of two semi real quaternions is defined by

$$p \times q = S_p S_q + S_p V_q + S_q V_p + h(V_p, V_q) + V_p \wedge V_q$$

for every $p, q \in Q_H$ by using the dot and cross products in $\mathbb{R}_1^3$.

The conjugate of q is defined by $\alpha q = -ae_1 - be_2 - ce_3 + d$ for a semi real quaternion $q = ae_1 + be_2 + ce_3 + d \in Q_H$. Now we define a semi-Riemannian metric h as follows:

$$h: Q_H \times Q_H \to \mathbb{R}$$

$$h(p,q) = \frac{1}{2}[\varepsilon(p)\varepsilon(\alpha q)(p \times \alpha q) + \varepsilon(q)\varepsilon(\alpha p)(q \times \alpha p)]$$

where the semi-real quaternion inner product is defined by,

$$h(q,q) = a^2\varepsilon(e_1) + b^2\varepsilon(e_2) + c^2\varepsilon(e_3) + d^2$$

The vector product of two semi real quaternions $p = a_1e_1 + b_1e_2 + c_1e_3 + d_1$ and $q = ae_1 + be_2 + ce_3 + d$ is given as

$$V_p \wedge V_q = \varepsilon(e_2)\varepsilon(e_3)(b_1c - bc_1)e_1 - \varepsilon(e_1)\varepsilon(e_3)(a_1c - ac_1)e_2 + \varepsilon(e_1)\varepsilon(e_2)(a_1b - ab_1)e_3$$

Also, the norm of a semi real quaternion is defined by

$$\| q \|^2 = |h(q,q)| = |a^2\varepsilon(e_1) + b^2\varepsilon(e_2) + c^2\varepsilon(e_3) + d^2|$$

and q is called a spatial or temporal quaternion iff $q + \alpha q = 0$ or $q - \alpha q = 0$, respectively.

B. Null Quaternionic Curves in $\mathbb{R}_1^4$

Without loss of generality, we choose e_1 as a timelike vector. Then $\{e_1, e_2, e_3, e_4 = 1\}$ is an orthonormal basis of $\mathbb{R}_1^4$. Let $\mathbb{R}_1^4$ be endowed with the quaternionic metric h. The dimension four semi-Euclidean space can be identified with the space of null spatial quaternionic curves in an obvious manner and we define a null quaternionic curve as,

$$C: I \subset \mathbb{R} \to \mathbb{Q}_H$$

$$C(s) = \gamma_1(s)e_1 + \gamma_2(s)e_2 + \gamma_3(s)e_3 + \gamma_4(s)e_4$$

where the tangent $L(s) = \sum_{i=1}^{4} \gamma_i'(s)e_i$ has zero length for a distinguished parameter s.

We consider a Cartan null quaternionic curve β in the dimension four Minkowski space (R_1^4, h) with a Cartan frame $\{L, N, U, W\}$ with respect to a pseudo-arc parameter s together with the Cartan curvatures p and τ. Then the Cartan equations are;

$$L' = W, N' = (p - \tau)U + pW$$
$$U' = (p - \tau)L, W' = pL + N$$

or

$$L' = W, N' = (\tau + p)U + pW$$
$$U' = (\tau + p)L, W' = pL + N$$

where

$$h(L,L) = h(N,N) = h(L,U) = h(N,U) = h(W,U) = h(N,W) = h(L,W) = 0$$
$$h(U,U) = h(W,W) = +1, h(L,N) = -1.$$

C. Canal surfaces in $\mathbb{R}_1^3$

A canal surface is described by the envelope of a one parameter family of spheres which are centered at a spine curve $C(s)$ with a radius given by the function $r(s)$ in the Euclidean space. In the case of $r(s)$ is constant, it is called a tubular surface. When we take the spine curve as a straight line, it is actually the surface of revolution. The great circles of a canal surface lie in the normal plane of the spine curve at every point.

Similar to the Euclidean case, in the Minkowski space $\mathbb{R}_1^3$, canal surfaces are formed as the envelopes of a family of pseudo spheres such as de-Sitter space, hyperbolic sphere or null cone (S_1^2, H_1^2 or Q^2, respectively).

Assume that $C(s)$ is a space curve in $\mathbb{R}_1^3$. Then the canal surface $X(s,\theta)$ admitting $C(s)$ as the spine curve which can be given in the following form:

$$X(s,\theta) = C(s) + \lambda(s,\theta)\alpha(s) + \mu(s,\theta)\beta(s) + \omega(s,\theta)\gamma(s)$$

where $\{\alpha(s), \beta(s), \gamma(s)\}$ is the attached orthonormal frame on $C(s)$ and $\{\lambda, \mu, \omega\}$ are second order differentiable functions on s and θ. The following equation is satisfied;

$$\| X(s,\theta) - C(s) \|^2 = \epsilon r(s)^2$$

where $\epsilon = \pm 1$ or 0.

When $C(s)$ is a null curve, it arises three cases depending on the type of the pseudo spheres. The cases are;

- The canal surface $X(s,\theta)$ is foliated by S_1^2 (i.e. $\epsilon = 1$)
- The canal surface $X(s,\theta)$ is foliated by H_1^2 (i.e. $\epsilon = -1$)
- The canal surface $X(s,\theta)$ is foliated by Q^2 (i.e. $\epsilon = 0$)

III. CANAL SURFACES THROUGH A NULL QUATERNIONIC SPINE CURVE

We construct the canal surfaces by using a null quaternionic curve as the spine curve and have three cases depending on the type of the foliating pseudo-spheres.

Let $\{L, N, W, U\}$ be the Cartan frame attached to the null quaternionic curve $C(s)$. Then the canal surface can be defined as

$$X(s,\theta) = C(s) + \alpha(s,\theta)L(s) + \beta(s,\theta)N(s) + \gamma(s,\theta)W(s) + \mu(s,\theta)U(s) \tag{1}$$

Case 1.

Let the canal surface $X(s,\theta)$ be foliated by the pseudo sphere S_1^3. Since $X(s,\theta) - C(s)$ is a normal vector to the surface, we have

$$h(X(s,\theta) - C(s), X_s) = 0 \tag{2}$$

$$h(X(s,\theta) - C(s), X_\theta) = 0 \tag{3}$$

so that X_θ and X_s are tangent to the pseudo sphere S_1^3 [29]. Since $\epsilon = 1$,

$$\| X(s,\theta) - C(s) \|^2 = r^2(s) \tag{4}$$

Then we have following two equations by using the equation (1):

$$\gamma^2 + \mu^2 - 2\alpha\beta = r^2 \tag{5}$$

$$\gamma\gamma_s + \mu\mu_s - \alpha_s\beta - \alpha\beta_s = rr' \tag{6}$$

On the other hand, we differentiate the equation (1) with respect to s and use the equation (6) to obtain

$$rr' = \beta \tag{7}$$

Since $X(s,\theta) - C(s)$ is the normal vector it can be parallel to N or perpendicular to L hence it is a lightlike or spacelike vector, respectively.

i. Suppose that the normal vector is lightlike, i.e. parallel to N. Then we have

$$h(X(s,\theta)-C(s),N)=-\alpha=0 \tag{8}$$

$$h(X(s,\theta)-C(s),L)=-\beta=-1 \tag{9}$$

Using the equations (7), (8) and (9), we calculate

$$r^2(s)=2(s\pm k)$$

where k is a constant number. If we substitute the last equation in (5), we get

$$\gamma(s,\theta)=\sqrt{2(s\pm k)}\cos\theta$$

$$\mu(s,\theta)=\sqrt{2(s\pm k)}\sin\theta$$

where $\theta\in[0,2\pi]$. Therefore, we obtain the parameterization of the surface as:

$$X(s,\theta)=C(s)+N(s)+\sqrt{2(s\pm k)}\cos\theta W(s)+\sqrt{2(s\pm k)}\sin\theta U(s) \tag{10}$$

ii. Now suppose that the normal vector is spacelike. Then we have

$$h(X(s,\theta)-C(s),L)=-\beta=0 \tag{11}$$

$$h(X(s,\theta)-C(s),N)=\alpha=0 \tag{12}$$

Hence we find r as a constant. It means $X(s,\theta)$ is a tubular surface and in the following form:

$$X(s,\theta)=C(s)+r\cos\theta W(s)+r\sin\theta U(s) \tag{13}$$

Theorem 3.1 *The tubular surface defined in the equation (13) is a surface of revolution.*

Proof. Since $X(s,\theta)-C(s)\perp L$ and r is constant, we have $\beta=0$ and

$$rr'=0$$

$$\Leftarrow rr'-\alpha\gamma+\alpha\gamma=0$$

$$\Leftarrow -\alpha\beta_s-\beta\alpha_s-\alpha\gamma+\alpha\gamma+\gamma\gamma_s+\mu\mu_s-\beta=h(X(s,\theta)-C(s),X_s)=0$$

The last equation implies $p=\tau=0$, hence $C(s)$ is a straight line.

Case 2.

Suppose that the canal surface $X(s,\theta)$ is foliated by the pseudo sphere H_1^3. Since $\epsilon=-1$. Then we have

$$\| X(s,\theta)-C(s) \|^2=-r^2(s)$$

$$h(X(s,\theta)-C(s),X_s)=0$$

$$h(X(s,\theta)-C(s),X_\theta)=0$$

so that X_θ and X_s are tangent to the pseudo sphere H_1^3. Following similar steps we obtain $rr'=\beta$. Then we find

$$r^2(s)=2(-s\pm k)$$

where k is a constant number and $\alpha=0$.

i. If the normal vector is lightlike, then we have the canal surface which can be written in the following form:

$$X(s,\theta)=C(s)+N(s)+\sqrt{2(-s\pm k)}\cos\theta W+\sqrt{2(-s\pm k)}\sin\theta U \tag{14}$$

ii. Now suppose that the normal vector is spacelike, since $\gamma^2+\mu^2=-r^2$, this is a contradiction.

Case 3.

If the canal surface $X(s,\theta)$ is foliated by the pseudo sphere Q^3, we have following subcases:

i. Let the normal vector be parallel to N. Then $\alpha=0$ and $\beta=1$ hence $\gamma=\mu=0$. We obtain

$$X(s,\theta) = C(s) + N(s) \tag{15}$$

However, the equation (15) does not imply a surface.

ii. Once we choose the normal vector as spacelike, we get $\gamma = \mu = 0$. Hence it does not construct a surface.

Theorem 3.2 *Canal surface defined in the equation (10) has singularities if the principal curvatures p and τ of the curve $C(s)$ satisfy one of the following equations:*

i.

$$\begin{aligned}\tau(s) &= \frac{1}{2\sin(\theta)^2(s+k)}\\ &(\pm 4\sqrt{\left(\left(\frac{\cos(\theta)^2}{2}+\frac{\sin(\theta)}{2}-\frac{1}{2}\right)\sqrt{2s+2k}+(-p(s)\cos(\theta)^2+(-\sin(\theta)+1)p(s)+s+k)\cos(\theta)(s+k)\,(s)\cos(\theta)(s+k)^2}\\ &+\sqrt{2s+2k}\sin(\theta)+4\left(-\frac{p(s)\cos(\theta)^2}{2}+\left(-\frac{p(s)\sin(\theta)}{2}+k+s\right)\cos(\theta)+\frac{p(s)}{2}\right)(s+k))\end{aligned}$$

ii.

$$\begin{aligned}\tau(s) &= \frac{1}{2\sin(\theta)^2(s+k)}\\ &\left(\pm 4\sqrt{\left(\left(-\frac{\cos(\theta)^2}{2}-\frac{\sin(\theta)}{2}+\frac{1}{2}\right)\sqrt{2s+2k}+(-p(s)\cos(\theta)^2+(-\sin(\theta)+1)p(s)+s+k)\cos(\theta)(s+k)\right)\cos(\theta)(s+k)^2}\right.\\ &\left.+\sqrt{2s+2k\sin(\theta)}-4\left(\frac{p(s)\cos(\theta)^2}{2}+\left(-\frac{p(s)\sin(\theta)}{2}+k+s\right)\cos(\theta)-\frac{p(s)}{2}\right)(s+k)\right)\end{aligned}$$

Proof. Using the equation (10) we calculate the partial derivatives X_s and X_θ. Then we have

$$E = h(X_s, X_s) = -2\sqrt{2(s\pm k)}\cos\theta(1+\sqrt{2(s\pm k)}(p(s)-\tau(s)+p(s)\cos\theta)) + (p(s) + \frac{\cos\theta}{\sqrt{2(s\pm k)}})^2 + (p(s)-\tau(s)+\frac{\sin\theta}{\sqrt{2(s\pm k)}})^2$$

$$G = h(X_\theta, X_\theta) = 2(s\pm k)$$

$$F = h(X_s, X_\theta) = -\sqrt{2(s\pm k)}(\sin\theta(p(s)+\frac{\cos\theta}{\sqrt{2(s\pm k)}}) - \cos\theta(p(s)-\tau(s)+\frac{\sin\theta}{\sqrt{2(s\pm k)}}))$$

We obtain the mentioned cases in the Theorem3.2 for $EG - F^2 = 0$ by using a symbolic programming language.

Corollary 3.1 *If the spine curve is a planar null quaternionic one, then the corresponding canal surface has singularities for:*

$$\begin{aligned}p(s) &= \frac{1}{4\left(\cos(\theta)\sin(\theta)+\frac{1}{2}\right)(s+k)}\\ &\left(\pm 4\sqrt{\cos(\theta)\left(\left(-\frac{\cos(\theta)^2}{2}+\left(\frac{\sin(\theta)}{2}-\frac{1}{2}\right)\cos(\theta)-\frac{\sin(\theta)}{2}+\frac{1}{2}\right)\sqrt{2s+2k}+\cos(\theta)(\cos(\theta)+1)^2(s+k)^2\right)(s+k)^2}\right.\\ &\left.-(\cos(\theta)+\sin(\theta))\sqrt{2s+2k}+4\cos(\theta)(s+k)^2(\cos(\theta)+1)\right)\end{aligned}$$

Now assume that the canal surface is regular. We can give following theorems:

Theorem 3.3 *Gauss curvature of the canal surface defined in equation (10) is*

$$\begin{aligned}K = &\left(\left(\left(\cos(\theta)\sqrt{2s+2k}(p(s)-\tau(s))+\frac{d}{dx}p(s)-\frac{d}{ds}v(s)-\frac{\sin(\theta)}{(2s+2k)^{3/2}}\right)^2+\left(1+\sqrt{2s+2k}(p(s)-\tau(s)+p(s)\cos(\theta))+\sqrt{2s+2k}\cos(\theta)p(s)+\frac{d}{ds}p(s)-\frac{\cos(\theta)}{(2s+2k)^3(2)}\right)^2\right.\right.\\ &-\left(\frac{p(s)-t(t)+p(t)\cos(\theta)}{\sqrt{2x+2k}}+\sqrt{2t+2k}\left(\frac{d}{dt}p(s)-\frac{d}{dt}t(s)+\left(\frac{d}{dt}p(t)\right)\cos(\theta)\right)+\left(p(t)+\frac{\cot(\theta)}{\sqrt{2t+2k}}\right)p(t)+\left(p(t)-t(t)+\frac{\sin(\theta)}{\sqrt{2t+2k}}\right)(p(t)-t(s))\right)\left(\frac{2\cos(\theta)}{\sqrt{2t+2k}}+p(t)\right)\Big)\\ &\Big((2s+2k)\sin(\theta)^2+\cos(\theta)^2(2s+2k)-\Big(-\left(1+\sqrt{2s+2k}(p(s)-\tau(s)+p(s)\cos(\theta))+\sqrt{2s+2k}\cos(\theta)p(s)+\frac{d}{ds}p(s)-\frac{\cos(\theta)}{(2s+2k)^{2s)}}\right)\cos(\theta)\sqrt{2s+2k}\\ &-\left(\cos(\theta)\sqrt{2z+2k}(p(s)-u(s))+\frac{d}{dt}p(s)-\frac{d}{dt}t(s)-\frac{\sin(\theta)}{(2x+2kd^3)}\right)\sqrt{2z+2k}\sin(\theta)\Big)^2\Big)/(((-4k-4z+2p(s))\cos(\theta)+2\sin(\theta)(p(s)-2(s)))\sqrt{2z+2k}-3((k+t\\ &-\frac{\pi(s)}{2})p(s)+\frac{\pi(s)^2}{4})(s+k)\cos(\theta)^2-s(p(s)-\tau(s))\left(-\frac{p(s)\sin(\theta)}{2}+k+s\right)(s+k)\cos(\theta)+(2s+2k)p(s)^2-4\pi(s)(s+k)p(s)+1+(2s+2k)\tau(s)^2)\end{aligned}$$

where $p(s)$ and $\tau(s)$ are the curvature and torsion of the null quaternionic spine curve, respectively.

Theorem 3.4 *Mean curvature of the canal surface defined in equation (10) is*

$$H = (32(s+k)((-(p(s)^2 - 2p(s)\tau(s) + \frac{\tau(s)^2}{2})(s+k)^2\cos(\theta)^3 + (s+k)^2\sin(\theta)(-\frac{\tau(s)^2}{2} + p(s)^2)\cos(\theta)^2 + (\frac{(s+k)^2 p(s)^2}{2} - (\tau(s) + \frac{s}{4})(s+k)^2 p(s) + \frac{1}{8} + \frac{\tau(s)(s+k)^2}{2})\cos(\theta) - \frac{\sin(\theta)(s+k)^2 p(s)^2}{2}$$

$$- \frac{p(s)(s+k)^2}{4} + \frac{\tau(s)(s+k)^2}{2} + \frac{\sin(\theta)}{8})(\frac{d}{d}p(s)) + (p(s)(s+k)^2(p(s)-\tau(s))\cos(\theta)^3 - (s+k)^2\sin(\theta)(p(s) - \frac{\pi(s)}{2})\tau(s)\cos(\theta)^2 - (p(s)-\tau(s))(s+k)^2(p(s) - \frac{1}{2})\cos(\theta) + \frac{\sin(\theta)(s+k)^2 p(s)^2}{2}$$

$$- \frac{p(s)(s+k)^2}{4} - \frac{\sin(\theta)}{8})(\frac{d}{dt}\tau(s)) + (s+k)^2(p(s) - \frac{\tau(s)}{2})\tau(s)\cos(\theta)^3 + p(s)\sin(\theta)(s+k)^2(p(s)-\tau(s))\cos(\theta)^2 + (-\frac{(k+s-\frac{3}{2})(s+k)p(s)^2}{2} - (s+k)(k+s+\frac{\tau(s)}{2} + \frac{1}{4})p(s) + (\frac{k}{4} + \frac{s}{4})\tau(s)^2$$

$$+ \frac{1}{8} + \frac{k^2}{2} + sk + \frac{s^2}{2})\cos(\theta) - \frac{(p(s)-\tau(s))(s+k)((-\frac{\sin(\theta)}{4} - \frac{1}{4})p(s) + k + s + \frac{\sin(\theta)}{2})}{2})\sqrt{2s+2k} - 16(s+k)^3(\frac{d}{d}p(s))^2 + 16(s+k)^3(\cos(\theta)^2 + \frac{d}{d}\tau(s) + \cos(\theta) - 1)(\frac{d}{d}p(s)) - 8(s$$

$$+k)^3(\frac{d}{d}\tau(s))^2 - 16\cos(\theta)(s+k)^3(\frac{d}{d}\tau(s)) - 64p(s)(p(s)^2 - 4p(s)\tau(s) + 2\tau(s)^2)(s+k)^4\cos(\theta)^4 + 128(p(s)-\tau(s))(\sin(\theta)p(s)^2 + \frac{\tau(s)(\sin(\theta)+1)p(s)}{2} - \frac{\tau(s)^2(\sin(\theta)+1)}{4})(s+k)^4\cos(\theta)^3$$

$$+64(s+k)^2(\sin(\theta)(s+k)^2 p(s)^3 - 2(\frac{s}{8} + \tau(s)(\sin(\theta)+1))(s+k)^2 p(s)^2 + ((s+k)^2(\sin(\theta)+1)\tau(s)^2 + (sk + \frac{1}{2}s^2 + \frac{1}{2}k - \frac{1}{4})\tau(s) + \frac{1}{2} + \frac{s^2}{2} + \frac{k^2}{2} + sk)p(s) - \frac{\tau(s)^2(2sk + s^2 + k^2 - \frac{1}{2})}{4})\cos(\theta)^2$$

$$-32((s+k)^2(\sin(\theta)+1)p(s)^2 + (2k^2 + 4sk + 2s^2 + \frac{\sin(\theta)}{2})p(s) - k^2 - 2sk - s^2 - \frac{\sin(\theta)}{2} - \frac{1}{2})(p(s)-\tau(s))(s+k)^2\cos(\theta) + 16(s+k)^3 p(s)^3 - 16(s+k)^2((s+i)\tau(s) + i + (2s+1)k + s^2 + s$$

$$-\frac{1}{2})p(s)^2 + 32(k+s+\frac{\tau(s)}{4} + \frac{1}{2})(s+k)^3\tau(s)p(s) - 16(s+k)^3(k+s+\frac{1}{2})\tau(s)^2 - 8k^3 + (-24s-4)k^2 + (-24s^2-8s)k - 8s^3 - 4s^2 - 1)/(64(((-\frac{p(s)}{4} + \frac{s}{2} + \frac{k}{2})\cos(\theta)$$

$$- \frac{\sin(\theta)(p(s)-\tau(s))}{4})\sqrt{2s+2k} + (s+k)((k+s-\frac{\tau(s)}{2})p(s) + \frac{\tau(s)^2}{4})\cos(\theta)^2 + (p(s)-\tau(s))(s+k)(-\frac{p(s)\sin(\theta)}{2} + k + s)\cos(\theta) + (-\frac{s}{4} - \frac{k}{4})p(s)^2 + \frac{\pi(s)(s+k)p(s)}{2} - \frac{1}{8} + (-\frac{s}{4}$$

$$-\frac{k}{4})\tau(s)^2)(s+k)^2)$$

where $p(s)$ and $\tau(s)$ are the curvature and torsion of the null quaternionic spine curve, respectively.

Theorem 3.5 *Tubular surface defined in the equation (13) has singularities if the principal curvatures $p(s)$ and $\tau(s)$ of the curve $C(s)$ satisfy the following equation:*

$$p(s) = \frac{\tau(s)}{2} - \frac{1}{2r\cos\theta}$$

Suppose that the tubular surface in (13) is regular. Then we can give the following theorems:

Theorem 3.6 *Gauss curvature of the tubular surface defined in equation (13) is*

$$K = \frac{1}{4\left(\frac{1}{2} + r\left(p(s) - \frac{\tau(s)}{2}\right)\cos(\theta)\right)\cos(\theta)r}\Big(-1 + 8\left(p(s) - \frac{\tau(s)}{2}\right)p(s)r^2\cos(\theta)^4 + 6(p(s)\sin(\theta)r - \tau(s)\sin(\theta)r + 1)\left(p(s) - \frac{\tau(s)}{3}\right)r\cos(\theta)^3$$
$$+(-9p(s)^2r^2 + (6r^2\tau(s) + 2r\sin(\theta))p(s) - \tau(s)^2r^2 - 2\tau(s)\sin(\theta)r + 1)\cos(\theta)^2 - 6\left(p(s) - \frac{\tau(s)}{3}\right)r\cos(\theta)\Big)$$

Theorem 3.7 *Mean curvature of the tubular surface defined in equation (13) is*

$$H = \frac{-1 - 10\left(\frac{\tau(s)^2}{5} + \left(-\frac{4p(s)}{5} + \frac{1}{5}\right)\tau(s) + p(s)^2 - \frac{2p(s)}{5}\right)r^2\cos(\theta)^2 - 6\left(p(s) - \frac{\tau(s)}{3} - \frac{1}{3}\right)r\cos(\theta)}{8\left(\frac{1}{2} + r\left(p(s) - \frac{\tau(s)}{2}\right)\cos(\theta)\right)\cos(\theta)r}$$

Corollary 3.2 *The tubular surface defined in equation (13) is minimal if the following relation holds for the curvature and torsion of the null quaternionic spine curve:*

$$p(s) = \frac{\frac{2r\cos(\theta)\tau(s)}{5} + \frac{r\cos(\theta)}{5} - \frac{3}{10} \pm \frac{\sqrt{-4\tau(s)^2\cos(\theta)^2r^2 - 4\tau(s)\cos(\theta)^2r^2 + 4r^2\cos(\theta)^2 - 4r\cos(\theta)\tau(s) + 8r\cos(\theta) - 1}}{10}}{\cos(\theta)r}$$

Proof. We solve the equation $H = 0$ for the mean curvature given in Theorem 3.7 by using a symbolic programming language such as Maple.

Corollary 3.3 *The tubular surface defined in equation (13) is developable if*

$$\tau(s) = \frac{1}{(\sin(\theta)^2 - 1)(2\sin(\theta)^2 - 1)r}\Big(4r\sin(\theta)^4 p(s) \pm 2\sqrt{1 - \sin(\theta)^2}r\sin(\theta)^3 p(s) -$$
$$5r\sin(\theta)^2 p(s) \pm 2\sqrt{1 - \sin(\theta)^2}r\sin(\theta)p(s) + \sin(\theta)^3 \pm \sqrt{1 - \sin(\theta)^2}\sin(\theta)^2 +$$
$$rp(s) - \sin(\theta)\Big)$$

where $p(s)$ and $\tau(s)$ are the curvature and torsion of the null quaternionic spine curve, respectively.

IV. DISCUSSION AND CONCLUSION

The semi real quaternions and null quaternionic curves are developing subjects and attract attention of many scientists. Hence, in this work, we introduce the canal surfaces that admit the spine curve as a null quaternionic curve. It is obtained two kinds of canal surfaces that one is actually a tubular surface. We give the singularity conditions for that kind of a surfaces by using a symbolic programming language. When we assume that the surface is regular, we can interpret the Gauss and Mean curvatures by the means of the Cartan curvatures of the null quaternionic spine curve. Lastly, we give the relations between the Cartan curvatures of the spine curve for the pre-defined tubular surface to be a minimal or developable one.

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REFERENCES

[1] Hamilton, W. R. (1844). On quaternions; or on a new system of imagniaries in algebra. *Lond. Edinb. Dublin. Philos. Mag. J. Sci.*, 25 (3), 489-495.
[2] Aslan, S & Yaylı, Y. (2016). Canal surfaces with quaternions. *Adv. Appl. Cliffor.d Algebras*, 26, 31–8.
[3] Gök, İ. (2017). Quaternionic approach of canal surfaces constructed by some new ideas. *Adv. Appl. Clifford Algebras*, 27, 1175-1190.
[4] Shoemake, K. (1985). Animating rotation with quaternion curves. *Proceedings of the 12th annual conference on computer graphics and interactive techniques (SIG-GRAPH 85),* 19, New York, NY, USA, 245-254.
[5] Haralick, R. M. (2017). Longitudinal & Scalar Waves: Biquaternion generalized maxwell equations. Computer Science, Graduate Center, 172.
[6] Rosenfeld, B. (1997). Geometry of Lie groups. Kluwer Academic Publishers, Netherlands, 397.
[7] Dyachkova, M. (2007). On Hopf bundle analogue for semiquaternion algebra. *10th International Conference DGA*, Olomouc, Czech Republic, 45-47.
[8] Bharathi, K. & Nagaraj, M. (1987). Quaternion valued function of a real variable Serret-Frenet formula. *Indian J. Pure Appl. Math*., 18, 507-511.
[9] Çöken, A. C. & Tuna, A. (2004). On the quaternionic inclined curves in the semi-Euclidean space E42. *Appl. Math. Comput.,* 155, 373–389.
[10] Gök, İ., Okuyucu, O. Z., Kahraman, F. & Hacisalihoglu, H. H. (2011). On the quaternionic B2-slant helix in the Euclidean space E4. *Adv. Appl. Clifford Algebras*, 21, 707–719.
[11] Körpınar, T. & Baş, S. (2016). Characterization of Quaternionic Curves by Inextensible Flows. *Prespacetime Journal*, 7, 1680-1684.
[12] Okuyucu, O. Z. (2013). Characterizations of the quaternionic Mannheim curves in Euclidean space E4. *Mathematical Combinatorics*, 2, 44-53.
[13] Kecilioglu, O. & Ilarslan, K. (2013). Quaternionic Bertrand Curves in Euclidean 4-Space. *Bull. Math. Anal. Appl.*, 5 (3), 27-38.
[14] Önder, M. (2020). Quaternionic Salkowski Curves and Quaternionic Similar Curves. *Proc. Natl. Acad. Sci. India, Sect. A Phys. Sci.*, 90 (3), 447-456.
[15] Shoemake, K . (1985). Animating rotation with quaternion curves. *Proceedings of the 12th annual conference on computer graphics and interactive techniques (SIG-GRAPH 85),* 19, New York, NY, USA, 245-254 .
[16] Duggal, K. L. & Bejancu, A. (1996), Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications. Kluwer Academic Publisher, Dordrecht / Boston / London, 303.
[17] Ferrandez A., Gimenez A. & Lucas P. (2001). Null Helices in Lorentzian Space Forms. *Int. J. Mod. Phys. A*, 16 (30), 4845-4863.
[18] Çoken, A. C. & Ciftci, U. (2005). On the Cartan curvatures of a null curve in Minkowski spacetime. *Geom. Dedicata*, 114, 71-78.
[19] Kahraman, T. (2019). Null Quaternionic Slant Helices in Minkowski Spaces. *International J.Math. Combin.,* 1, 45-52.
[20] Aksoy, A. T. (2016). Pseudo-Spherical Null Quaternionic Curves in Minkowski Space R^4. *Acta Physica Polonica A*, 130, 259-261.
[21] Özel, Ş., Külahçı, M. A. & Bektaş, M. (2021). The Characterizations Of Null Quaternionic Curves In Minkowski 3-Space. *Turkish Journal of Science and Technology*, 16 (2), 261-267.
[22] Aksoy, A. T. & Çöken, A. C. (2015). Null Quaternionic Curves in Semi-Euclidean 3-Space of Index ν. *Acta Physica Polonica A*, 128, B-286-B-289.
[23] Maekawa, T., Patrikalakis, M. N., Sakkalis, T. & Yu, G. (1998). Analysis and applications of pipe surfaces. *Comput. Aided Geom. Des*., 15, 437–58 .

[24] Shani, U. & Ballard, D. H. (1984). Splines as embeddings for generalized cylinders. *Computer Vision , Graphics and Image Processing*, 27, 129-156.
[25] Wang, L., Ming, C. L. & Blackmore, D. (1995). Generating sweep solids for NC verification using the SEDE method. *Proceedings of the Fourth Symposium on Solid Modeling and Applications*, Atlanta, 364-375.
[26] Farouki, R. T. & Sverrissor, R. (1996). Approximation of rolling-ball blends for free-form parametric surfaces. *Computer-Aided Design*, 28, 871-878.
[27] Xu, Z., Feng, R. & Sun, JG. (2006). Analytic and algebraic properties of canal surfaces. *Journal of Computational and Applied Mathematics,* 195 (1-2), 220-228.
[28] Jinhua Qian, J., Tian, X., Fu, X. & Kim, Y. H. (2020). Classifications of Canal Surfaces with the Gauss Maps in Minkowski 3-Space. *Mathematics,* 8, 1453.
[29] Gray, A., Abbana, E. & Salamon, S. (2006). Modern Differential Geometry of Curves and Surfaces with Mathematica (3rd edition), Studies in Advanced Mathematics, Chapman and Hall/CRC, Boca Raton, FL, 1016.
[30] Ates, F. (2021). Tubular Surfaces Around a Null Curve and Its Spherical Images, *FUJMA*, 4 (3), 210-220.