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# On the integration of first order nonlinear differential equations and the conditions of Fuchs' theorem

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**Abstract**

In this paper we give the general solutions of a class of first order nonlinear Fuchs ordinary differential equations. This leads us to show by an example that the necessary conditions of Fuchs' theorem are not sufficient.

*Keywords:* Fuchs' theorem movable critical points sufficient conditions.
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## 1. Introduction

Among the nonlinear ordinary differential equations of the first order, easily integrable, we can cite Bernoulli's equations, Lagrange's equations, Clairaut's equations, Darboux's equations, ... [17]. Elliptic equations and Riccati's equations, although they are not integrable in general, are often encountered in the study of ordinary differential equations [17, 18]. The integration of several equations of higher order is reduced to the integration thereof by substitution [17].

The study of first order nonlinear Fuchs ordinary differential equations is motivated by their importance due to their appearance in many mathematical problems and their application in physics, see, for example, [16, 4, 2, 3, 10].

In the first part of this paper we introduce a class of these Fuchs equations that can be integrated. We give their solutions using quadratures, Riccati equations or elliptic functions.
Fuchs' theorem gives the necessary conditions so that the first order equation does not admit movable (contrary fixed) critical singular points [1, 8, 9, 5]. Painlevé's theorem guarantees that the equation studied

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in Fuchs' theorem does not admit essential movable critical singular points [15, 6]. Hermite's theorem ensures that the equation studied in Fuchs' theorem, when it is independent of $z$, is without movable critical singular points only if it is of genus zero or one [7, 6].

In the second part of this paper we give an example of first order ordinary differential equations which satisfies the conditions of Fuchs' theorem, but which is not with fixed critical points. This allows us to conclude that the conditions of Fuchs' theorem are not sufficient.

## 2. Integration of certain Fuchs equations

The work of [12, 11, 13, 14] focuses on first order ordinary differential equations with fixed critical points. Among their results, they showed that the equation

$$\left(y' - k\left(ay^2 + by + c\right)\right)^k \left(y' + m\left(ay^2 + by + c\right)\right)^m - \alpha = 0, \tag{1}$$

is satisfying the necessary Fuchs' conditions to be with fixed critical points when $k + m = 1, 2, 3, 4$ and $5$, where $k$ and $m$ are positive integers, and $a, b, c$ and $\alpha$ are complex numbers.
In this paper we give the general solution for some equations of the form (1). In the case where we have $a = 0$, we find the general solution of equation (1) for all $k$ and $m$ arbitrary positive integers. For the case $a \neq 0$, we give the general solution of equation (1) only when $k = 1$ and $m$ is even positive integer ($m = 2n$). The method used consists in finding a rational parameterization of the algebraic curve (1) where $y'$ and $y$ are assumed to be independent variables, and then to find the differential equation verified by the parameter. In the case where the initial equation (1) has fixed critical points, the equation verified by the parameter is either a linear equation, a Riccati equation or else it is integrable using elliptic functions.

### *2.1. Case where $a = 0$*

In this case, equation (1) is written in the form

$$\left(y' - k\left(by + c\right)\right)^k \left(y' + m\left(by + c\right)\right)^m = \alpha. \tag{2}$$

Let $D$ be the greatest common divisor (GCD) of $k$ and $m$. If $D > 1$, then by taking the $D$-th root of the two sides of equation (2) we obtain as powers in the left hand side coprime integers. So we can assume $k$ and $m$ to be coprime.
By setting $u = y + \frac{c}{b}$, equation (2) is written in the form

$$\left(u' - kbu\right)^k \left(u' + mbu\right)^m = \alpha. \tag{3}$$

In conclusion, it suffices to study the equation of the form

$$\left(y' - kby\right)^k \left(y' + mby\right)^m = \alpha, \tag{4}$$

with $k$ and $m$ are positive coprime integers, and $b$ and $\alpha$ are complex numbers.
Knowing that $k$ and $m$ are coprime, then according to Bezout's lemma there exist two positive integers $p$ and $q$ such that $pk - qm = 1$ or $qm - pk = 1$.

Suppose that we have the existence of $p$ and $q$ such that $pk - qm = 1$. Let's pose

$$\begin{cases} y' - kby = t\tau^p, \\ y' + mby = t\tau^{-q}. \end{cases} \tag{5}$$

By solving the system (5) with respect to $y'$ and $y$, while taking into account equation (4) we obtain

$$\begin{cases} y = \dfrac{t}{b\,(k+m)} \left( \dfrac{t^{q(k+m)}}{\alpha^q} - \dfrac{\alpha^p}{t^{p(k+m)}} \right), \\[2ex] y' = \dfrac{1}{k+m} t \left( \dfrac{m\alpha^p}{t^{p(k+m)}} + \dfrac{kt^{q(k+m)}}{\alpha^q} \right). \end{cases} \tag{6}$$

By taking the derivative of the function $y$, given by formula (6), we get

$$y' = \frac{1}{b(k+m)}\left(\frac{(q(k+m)+1)\,t^{q(k+m)}}{\alpha^q} + \frac{(p(k+m)-1)\,\alpha^p}{t^{p(k+m)}}\right)t'. \tag{7}$$

Knowing that $pk - qm = 1$, then equation (7) can be written in the form

$$y' = \frac{1}{b(k+m)}\left(\frac{k(p+q)\,t^{q(k+m)}}{\alpha^q} + \frac{m(p+q)\,\alpha^p}{t^{p(k+m)}}\right)t'. \tag{8}$$

From the formulas (6) and (8) we deduce the equation

$$t' = \frac{b}{p+q}t. \tag{9}$$

By integrating equation (9) we obtain

$$t = \gamma e^{\frac{b}{p+q}z}, \quad \gamma \in \mathbb{C}. \tag{10}$$

According to formula (6), the solution of equation (4) is therefore of the form

$$y = \frac{1}{b(k+m)}\left(\frac{1}{\alpha^q}\gamma^{q(k+m)+1}e^{\frac{b}{p+q}(q(k+m)+1)z} - \frac{\alpha^p}{\gamma^{p(k+m)-1}e^{\frac{b}{p+q}(p(k+m)-1)z}}\right). \tag{11}$$

If $m$ and $k$ are such that $qm - pk = 1$, then using the same reasoning we obtain the general solution in the form

$$y = \frac{1}{b(k+m)}\left(-\frac{1}{\alpha^q}\gamma^{q(k+m)+1}e^{-\frac{b}{p+q}(q(k+m)+1)z} + \frac{\alpha^p}{\gamma^{q(k+m)-1}e^{-\frac{b}{p+q}(q(k+m)-1)z}}\right). \tag{12}$$

If we look for constant solutions of equation (4) we end up with

$$(-kby)^k (mby)^m = (-kb)^k (mb)^m y^{m+k} = \alpha.$$

Hence

$$y^{m+k} = \frac{\alpha}{(-k)^k m^m b^{k+m}},$$

which gives $m+k$ constant solutions. The general solution therefore consists of these $m+k$ constant solutions and the family of solutions (11).

**Example 2.1.** *Let's consider the differential equation*

$$(y' - 2y)^2 (y' + 3y)^3 = 108. \tag{13}$$

*We have* $k = 2, m = 3, b = 1, \alpha = 108$ *and* $3m - 4k = 1$. *The general solution of equation (13) is therefore given by*

$$y = \frac{1}{5}\left(\frac{1}{108}\gamma^6 e^{2z} - \frac{108^2}{\gamma^9 e^{3z}}\right),$$

*to which is added the constant solutions* $y = a_j \quad j = 1, 2, \ldots, 5$, *where* $a_j$*'s are the fifth roots of the unit, i.e.* $(a_j)^5 = 1$.

*2.2. Case where $a \neq 0$*

In this case we only consider the case where $k = 1$ and $m = 2n$. Equation (1) is then written in the form

$$\left(y' - \left(ay^2 + by + c\right)\right)\left(y' + 2n\left(ay^2 + by + c\right)\right)^{2n} - \alpha = 0. \tag{14}$$

Equation (14) can also be written in the form

$$\left(y' - \left(a\left(y + \tfrac{b}{2a}\right)^2 + \tfrac{c}{a} - \left(\tfrac{b}{2a}\right)^2\right)\right)\left(y' + 2n\left(a\left(y + \tfrac{b}{2a}\right)^2 + \tfrac{c}{a} - \left(\tfrac{b}{2a}\right)^2\right)\right)^{2n} - \alpha = 0.$$

It is therefore sufficient to study the equation of the form

$$\left(y' - ay^2 - b\right)\left(y' + 2nay^2 + 2nb\right)^{2n} - \alpha = 0. \tag{15}$$

By setting $y' - ay^2 - b = \tau$ and $y' + 2nay^2 + 2nb = t$ we obtain

$$y' = \frac{1}{2n+1}\frac{t^{2n+1} + 2n\alpha}{t^{2n}}, \tag{16}$$

and by setting

$$y = \frac{u}{t^n}, \tag{17}$$

we get

$$u^2 = \frac{1}{(2n+1)\,a}t^{2n+1} - \frac{b}{a}t^{2n} - \frac{1}{(2n+1)\,a}\alpha. \tag{18}$$

From equation (18) we deduce

$$\frac{\partial u}{\partial t} = \frac{1}{au}\left(\frac{t^{2n}}{2} - nbt^{2n-1}\right). \tag{19}$$

By taking the derivative of the function $y$ given by formula (17) we obtain

$$\begin{aligned} y' &= -n\frac{u}{t^{n+1}}t' + \tfrac{1}{t^n}\left(\left(\frac{1}{au}\left(\tfrac{t^{2n}}{2} - nbt^{2n-1}\right)\right)t'\right) \\ &= -\frac{1}{2at^{n+1}u}t'\left(2anu^2 - tt^{2n} + 2bnt^{2n}\right). \end{aligned} \tag{20}$$

Identifying the values of $y'$ given by the formulas (16) and (20) we obtain

$$t' = 2at^{1-n}u = 2at^{1-n}\sqrt{\frac{1}{(2n+1)\,a}t^{2n+1} - \frac{b}{a}t^{2n} - \frac{1}{(2n+1)\,a}\alpha}. \tag{21}$$

The integration of equation (15) therefore leads to the integration of equation (21). In the case where we have $n = 1$, the solution of equation (15) is expressed using elliptic functions. In parametric form, the solution of equation (15) is given by the formula

$$z = \frac{1}{2a}\int \frac{t^{n-1}dt}{\sqrt{\frac{1}{(2n+1)a}t^{2n+1} - \frac{b}{a}t^{2n} - \frac{1}{(2n+1)a}\alpha}},$$

$$y = \frac{\sqrt{\frac{1}{(2n+1)a}t^{2n+1} - \frac{b}{a}t^{2n} - \frac{1}{(2n+1)a}\alpha}}{t^n}.$$

## 3. Example of equation with movable critical point

Before giving an example of an equation which satisfies the conditions of Fuchs' theorem, but which is not with fixed critical points, we will state the three theorems that we mentioned in the introduction.

**Theorem 3.1** (of Fuchs). *In order for the Fuchs equation*

$$\sum_{k=0}^{k=n} A_k(z,w)\left(w'\right)^{n-k}=0,\ n\in\mathbb{N}, \tag{22}$$

*where $A_k(z,w)$ are polynomials in $w$ with coefficients that are analytic in $z$, have fixed critical points, it is necessary that the following conditions must be satisfied.*

1. *The coefficient $A_0(w,z)$ is independent of $w$. It can therefore be considered $A_0(w,z)=1$.*
2. *The degree of $A_k(z,w)$ as a polynomial in $w$ is less than or equal to $2k$.*
3. *The roots in $w$ of the discriminant $D(w,z)$ must be solutions of equation (22).*
4. *If the expansion of $w'$ in the neighborhood of a solution $w_0$ of $D(w,z)=0$ is written in the form*

$$w'=s_0+b_k\left(w-w_0\right)^{\frac{k}{m}}+b_{k+1}\left(w-w_0\right)^{\frac{k+1}{m}}+\cdots,$$

*then we must have $k\geq m-1$.*

**Theorem 3.2** (of Painlevé). *The solutions of the equations of the form*

$$P\left(z,w,w'\right)=0,$$

*where $P$ is a polynomial in $w$ and $w'$ with analytic coefficients in $z$, do not admit movable critical essential singular points.*

**Theorem 3.3** (of Hermite). *If the equation*

$$P\left(w,w'\right)=0,$$

*where $P$ is a polynomial in $w$ and $w'$ with constant coefficients, does not admit movable critical points then the algebraic curve defined by $P(w,s)=0$ is of genus equal to $0$ or $1$.*

Let's now consider the equation

$$F\left(y,y'\right)=\left(y'-y^2\right)\left(y'+4y^2\right)^4+256=0. \tag{23}$$

**Lemma 3.4.** *The equation (23) is satisfying the conditions of Fuchs' theorem.*

*Proof.* The equation (23) can be written in the general form (22) of Fuchs equation as

$$F\left(y,y'\right)=\left(y'\right)^5+15y^2\left(y'\right)^4+80y^4\left(y'\right)^3+160y^6\left(y'\right)^2-256y^{10}+256=0. \tag{24}$$

Thus, obviously, the first two conditions of Fuchs' theorem are satisfied.
By eliminating $y'$ between the two equations $F(y,y')=0$ and $\dfrac{\partial F}{\partial y'}(y,y')=0$ we obtain the discriminant of equation (23):

$$D(y)=256\left(1-y^{10}\right). \tag{25}$$

It is clear that each root of the discriminant (25) is a solution of equation (23), so the third condition of Fuchs' theorem is checked.
Let $y_0$ be a root of the discriminant (25). Substituting $y$ with $u+y_0$ in equation (23), we obtain:

$$F(y',u)=\left(y'-(u+y_0)^2\right)\left(y'+4\left(u+y_0\right)^2\right)^4+256=0. \tag{26}$$

Since $\dfrac{\partial F}{\partial u}(0,0) \neq 0$, $\dfrac{\partial F}{\partial y'}(0,0) = 0$ and $F(0,0) = 0$, according to the implicit function theorem, $u$ can be written in the neighborhood of $y' = 0$ as

$$u = A_2 \left(y'\right)^2 + A_3 \left(y'\right)^3 + A_4 \left(y'\right)^4 + \dots. \tag{27}$$

As a result $y'$ can be written in the neighborhood of $y = y_0$ as

$$y' = B_1 \left(y - y_0\right)^{\frac{1}{2}} + B_2 \left(y - y_0\right)^{\frac{2}{2}} + B_3 \left(y - y_0\right)^{\frac{3}{2}} + \dots. \tag{28}$$

So, the fourth condition of Fuchs' theorem, is also verified. □

**Lemma 3.5.** *The equation (23) has movable critical points.*

*Proof.* By setting $t = y' + 4y^2$ and $\tau = y' - y^2$ we get $y' = \frac{t^5 - 1024}{5t^4}$. If we set $y = \frac{u}{t^2}$ we deduce that $u^2 = \frac{1}{5}\left(t^5 + 256\right)$, which gives $\frac{\partial u}{\partial t} = t^5$. And so by differentiating we get

$$y' = \frac{-1}{2ut^3}\left(4u^2 - t^5\right) t'.$$

The parameter $t$ therefore satisfies the differential equation

$$t' = \frac{\frac{t^5 - 1024}{5t^4}}{\frac{-1}{2ut^3}\left(4u^2 - t^5\right)} = \frac{2}{t}\sqrt{\frac{1}{5}\left(t^5 + 256\right)}.$$

The equation thus obtained has movable critical points and therefore equation (23) is also.
Alternate proof to show that the equation (23) has movable critical points is by calculating its genus, which is equal to 2 and then by applying Hermit's theorem. □

As an immediate result of the last two lemmas we have the following theorem.

**Theorem 3.6.** *The four necessary conditions of Fuchs' theorem 3.1 are not sufficient.*

## 4. Conclusion

In conclusion, we have given the general solution for a class of Fuchs differential equation

$$\left(y' - k\left(ay^2 + by + c\right)\right)^k \left(y' + m\left(ay^2 + by + c\right)\right)^m - \alpha = 0,$$

which leads us to show that the equation

$$\left(y' - y^2\right)\left(y' + 4y^2\right)^4 + 256 = 0$$

has movable critical points while it satisfies all the conditions of Fuchs' theorem.
As a result we have shown that the necessary conditions of Fuchs' theorem are not sufficient.

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