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Oscillation Criteria for Fourth Order Differential Equations

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Abstract: Oscillation theory is one of the important and striking subjects of applied mathematics. Therefore, it has been meticulously studied by many researchers for many years. Many results have been obtained concerning the oscillation of differential equations of various orders. Some of the oscillation criteria obtained are related to fourth order differential equations. In this study, new oscillation criteria are given for a special type of fourth order differential equation. The importance of these criteria is due to the fact that the known results are expanded and have not been used before.

Dördüncü Mertebeden Diferensiyel Denklemler için Salınım Kriterleri

Anahtar Kelimeler
Salınım,
Dördüncü mertebe,
Diferensiyel denklem

Öz: Salınım teorisi, uygulamalı matematiğin önemli ve ilgi çekici konularından biridir. Bu nedenle birçok araştırmacı tarafından uzun yıllar yoğun bir şekilde incelenmiştir. Çeşitli mertebeden diferensiyel denklemlerin salınımı ile ilgili birçok sonuç elde edilmiştir. Elde edilen salınım kriterlerinin bir kısmı dördüncü mertebeden diferensiyel denklemlerle ilgilidir. Bu çalışmada dördüncü mertebeden belirli tipteki diferensiyel denklemler için yeni salınım kriterleri verilmiştir. Burada elde edilen kriterlerin önemi, bilinen sonuçların genişletilmiş hali ve daha önce kullanılmamış olmasından kaynaklanmaktadır.

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1. Introduction

This paper concerns the oscillatory behaviour of solutions to a fourth order linear delay differential equation

$$(r_2(r_1x')')'' + q_1(t)x(\tau_1(t)) = 0, \qquad t \geq t_0 > 0. \tag{1}$$

The following properties are assumed to be provided during this study

(A_1) $r_1,\ r_2 \in C([t_0,\infty),\mathbb{R})$ are positive and satisfy

$$\pi_1(t_0) = \int_{t_0}^{\infty} \frac{ds}{r_1(s)} < \infty \quad \text{and} \quad \pi_2(t_0) = \int_{t_0}^{\infty} \frac{ds}{r_2(s)} < \infty,$$

(A_2) $q_1 \in C([t_0,\infty),\mathbb{R})$, $q_1(t) \geq 0$ and does not vanish for all large t for this interval $[t_*,\infty)$ for some $t_* \in [t_0,\infty)$,

(A_3) $\tau_1(t) \leq t$, $\lim_{t\to\infty}\tau_1(t) = \infty$ such that $\tau_1 \in C^1([t_0,\infty),\mathbb{R})$ is a strictly increasing function.

Further, equation (1) is called delay differential equation, since $\tau_1(t) \leq t$.

Define the operators

$$L_0x = x,\ L_1x = r_1x',\ L_2x = r_2(r_1x')',\ L_3x = (r_2(r_1x')')',\ L_4x = (r_2(r_1x')')''.$$

x is said to be a solution of (1); if function x four times continuously differentiable and satisfies equation (1) on $[T_x,\infty)$. If a solution of (1) has no largest zero for all large t then this solution is termed oscillatory. Otherwise, a solution of (1) is termed nonoscillatory. If all of the solutions of equation (1) are oscillates, (1) is called oscillatory.

Dzurina and Jadlovska[5] used the definition of property (A) in their work. The definition of property A is if any solution x of (1) is either oscillatory or satisfies $\lim\limits_{t\to\infty} x(t) = 0$. We can also see this in the results of Kiguradze's work[6]. In place of property (A), some researchers prefer to use that the equation is almost oscillatory. Dzurina and Jadlovska's conclusions[5] include new results obtained on property (A) and oscillation of the form

$$(r_2(r_1y')')' + q_1(t)y(\tau_1(t)) = 0 \qquad (2)$$

third order delay differential equation. Research of the qualitative behavior of canonical third order differential equations especially in point of oscillation and nonoscillation have been the main topic of wide research. And a lot of studies have been done on this point. Among these studies we can refer [1-5].

In addition, the oscillation theory for higher order differential equations call attention of a great number of authors. Thus, a lot of work have been conducted on the oscillation theory of higher order differential equations, especially the fourth order, which we can refer to as reference [7-11].

In summary, we can say the following for our work: First of all, we will examine the applicability of the new criteria obtained in [5] and the results of the oscillation of (2) to the fourth order differential equation in form (1), which we have discussed.

[5] forms the basis of our ideas. However, we would like to point out that the results we will obtain in this paper are more general and improved than those obtained in [5].

2. Material and Method

The whole of the functional inequalities used in our work are supposed to hold, which are satisfied for all large t. As usual, we may consider solely not negative solutions of (1). We begin with the main lemma, which is used in our theorems.

Lemma 1. Let $(A_1) - (A_3)$ hold and x be a solution of equation (1) such that $x(t) > 0$. Then there are eight cases

for x:

Case (a): $x > 0,\ L_1x < 0,\ L_2x < 0,\ L_3x < 0,\ L_4x < 0,$
Case (b): $x > 0,\ L_1x < 0,\ L_2x > 0,\ L_3x < 0,\ L_4x < 0,$
Case (c): $x > 0,\ L_1x > 0,\ L_2x > 0,\ L_3x > 0,\ L_4x < 0,$
Case (d): $x > 0,\ L_1x > 0,\ L_2x < 0,\ L_3x < 0,\ L_4x < 0,$
Case (e): $x > 0,\ L_1x > 0,\ L_2x < 0,\ L_3x > 0,\ L_4x < 0,$
Case (f): $x > 0,\ L_1x > 0,\ L_2x > 0,\ L_3x < 0,\ L_4x < 0,$
Case (g): $x > 0,\ L_1x < 0,\ L_2x < 0,\ L_3x > 0,\ L_4x < 0,$
Case (h): $x > 0,\ L_1x < 0,\ L_2x > 0,\ L_3x > 0,\ L_4x < 0,$

for $t \geq t_1$ with sufficiently large t.

Proof. The proof is obvious and hence is neglected.

At this time, we will build a new criterion for almost oscillatory, that is property (A) of (1).

Theorem 1. Let (A_1)-(A_3) hold. If

$$\int_{t_0}^{\infty} \frac{1}{r_1(z_1)} \left(\int_{t_0}^{z_1} \frac{1}{r_2(s_1)} \left(\int_{t_0}^{s_1} \int_{t_0}^{u_1} q_1(s) ds du_1 \right) ds_1 \right) dz_1 = \infty, \tag{3}$$

then all of the solutions of (1) are almost oscillatory.

Proof.

Firstly, we note that if both (A_1) and (3) hold, then

$$\int_{t_0}^{\infty} \frac{1}{r_2(s_1)} \int_{t_0}^{s_1} \int_{t_0}^{u_1} q_1(s) ds du_1 ds_1 = \int_{t_0}^{\infty} q_1(s) ds = \infty. \tag{4}$$

If x is a not an oscillatory solution of (1) on $[t_0, \infty)$, for $t \geq t_1$ we get $t_1 \geq t_0$ such that $x(t) > 0$ and $x(\tau_1(t)) > 0$. From Lemma 1, eight possible cases may emerge for $t \geq t_1$. Each of these cases will be taken into account individually.

Suppose that case (a) holds. As $L_1 x < 0$, we view that x is not increasing, namely, there is a constant $m \geq 0$ such that $\lim_{t\to\infty} x(t) = m$. It is assert that $m = 0$. Vice versa, suppose that $m > 0$. So there is a $t_2 \geq t_1$ such that $x(\tau_1(t)) \geq m$ for $t \geq t_2$. Hence, for $t \geq t_2$

$$-L_4 x(t) = q_1(t) x(\tau_1(t)) \geq m q_1(t). \tag{5}$$

Integrating (5) from t_2 to t twice, we acquire

$$-L_2 x(t) \geq -L_2 x(t_2) + m \int_{t_2}^{t} \int_{t_2}^{u_1} q_1(s) ds du_1 \geq m \int_{t_2}^{t} \int_{t_2}^{u_1} q_1(s) ds du_1.$$

Therefore

$$-(L_1 x)'(t) \geq \frac{m}{r_2(t)} \int_{t_2}^{t} \int_{t_2}^{u_1} q_1(s) ds du_1. \tag{6}$$

Integrating (6) again from t_2 to t, we obtain

$$-x'(t) \geq \frac{m}{r_1(t)} \int_{t_2}^{t} \frac{1}{r_2(s_1)} \left(\int_{t_2}^{s_1} \int_{t_2}^{u_1} q_1(s) ds du_1 \right) ds_1. \tag{7}$$

Integrating (7) from t_2 to t and because of (3), we get

$$x(t) \leq x(t_2) - m \int_{t_2}^{t} \frac{1}{r_1(z_1)} \left(\int_{t_2}^{z_1} \frac{1}{r_2(s_1)} \left(\int_{t_2}^{s_1} \int_{t_2}^{u_1} q_1(s) ds du_1 \right) ds_1 \right) dz_1 \to -\infty$$

as $t \to \infty$, this contradicts with our assumption. So $\lim_{t\to\infty} x(t) = 0$.

Suppose that case (b) exists. If proceeds same way with case (a), (5) is obtained. Integrating (5) from t_2 to t twice, we get

$$L_2 x(t) \leq -L_2 x(t_2) - m \int_{t_2}^{t} \int_{t_2}^{u_1} q_1(s) ds du_1 \to -\infty \quad as \quad t \to \infty \tag{8}$$

we used (4), thus a contradiction is obtained and $\lim_{t\to\infty} x(t) = 0$.

Suppose that case (c) applies. So

$$w_1(t) = \frac{L_3 x(t)}{x(\tau_1(t))}, \quad t \geq t_1$$

is defined. Certainly, $w_1(t) > 0$ for $t \geq t_1$. By (1), we acquire

$$w_1'(t) = \frac{L_4 x(t)}{x(\tau_1(t))} - \frac{L_3 x(t) x'(\tau_1(t))\tau_1'(t)}{x^2(\tau_1(t))}$$

$$\geq \frac{L_4 x(t)}{x(\tau_1(t))} = -q_1(t).$$

If the integral taken from t_2 to t and using equality (4), we attain

$$w_1(t) \leq w_1(t_2) - \int_{t_2}^{t} q_1(s) ds \to -\infty \quad \text{as} \quad t \to \infty,$$

which contradicts with $w_1(t) > 0$.

Suppose that case (d) applies. Because of x is an increasing function, integration (1) from t_1 to t gives,

$$-(L_2 x)'(t) \geq k \int_{t_1}^{t} q_1(s) ds. \tag{9}$$

Integrating (9) from t_1 to t twice and using equality (4), we obtain

$$r_1 x'(t) \leq r_1 x'(t_1) - k \int_{t_1}^{t} \frac{1}{r_2(s_1)} \left(\int_{t_1}^{s_1} \int_{t_1}^{u_1} q_1(s) ds du_1 \right) ds_1 \to -\infty$$

as $t \to \infty$, and this is a contradiction.

Proof of case (e) and case (f) are similar to proof of case (c) and case (b). Hence these cases are omitted. Now, we suppose that case (g) applies. From (1), we acquire

$$-L_4 x(t) = q_1(t) x(\tau_1(t)) \geq m q_1(t). \tag{10}$$

Integrating (10) from t_2 to t, we attain

$$L_3 x(t) \leq L_3 x(t_2) - m \int_{t_2}^{t} q_1(s) ds \to -\infty$$

as $t \to \infty$. Thus a contradiction is obtained and $\lim_{t\to\infty} x(t) = 0$.

Case (h) can be proved similar to case (g).

Hence, the proof is complete.

3. Results

Theorem 2. Assume $(A_1)-(A_3)$. If

$$\liminf_{t\to\infty}\int_{\tau_1(t)}^{t}\frac{1}{r_1(z_1)}\left(\int_{t_0}^{z_1}\frac{1}{r_2(s_1)}\left(\int_{t_0}^{s_1}\int_{t_0}^{u_1}q_1(s)dsdu_1\right)ds_1\right)dz_1>\frac{1}{e} \tag{11}$$

and

$$\limsup_{t\to\infty}\int_{\tau_1(t)}^{t}\frac{1}{r_1(s_1)}\int_{s_1}^{t}\frac{1}{r_2(u_1)}\int_{u_1}^{t}\int_{z_1}^{t}q_1(s)dsdz_1du_1ds_1>1 \tag{12}$$

then (1) is almost oscillatory.

Proof. Let x is not oscillatory solution of (1) on $[t_0,\infty)$**.** As usual, we may get a value of $t_1\geq t_0$ such that $x(t)>0$ and $x(\tau_1(t))>0$ for $t\geq t_1$. At the time there exist eight possible cases (a)-(h), as Lemma 1. Let case (a) holds. If the integral of equation (1) taken from t_1 to t with x is not increasing, we have

$$-L_3x(t)=-L_3x(t_1)+\int_{t_1}^{t}q_1(s)x(\tau_1(s))ds\geq x(\tau_1(t))\int_{t_1}^{t}q_1(s)ds \tag{13}$$

namely,

$$-(L_2x)'(t)\geq x(\tau_1(t))\int_{t_1}^{t}q_1(s)ds. \tag{14}$$

If the integral of (14) taken from t_1 to t again, we acquire

$$-L_2x(t)+L_2x(t_1)\geq\int_{t_1}^{t}x(\tau_1(u_1))\int_{t_1}^{u_1}q_1(s)dsdu_1\geq x(\tau_1(t))\int_{t_1}^{t}\int_{t_1}^{u_1}q_1(s)dsdu_1.$$

If the integral taken from t_1 to t, we attain

$$-L_1x(t)\geq\int_{t_1}^{t}\frac{x(\tau_1(s_1))}{r_2(s_1)}\int_{t_1}^{s_1}\int_{t_1}^{u_1}q_1(s)dsdu_1ds_1 \tag{15}$$

$$\geq x(\tau_1(t))\int_{t_1}^{t}\frac{1}{r_2(s_1)}\int_{t_1}^{s_1}\int_{t_1}^{u_1}q_1(s)dsdu_1ds_1$$

or

$$x'(t)+\left(\frac{1}{r_1(t)}\int_{t_1}^{t}\frac{1}{r_2(s_1)}\int_{t_1}^{s_1}\int_{t_1}^{u_1}q_1(s)dsdu_1ds_1\right)x(\tau_1(t))\leq 0.$$

But, by [6], condition (11) provides that this inequality have not a positive solution, this is a contradiction with our primary supposition.

Suppose that case (b) applies. Integrating (1) from u_1 to $t(>u_1)$, twice and from the fact the monotony of x, we acquire

$$-L_2x(t)+L_2x(u_1)\geq\int_{u_1}^{t}x(\tau_1(s_1))\int_{u_1}^{s_1}q_1(s)dsds_1,$$

and

$$(L_1x)'(u_1) \geq \frac{x(\tau_1(t))}{r_2(u_1)} \int_{u_1}^{t} \int_{u_1}^{s_1} q_1(s)dsds_1.$$

Repeating the steps above, integrating from u_1 to $t(> u_1)$ twice, we attain

$$x(u_1) \geq x(\tau_1(t)) \int_{u_1}^{t} \frac{1}{r_1(z_1)} \int_{u_1}^{z_1} \frac{1}{r_2(s_1)} \int_{u_1}^{s_1} \int_{u_1}^{x} q_1(s)dsdxds_1dz_1. \tag{16}$$

Substitute of $u_1 = \tau_1(t)$ in (16), we get a contradiction with (12).

Pointing that (3) is required for the validation of (11), it pursue right away that cases (c)-(f) are not possible.

Suppose that case (g) holds. Integrating equation (1) from u_1 to $t(> u_1)$ and from the fact that the monotony of x we get

$$L_3x(t) - L_3x(u_1) = -\int_{u_1}^{t} q_1(s)x(\tau_1(s))ds$$

namely

$$L_3x(u_1) \geq x(\tau_1(t)) \int_{u_1}^{t} q_1(s)ds.$$

Integrating again from u_1 to t; $(t > u_1)$, we have

$$L_2x(u_1) \geq x(\tau_1(t)) \int_{u_1}^{t} \int_{u_1}^{s_1} q_1(s)dsds_1$$

that is

$$(L_1x)'(u_1) \geq \frac{x(\tau_1(t))}{r_2(t)} \int_{u_1}^{t} \int_{u_1}^{s_1} q_1(s)dsds_1.$$

Integrating again from u_1 to t; $(t > u_1)$ twice, we acquire

$$x(u_1) \geq x(\tau_1(t)) \int_{u_1}^{t} \frac{1}{r_1(z_1)} \int_{u_1}^{z_1} \frac{1}{r_2(s_1)} \int_{u_1}^{s_1} \int_{u_1}^{x} q_1(s)dsdxds_1dz_1. \tag{17}$$

Substitute of $u_1 = \tau_1(t)$ in (17), this contradicts with (12).

Let case (h) holds. Integrating (1) from u_1 to t twice, we get

$$-(L_1x(u))' \geq \frac{x(\tau_1(t))}{r_2(t)} \int_{u_1}^{t} \int_{u_1}^{s_1} q_1(s)dsds_1.$$

Integrating the last inequality from u_1 to t; $(t > u_1)$, we gain

$$0 \geq -L_1x(t) + L_1x(u_1) \geq \int_{u_1}^{t} \frac{x(\tau_1(z_1))}{r_2(z_1)} \int_{u_1}^{z_1} \int_{u_1}^{s_1} q_1(s)dsds_1dz_1.$$

Repeating the steps above, integration from u_1 to t; $(t > u_1)$ once, we have inequality (17) and a contradiction with (12). This situation is similar to the last part of the proof of the case (g). So the proof is completed.

Theorem 3. Let $(A_1) - (A_3)$. If

$$\limsup_{t\to\infty} \pi_1(t) \int_{t_0}^{t} \frac{1}{r_2(u_1)} \int_{t_0}^{s_1} \int_{t_0}^{u_1} q_1(s) ds du_1 ds_1 > 1 \tag{18}$$

and (12) hold, then equation (1) is oscillatory.

Proof. Let x is a not oscillatory solution of equation (1) on $[t_0, \infty)$. As usual, for $t \geq t_1$ we may get $t_1 \geq t_0$ such that $x(t) > 0$ and $x(\tau_1(t)) > 0$. At that time there are eight possible cases (a)-(h), as Lemma 1. Suppose that case (a) applies. In that case

$$x(t) = x(\infty) - \int_{t}^{\infty} \frac{1}{r_1(s)} L_1 x(s) ds \geq -L_1 x(t) \pi_1(t). \tag{19}$$

Using the monotony of x and (19) in (15), we see that

$$\begin{aligned} -L_1 x(t) \geq \quad & x(t) \int_{t_1}^{t} \frac{1}{r_2(s_1)} \int_{t_1}^{s_1} \int_{t_1}^{u_1} q_1(s) ds du_1 ds_1 \\ \geq \quad & -L_1 x(t) \pi_1(t) \int_{t_1}^{t} \frac{1}{r_2(s_1)} \int_{t_1}^{s_1} \int_{t_1}^{u_1} q_1(s) ds du_1 ds_1. \end{aligned}$$

If this inequality are taking limsup on both sides, and this contradicts with (18). The proof of case (b) keeps going in the same way as the case of Theorem 4. To prove that cases (c)-(f) are not possible, pointing that (4) is required for the validity of (18). The proof of the other cases keeps going in the same way as that of Theorem 2. Thus, the proof is completed.

Example 1. We take account of the fourth order delay differential equation form of

$$(t^3(t^2 x'(t))')'' + 2t^3 x(\frac{t}{2}) = 0, \qquad t \geq 0, \tag{20}$$

where $r_1(t) = t^2, r_2(t) = t^3, q_1(t) = 2t^3, \tau_1(t) \leq t, \lim_{t\to\infty} \tau_1(t) = \infty$.

And the equation (20) has the main assumptions $(A_1) - (A_3)$. Also condition (3), i.e.,

$$\int_{t_0}^{\infty} \frac{1}{{z_1}^2} \left(\int_{t_0}^{z_1} \frac{1}{{s_1}^3} \left(\int_{t_0}^{s_1} \int_{t_0}^{u_1} 2s^3 \, ds du_1 \right) ds_1 \right) dz_1 = \infty$$

is supplied and by Theorem 2, we infer from all of the solutions of Eq. (20) are almost oscillatory without any additional requirement.

4. Discussion and Conclusion

In this paper, three theorem on oscillation for fourth order differential equations with noncanonical operators has been obtained and an example has been given for intelligibility of the theorems. Furthermore, obviously in Theorem 2 any nonoscillatory solution satisfies either case (a)-(b) or case (g)-(h) of Lemma 1. It has been shown that the oscillation results given for third order noncanonical differential equations can be applied to fourth order differential equations.

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