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Asymptotic Approach to Boundary Functionals of a Semi-Markovian Random Walk With Generalized Delaying Barrier

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Abstract. In this study, a semi-Markovian random walk process with a generalized delaying barrier and its three boundary functionals (N_1, τ_1, γ_1) are mathematically constructed. Under some weak conditions, exact expressions and asymptotic expansions for the expected values and variances of these boundary functionals are obtained.

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1. INTRODUCTION

A great deal of problems on the stochastic finance, reliability, mathematical biology, queuing theory, stock control theory and mathematical insurance can be clarified by means of random walk process or its various modifications. Some of the application fields of these kind of problems are control of military stocks, refinery stocks, mathematical insurance, etc. Many studies have been done related to semi-Markovian random walk with a delaying barrier due to their practical and theoretical importance (see, Aliyev et. al. [1], [2]; Alsmeyer [3]; Borovkov [4]; Feller [5]; Gihman and Skorohod [6]; Janssen and Leeuwaarden [7]; Khaniyev [8]; Khaniyev and Kucuk [10]; Khaniyev and Mammadova [11]; Khaniyev et. al. [9], [12]; Lotov [13]; Rogozin [14]). As known, it is very important to investigate the boundary functionals of the stochastic processes. There are many studies on investigation of the boundary functionals in literature (see, Alsmeyer [3]; Janssen and Leeuwaarden [7]; Khaniyev [8]; Khaniyev and Mammadova [11]; Lotov [13]; Rogozin [14]; etc.). In this study, a semi-Markovian random walk process with a generalized delaying barrier and its three boundary functionals (N_1, τ_1, γ_1) are mathematically constructed. With the use of asymptotic approach, the approximation formulas are obtained for the expected values and variances of these boundary functionals.

2. Mathematical Construction of the Process $X(t)$ and Its Boundary Functionals

Let $\{(\xi_n, \eta_n)\}$, $n = 1, 2, 3, \cdots$ be a sequence of independent and identically distributed random pairs defined on a probability space $(\Omega, \mathcal{F}, P)$, where ξ_n takes only positive, η_n takes both negative and positive values. Suppose that ξ_n and η_n are mutually independent random variables and distribution functions of these random variables are known:

$$\Phi(t) = P\{\xi_n \leq t\}, F(x) = P\{\eta_n \leq x\}, t \geq 0, x \in \mathbb{R}.$$

Define renewal sequence $\{T_n\}$ and random walk $\{S_n\}$ as follows:

$$T_n = \sum_{i=1}^{n} \xi_i,\ S_n = \sum_{i=1}^{n} \eta_i, \quad n \geq 1, \quad T_0 = S_0 = 0, \quad n = 1, 2, \cdots .$$

Moreover, define the sequences of random variables $N_n, L_n, \zeta_n, \tau_n, \theta_n$ and γ_n as follows:

$$\begin{aligned}
N_0 =& 0, L_0 = 0, \zeta_0 = z, \tau_0 = 0, \theta_0 = 0 \ and\ \gamma_0 = 0,\\
N_1 \equiv& N_1(\lambda z) = \inf\{n \geq 1 : \lambda z - S_n < 0\},\\
L_1 =& \inf\{n \geq 1 : -\eta_{N_1+n} > 0\}, \quad \zeta_1 = -\eta_{N_1+L_1},\\
\tau_1 =& T_{N_1} = \sum_{i=1}^{N_1} \xi_i, \theta_1 = \sum_{i=1}^{L_1} \xi_{N_1+i}, \quad \gamma_1 = \tau_1 + \theta_1,\\
N_n \equiv& N_n(\lambda\zeta_{n-1})\\
&= \inf\left\{k \geq 1 : \lambda\zeta_{n-1} - \left(S_{N_1+L_1+\cdots+N_{n-1}+L_{n-1}+k} - S_{N_1+L_1+\cdots+N_{n-1}+L_{n-1}}\right) < 0\right\},\\
L_n =& \inf\{k \geq 1 : -\eta_{N_1+L_1+\cdots+N_n+k} > 0\}, \quad \zeta_n = -\eta_{N_1+L_1+\cdots+N_n+L_n},\\
\tau_n =& T_{N_1+L_1+\cdots+N_{n-1}+L_{n-1}+N_n},\\
\theta_n =& \sum_{i=1}^{L_n} \xi_{N_1+L_1+\cdots+N_{n-1}+L_{n-1}+i}, \quad \gamma_n = \tau_n + \theta_n, n = 1, 2, \cdots
\end{aligned}$$

Moreover, define $v(t) \equiv \max\{n \geq 1 : T_n > t\}$, $t > 0$. We can now construct desired stochastic process $X(t)$ in the following form:

$$X(t) = \max\left\{0, \lambda\zeta_n - \left(S_{v(t)} - S_{N_1+L_1+\cdots+N_n+L_n}\right)\right\},$$

for each $t \in [\gamma_n, \gamma_{n+1})$, $n = 0, 1, \cdots$, where λ is a positive constant.

One of trajectories for the process $X(t)$ is given as in Figure 1:

Note that, $\zeta_1, \zeta_2, \cdots$ are independent and identically distributed positive valued random variables having the following distribution function $\pi(z)$:

$$\pi(z) \equiv P\{\zeta_n \leq z\} = \frac{F(0) - F(-z)}{F(0)}, z \geq 0.$$

The process $X(t)$ is called as "Semi-Markovian Random Walk Process with a Generalized Delaying Barrier". The main purpose of this study is to obtain exact expressions and asymptotic expansions for the expected values and variances of the boundary functionals $(N_1(\lambda\zeta_1), \tau_1(\lambda\zeta_1), \gamma_1(\lambda\zeta_1))$ of the process $X(t)$, as $\lambda \to \infty$.

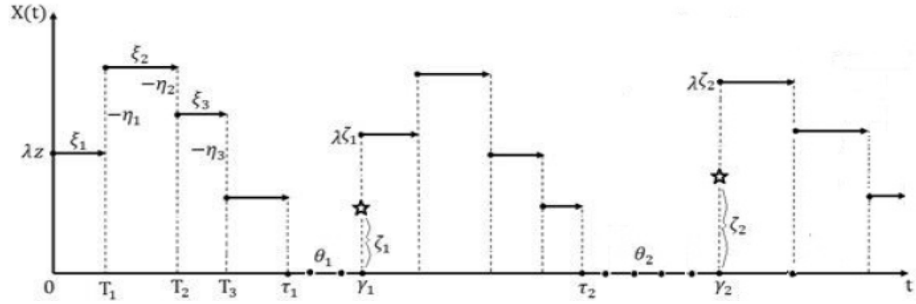

Figure 1. One of trajectories for the process

3. Exact Expressions for Expected Values and Variances of the Boundary Functionals

Define the first ladder variables of random walk $\{S_n\}$ before the exact expressions are obtained:

$$\nu_1^+ = \min\{n \geq 1 : S_n > 0\}, \quad \chi_1^+ \equiv \sum_{i=1}^{\nu_1^+} \eta_i = S_{\nu_1^+}.$$

Note that ν_1^+ and χ_1^+ are called as the first ladder epoch and ladder height of random walk $\{S_n\}$, respectively.

It is known that the random variables ν_1^+ and χ_1^+ are finite with probability 1, when $m_1 \equiv E(\eta_n) > 0$. Now, define the sequences of independent random pairs (ν_n^+, χ_n^+), $n = 2, 3, \cdots$ which have the same distribution as (ν_1^+, χ_1^+). With the use of the random pairs (ν_n^+, χ_n^+) and E. DynkinÔÇÖs Principle, boundary functional $N_1(x)$ can be represented as follows:

$$N_1(x) = \sum_{i=1}^{H(x)} \nu_i^+, x \geq 0. \tag{3.1}$$

Here, $H(x)$ is a renewal process which is generated by ladder heights $\{\chi_n^+\}$. The process $H(x)$ is defined as follows:

$$H(x) \equiv \min\left\{n \geq 1 : \sum_{i=1}^{n} \chi_i^+ > x\right\}, x \geq 0. \tag{3.2}$$

As it can be seen from Eq.(3.1), the boundary functional $N_1(x)$ is a renewal reward process.

Using Eq.(3.1) and Wald identity, the expected value of boundary functional $N_1(x)$ can be written as follows:

$$E(N_1(x)) = E\left\{\sum_{i=1}^{H(x)} \nu_i^+\right\} = E(\nu_1^+) E(H(x)) = \frac{\mu_1}{m_1} E(H(x)) \tag{3.3}$$

Here, $\mu_1 = E(\chi_1^+)$ and $m_1 = E(\eta_1)$.

Moreover, $E\left(H\left(x\right)\right)$ is renewal function generated by ladder heights $\{\chi_n^+\}$ which is defined as follows:

$$E\left(H\left(x\right)\right)=\sum_{n=0}^{\infty}F_+^{*n}\left(x\right) \tag{3.4}$$

Here, $F_+\left(x\right)\equiv P\left\{\chi_1^+\leq x\right\}$ is distribution function of first ladder height $\left(\chi_1^+\right)$.

With the use of Eq.(3.1) and Borovkov identity, the variance of boundary functional $N_1\left(x\right)$ can be written as follows:

$$\begin{aligned} Var\left(N_1\left(x\right)\right)=&Var\left(\sum_{i=1}^{H(x)}\nu_i^+\right) \\ =&E\left(H\left(x\right)\right)Var\left(\nu_1^+\right)+\left(E\left(\nu_1^+\right)\right)^2Var\left(H\left(x\right)\right). \end{aligned} \tag{3.5}$$

From definition of χ_1^+ and Wald identity, we can get the following equality:

$$E\left(\chi_1^+\right)=E\left(\eta_1\right)E\left(\nu_1^+\right). \tag{3.6}$$

From Eq.(3.6), we have

$$E\left(\nu_1^+\right)=\frac{E\left(\chi_1^+\right)}{E\left(\eta_1\right)}=\frac{\mu_1}{m_1}. \tag{3.7}$$

According to Borovkov Identity, following equation can be obtained:

$$Var\left(\chi_1^+\right)=Var\left(\sum_{i=1}^{\nu_1^+}\eta_i\right)=E\left(\nu_1^+\right)Var\left(\eta_1\right)+\left(E\left(\eta_1\right)\right)^2Var\left(\nu_1^+\right). \tag{3.8}$$

From Eq.(3.8), we have

$$Var\left(\nu_1^+\right)=\frac{Var\left(\chi_1^+\right)-E\left(\nu_1^+\right)Var\left(\eta_1\right)}{\left(E\left(\eta_1\right)\right)^2}=\frac{\sigma_+^2}{m_1^2}-\frac{\mu_1\sigma_\eta^2}{m_1^3}. \tag{3.9}$$

Here, $m_1=E\left(\eta_1\right)$, $\mu_1=E\left(\chi_1^+\right)$ and $\sigma_\eta^2=Var\left(\eta_1\right)$, $\sigma_+^2=Var\left(\chi_1^+\right)$.

Substituting Eq.(3.7) and Eq.(3.9) to Eq.(3.5) the following formula is obtained:

$$Var\left(N_1\left(x\right)\right)=\frac{\left(m_1\sigma_+^2-\mu_1\sigma_\eta^2\right)}{m_1^3}E\left(H\left(x\right)\right)+\frac{\mu_1^2}{m_1^2}Var\left(H\left(x\right)\right) \tag{3.10}$$

Here, $E\left(H\left(x\right)\right)$ and $Var\left(H\left(x\right)\right)$ denote expected value and variance of renewal process $H\left(x\right)$, respectively.

In summary, we obtain the following proposition.

Proposition 3.1. *Exact expressions for expected value and variance of boundary functional $N_1\left(x\right)$ are as follows:*

$$\begin{aligned} E\left(N_1\left(x\right)\right)=&\frac{\mu_1}{m_1}E\left(H\left(x\right)\right), \\ Var\left(N_1\left(x\right)\right)=&\frac{\left(m_1\sigma_+^2-\mu_1\sigma_\eta^2\right)}{m_1^3}E\left(H\left(x\right)\right)+\frac{\mu_1^2}{m_1^2}Var\left(H\left(x\right)\right). \end{aligned}$$

Define ζ_1 as positive part of random variable $(-\eta_1)$. So distribution function of ζ_1 can be written as follows:

$$\pi_\zeta(z) \equiv P\{\zeta_1 \le z\} = \frac{F(0) - F(-z)}{F(0)}, \quad z \ge 0.$$

Now, we can give the following proposition.

Proposition 3.2. *Exact expressions for expected value and variance of boundary functional $N_1(\lambda\zeta_1)$ are as follows:*

$$\begin{aligned} E(N_1(\lambda\zeta_1)) =& \frac{\mu_1}{m_1} E(H(\lambda\zeta_1)), \\ Var(N_1(\lambda\zeta_1)) =& \frac{\left(m_1\sigma_+^2 - \mu_1\sigma^2\right)}{m_1^3} E(H(\lambda\zeta_1)) + \frac{\mu_1^2}{m_1^2} Var(H(\lambda\zeta_1)). \end{aligned}$$

Here, $E(H(\lambda\zeta_1)) = \int_0^\infty E(H(\lambda z))\, d\pi_\zeta(z)$ *and* $Var(H(\lambda\zeta_1)) = \int_0^\infty Var(H(\lambda z))\, d\pi_\zeta(z)$.

Now, calculate exact expressions for expected value and variance of boundary functional $\tau_1(\lambda\zeta_1)$. Wald and Borovkov identities are used for the calculation.

Proposition 3.3. *Exact expressions for expected value and variance of boundary functional $\tau_1(\lambda\zeta_1)$ can be written as follows:*

$$\begin{aligned} E(\tau_1(\lambda\zeta_1)) =& \frac{\alpha_1\mu_1}{m_1} E(H(\lambda\zeta_1)), \\ Var(\tau_1(\lambda\zeta_1)) =& \frac{\left(\mu_1 m_1^2\sigma_\zeta^2 + m_1\sigma_+^2\alpha_1^2 - \mu_1\sigma_\eta^2\alpha_1^2\right)}{m_1^3} E(H(\lambda\zeta_1)) \\ &+ \left(\frac{\mu_1\alpha_1}{m_1}\right)^2 Var(H(\lambda\zeta_1)). \end{aligned}$$

Here, $\alpha_1 = E(\xi_1)$ *and* $\sigma_\xi^2 = Var(\xi_1)$.

In a similar manner, we can derive the exact expressions for expected value and variance of boundary functional $\gamma_1(\lambda\zeta_1)$.

Proposition 3.4. *Exact expressions for expected value and variance of boundary functional $\gamma_1(\lambda\zeta_1)$ can be written as follows:*

$$\begin{aligned} E(\gamma_1) =& \frac{\alpha_1\mu_1}{m_1} E(H(\lambda\zeta_1)) + E(\theta_1), \\ Var(\gamma_1) =& \frac{\left(\mu_1 m_1^2\sigma_\zeta^2 + m_1\sigma_+^2\alpha_1^2 - \mu_1\sigma_\eta^2\alpha_1^2\right)}{m_1^3} E(H(\lambda\zeta_1)) \\ &+ \left(\frac{\mu_1\alpha_1}{m_1}\right)^2 Var(H(\lambda\zeta_1)) + Var(\theta_1). \end{aligned}$$

Random variable θ_1 represents the delaying time in one period. The expected value and variance of θ_1 can be written as follows:

$$E(\theta_1) = \frac{\alpha_1}{F(0)}, \quad Var(\theta_1) = \frac{F(0)\,\sigma_\xi^2 + \overline{F}(0)\,\alpha_1^2}{(F(0))^2}.$$

Here, $\alpha_1 = E(\xi_1)$, $\sigma_\xi^2 = Var(\xi_1)$ *and* $\overline{F}(0) = 1 - F(0) = P\{\eta_1 \ge 0\}$.

As it can be seen from propositions, expected values and variances of N_1, τ_1 and γ_1 depend on expected value and variance of renewal process $H(x)$ which is generated by the sequence of ladder heights $\{\chi_n^+\}$. Calculation of moments of renewal process is so hard. Even if the calculation can be done, obtained formulas have very complicated mathematical structure. So, it is better using the asymptotic expansions than using exact expressions for boundary functionals N_1, τ_1 and γ_1.

4. The Asymptotic Expansions for Expected Values and Variances of the Boundary Functionals

In order to write the asymptotic expansions for the moments of boundary functionals, let compute the expected value and variance of renewal process $H(x)$ which are available in literature.

Proposition 4.1. *(Feller [5]) Suppose that, $E\left(\chi_1^{+2}\right) < \infty$ holds. Then, the following asymptotic relation for expected value of renewal process $H(x)$ can be written when $x \to \infty$:*

$$E(H(x)) = \frac{x}{\mu_1} + \frac{\mu_2}{2\mu_1^2} + g_1(x).$$

Here, $\mu_k = E\left(\chi_1^{+k}\right)$, $k = 1, 2, \cdots$ and $g_1(x)$ is a bounded function which goes to zero when $x \to \infty$, i.e., $\lim_{x\to\infty} g_1(x) = 0$.

Proposition 4.2. *(Feller [5]) Suppose that, $E\left(\chi_1^{+3}\right) < \infty$ holds. Then, the following asymptotic relation for variance of renewal process $H(x)$ can be written when $x \to \infty$:*

$$Var(H(x)) = \frac{\sigma_+^2}{\mu_1^3} x + B + g_2(x).$$

Here, $\sigma_+^2 = Var\left(\chi_1^{+2}\right)$, $B \equiv \frac{5\mu_2^2}{4\mu_1^4} - \frac{2\mu_3}{3\mu_1^3} - \frac{\mu_2}{2\mu_1^2}$ and $g_2(x)$ is a bounded function which goes to zero when $x \to \infty$, i.e., $\lim_{x\to\infty} g_2(x) = 0$.

Now, let give the following lemma.

Lemma 4.3. *Let $g : R^+ \to R$ be a bounded and measurable function. Besides, $\lim_{x\to\infty} g(x) = 0$. Then, the following asymptotic relation is true:*

$$\lim_{\lambda\to\infty} \int_0^\infty g(\lambda z)\, d\pi_\zeta(z) = 0.$$

With the use of Proposition 4.1, Proposition 4.2 and Lemma 4.1, main results of the study can be given by the following theorems.

Theorem 4.4. *Suppose that, $\mu_3 = E\left(\chi_1^{+3}\right) < \infty$ holds. Then, the following two term asymptotic expansions for expected value and variance of boundary functional N_1 can be written when $\lambda \to +\infty$:*

$$\begin{aligned} E(N(\lambda\zeta_1)) &= \frac{m_1^-}{m_1 F(0)}\lambda + \frac{\mu_2}{2m_1\mu_1} + o(1), \\ Var(N_1(\lambda\zeta_1)) &= \frac{\left(2m_1\sigma_+^2 - \mu_1\sigma_\eta^2\right) m_1^-}{F(0)\,\mu_1 m_1 3^1}\lambda \\ &\quad + \frac{1}{2\mu_1^2 m_1^3}\{m_1\mu_2\sigma_+^2 - \mu_1\mu_2\sigma_\eta^2 + 2m_1\mu_1^4 B_+\} + o(1). \end{aligned}$$

Here, $m_1^- = \int_{-\infty}^{0} |x| \, dF(x)$.

In order to state the following theorem we need to introduce notation $C_v(X)$ which represents the coefficient of variation of arbitrary random variable X, i.e., $C_v^2(X) \equiv \frac{Var(X)}{(E(X))^2}$.

Theorem 4.5. *Suppose that,* $\mu_3 = E\left(\chi_1^{+3}\right) < \infty$ *holds. Then, we can write the following asymptotic expansions for expected value and variance of boundary functional* $\tau_1(\lambda\zeta_1)$ *as* $\lambda \to +\infty$*:*

$$
\begin{aligned}
E\left(\tau_1(\lambda\zeta_1)\right) =& \alpha_1 \frac{m_1^-}{m_1 F(0)} \lambda + \alpha_1 \frac{\mu_2}{2\mu_1 m_1} + o(1), \\
Var\left(\tau_1(\lambda\zeta_1)\right) =& \alpha_1^2 \frac{\mu_1 m_1^-}{m_1^2 F(0)} \left(D + C_v^2\left(\chi_1^+\right)\right) \lambda \\
&+ \alpha_1^2 \frac{\mu_1^2}{2m_1^2} \left\{D + DC_v^2\left(\chi_1^+\right) + 2B_+\right\} + o(1).
\end{aligned}
$$

Here, $D \equiv C_v^2\left(\chi_1^+\right) + \frac{m_1}{\mu_1}\left(C_v^2(\xi_1) - C_v^2(\eta_1)\right)$.

In a similar manner, we can also give asymptotic expansions for the moments of the boundary functional $\gamma_1(\lambda\zeta_1)$.

Theorem 4.6. *Suppose that,* $\mu_3 = E\left(\chi_1^{+3}\right) < \infty$ *holds. Then, we can write the following asymptotic expansions for expected value and variance of boundary functional* $\gamma_1(\lambda\zeta_1)$ *as* $\lambda \to +\infty$*:*

$$
\begin{aligned}
E\left(\gamma_1(\lambda\zeta_1)\right) =& \alpha_1 \frac{m_1 + m_1^-}{m_1 F(0)} \lambda + \alpha_1 \frac{\mu_2}{2\mu_1 m_1} + o(1), \\
Var\left(\gamma_1(\lambda\zeta_1)\right) =& \alpha_1^2 \frac{\mu_1 m_1^-}{m_1^2 F(0)} \left(D + C_v^2\left(\chi_1^+\right)\right) \lambda \\
&+ \frac{\alpha_1^2}{2m_1^2 F^2(0)} \left\{2m_1^2\left[\overline{F}(0) + F(0)\, C_v^2(\xi_1)\right]\right. \\
&\left. + \mu_1^2 F^2(0)\left[D + DC_v^2\left(\chi_1^+\right) + 2B_+\right]\right\}.
\end{aligned}
$$

Here, $\overline{F}(0) = 1 - F(0) = P\{\eta_1 \geq 0\}$.

5. Conclusion

In this study, a semi-Markovian random walk $(X(t))$ with a generalized delaying barrier and its three boundary functionals (N_1, τ_1, γ_1) are mathematically constructed. Then, the exact expressions and asymptotic expansions for expected values and variances of boundary functionals N_1, τ_1 and γ_1 are calculated. Suggested methods and results can be used for investigation of the models that are expressed by random walk process with other types of barriers too. In the future studies, kurtosis and skewness and higher moments of the boundary functionals can be also calculated.

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