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Research Article

Berger-type Deformed Sasaki Metric and Harmonicity on Tangent Bundles

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Abstract
In this article, we present some results concerning the harmonicity on the tangent bundle equipped with the Berger-type deformed Sasaki metric. We establish necessary and sufficient conditions under which a vector field is harmonic with respect to the Berger-type deformed Sasaki metric and we construct some examples of harmonic vector fields. We also study the harmonicity of a vector field along a map between Riemannian manifolds, the target manifold being anti-paraKähler equipped with a Berger-type deformed Sasaki metric on its tangent bundle. Also, we discuss the harmonicity of the composition of the projection map of the tangent bundle of a Riemannian manifold with a map from this manifold into another Riemannian manifold, the source manifold being anti-paraKähler whose tangent bundle is endowed with a Berger-type deformed Sasaki metric. After that, we study the harmonicity of the identity map on the tangent bundle equipped with the Berger-type deformed Sasaki metric. Finally, we introduce the φ-unit tangent bundle and we also study the harmonicity of the projection map of the φ-unit tangent bundle.

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1. Introduction

In this field, the geometry of the tangent bundle equipped with Sasaki metric has been studied by Sasaki S. [1], Yano K, Ishihara S. [2], Dombrowski P. [3], Salimov A.A., Gezer A. [4]. The rigidity of Sasaki metric has incited some researchers to construct and study other metrics on TM. Musso E, Tricerri F. has introced the notion of Cheeger-Gromoll metric [5], this metric has been studied also by many authors (see [6–8]). The study of Riemannian metrics in the tangent bundle is not limited to the Sasaki metric or the Cheeger-Gromoll metric, but there are studies on deformations of the Sasaki metric on the tangent bundle. We refer to for example [9–11].

Consider a smooth map $\phi:(M^m,g)\to(N^n,h)$ between two Riemannian manifolds, then the second fundamental form of ϕ is defined by

$$(\nabla d\phi)(X,Y)=\nabla^{\phi}_{X}d\phi(Y)-d\phi(\nabla_X Y). \tag{1}$$

Here ∇ is the Riemannian connection on M and ∇^{ϕ} is the pull-back connection on the pull-back bundle $\phi^{-1}TN$, and

$$\tau(\phi)=trace_g\nabla d\phi, \tag{2}$$

is the tension field of ϕ.

The energy functional of ϕ is defined by

$$E(\phi)=\int_K e(\phi)dv_g, \tag{3}$$

such that K is any compact of M, where

$$e(\phi)=\frac{1}{2}trace_g h(d\phi,d\phi), \tag{4}$$

is the energy density of ϕ.

A map is called harmonic if it is a critical point of the energy functional E. For any smooth variation $\{\phi_t\}_{t\in I}$ of ϕ with $\phi_0=\phi$ and $V=\frac{d}{dt}\phi_t\Big|_{t=0}$, we have

$$\frac{d}{dt}E(\phi_t)\Big|_{t=0}=-\int_K h(\tau(\phi),V)dv_g. \tag{5}$$

Then ϕ is harmonic if and only if $\tau(\phi)=0$.

One can refer to [12–15] for background on harmonic maps.

The main idea in this note consists in study of the harmonicity with respect to the Berger-type deformed Sasaki metric on the tangent bundle [11]. We establish necessary and sufficient conditions under which a vector field is harmonic (Theorem 9 and Theorem 13). We also construct some examples of harmonic vector fields. After that we study the harmonicity of the map $\sigma:(M,g)\longrightarrow(TN,h^{BS}),\, x\to\big(\phi(x),v\big)$ (Theorem 20 and Theorem 21) and the map $\Phi:(TM,g^{BS})\longrightarrow(N,h),\,(x,u)\to\phi(x)$ (Theorem 23 and Theorem 24), where $\phi:(M,g)\longrightarrow(N,h)$ is a smooth map and (TN,h^{BS}) (resp (TM,g^{BS})) is a tangent bundle equipped with the Berger-type deformed Sasaki metric on N (resp. on M). After that we study the harmonicity of the identity map $I:(TM,g^{BS1})\to(TM,g^{BS2})$ on the tangent bundle equipped with the Berger-type deformed Sasaki metric (Proposition 25 and Theorem 26). In the last section, we introduce the φ-unit tangent bundle $T_1^{\varphi}M$ equipped with Berger-type deformed Sasaki metric $\hat{g}^{BS}$, where we presented the formulas of the Levi-Civita connection (Theorem 28) and we study the harmonicity of the projection map of the φ-unit tangent bundle (Lemma 29) and the map $\Phi_1:(T_1^{\varphi}M,\hat{g}^{BS})\longrightarrow(N,h),\,(x,u)\to\phi(x)$. (Theorem 30 and Theorem 31).

2. Preliminaries

Let TM be the tangent bundle over an m-dimensional Riemannian manifold (M^m,g) and the natural projection $\pi:TM\to M$. A local chart $(U,x^i)_{i=\overline{1,m}}$ on M induces a local chart $(\pi^{-1}(U),x^i,u^i)_{i=\overline{1,m}}$ on TM, where (u^i) is the Cartesian coordinates in each tangent space T_xM at $x\in M$ with respect to the natural base $\left(\frac{\partial}{\partial x^i}|_x\right)$, x being an arbitrary point in U whose coordinates are (x^i). Denote by Γ^k_{ij} the Christoffel symbols of g and by ∇ the Levi-Civita connection of g. Let $C^\infty(M)$ be the ring of real-valued C^∞ functions on M and $\Im^1_0(M)$ be the module over $C^\infty(M)$ of C^∞ vector fields on M.

The Levi Civita connection ∇ defines a direct sum decomposition

$$T_{(x,u)}TM=V_{(x,u)}TM\oplus H_{(x,u)}TM, \tag{6}$$

of the tangent bundle to TM at any $(x,u)\in TM$ into vertical subspace

$$V_{(x,u)}TM = Ker(d\pi_{(x,u)})=\{X^i\frac{\partial}{\partial u^i}|_{(x,u)},X^i\in\mathbb{R}\}, \tag{7}$$

and the horizontal subspace

$$H_{(x,u)}TM = \{X^i\frac{\partial}{\partial x^i}|_{(x,u)}-X^iu^j\Gamma^k_{ij}\frac{\partial}{\partial u^k}|_{(x,u)},X^i\in\mathbb{R}\}. \tag{8}$$

Let $X=X^i\frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$^VX = X^i\frac{\partial}{\partial u^i}, \tag{9}$$

$$^HX = X^i\{\frac{\partial}{\partial x^i}-u^j\Gamma^k_{ij}\frac{\partial}{\partial u^k}\}. \tag{10}$$

We have $^H(\frac{\partial}{\partial x^i})=\frac{\partial}{\partial x^i}-u^j\Gamma^k_{ij}\frac{\partial}{\partial y^k}$ and $^V(\frac{\partial}{\partial x^i})=\frac{\partial}{\partial u^i}$, then $(^H(\frac{\partial}{\partial x^i}),{}^V(\frac{\partial}{\partial x^i}))_{i=\overline{1,m}}$ is a local adapted frame on TTM.

In particular, the vertical lift $^Vu=u^i\frac{\partial}{\partial u^i}$ is called the canonical vertical vector field or Liouville vector field on TM.

Lemma 1. *[2] Let (M,g) be a Riemannian manifold and R its tensor curvature, then we have*

1. $[{}^HX, {}^HY] = {}^H[X,Y] - {}^V(R(X,Y)u)$,
2. $[{}^HX, {}^VY] = {}^V(\nabla_X Y)$,
3. $[{}^VX, {}^VY] = 0$.

for any vector fields X, Y on M.

3. Berger-type deformed Sasaki metric

Let M be a $2m$-dimensional Riemannian manifold with a Riemannian metric g. An almost paracomplex manifold is an almost product manifold (M^{2m}, φ), $\varphi^2 = id$, such that the two eigenbundles T^+M and T^-M associated to the two eigenvalues $+1$ and -1 of φ, respectively, have the same rank.

A paracomplex structure is an integrable almost paracomplex structure. Let (M^{2m}, φ) be an almost paracomplex manifold. A Riemannian metric g is said to be an anti-paraHermitian metric if

$$g(\varphi X, \varphi Y) = g(X,Y), \tag{11}$$

or equivalently (purity condition), (B-metric) [16]

$$g(\varphi X, Y) = g(X, \varphi Y), \tag{12}$$

for any vector fields X, Y on M.

If (M^{2m}, φ) is an almost paracomplex manifold with an anti-paraHermitian metric g, then the triple (M^{2m}, φ, g) is said to be an almost anti-paraHermitian manifold (an almost B-manifold) [16]. Moreover, (M^{2m}, φ, g) is said to be anti-paraKähler manifold (B-manifold) [16] if φ is parallel with respect to the Levi-Civita connection ∇ of g i.e. $(\nabla \varphi = 0)$.

A Tachibana operator ϕ_φ applied to the anti-paraHermitian metric (pure metric) g is given by

$$(\phi_\varphi g)(X,Y,Z) = \varphi X(g(Y,Z)) - X(g(\varphi Y, Z)) + g((L_Y \varphi)X, Z) + g((L_Z \varphi)X, Y), \tag{13}$$

for any vector fields X, Y et Z on M [17].

In an almost anti-paraHermitian manifold, an anti-paraHermitian metric g is called paraholomorphic if

$$(\phi_\varphi g)(X,Y,Z) = 0, \tag{14}$$

for any vector fields X, Y et Z on M [16].

In [16], Salimov and his collaborators proved that for an almost B-manifold,

$$\nabla \varphi = 0 \Leftrightarrow \phi_\varphi g = 0, \tag{15}$$

by virtue of this view, in an almost anti-paraHermitian manifold the anti-paraKähler condition are similar to paraholomorphicity condition of the anti-paraHermitian metric.

The purity conditions for a $(0,q)$-tensor field ω with respect to the almost paracomplex structure φ given by

$$\omega(\varphi X_1, X_2, \ldots, X_q) = \omega(X_1, \varphi X_2, \ldots, X_q) = \ldots = \omega(X_1, X_2, \ldots, \varphi X_q),$$

for any vector fields $X_1, X_2, \ldots, X_q$ on M [16].

It is well known that if (M^{2m}, φ, g) is a anti-paraKähler manifold, the Riemannian curvature tensor is pure [16], and

$$\begin{cases} R(\varphi Y, Z) &= R(Y, \varphi Z) = R(Y,Z)\varphi = \varphi R(Y,Z), \\ R(\varphi Y, \varphi Z) &= R(Y,Z), \end{cases} \tag{16}$$

for any vector fields Y, Z on M.

Definition 2. *[11] Let (M^{2m}, φ, g) be an almost anti-paraHermitian manifold and TM be its tangent bundle. A fiber-wise Berger-type deformation of the Sasaki metric g^{BS} on TM is defined by*

$$\begin{aligned} g^{BS}({}^HX, {}^HY) &= g(X,Y), \\ g^{BS}({}^HX, {}^VY) &= 0, \\ g^{BS}({}^VX, {}^VY) &= g(X,Y) + \delta^2 g(X, \varphi u) g(Y, \varphi u), \end{aligned} \tag{17}$$

for any vector fields X, Y on M, where δ is some constant.
Note that if $\delta = 0$, g^{BS} is the Sasaki metric [1].

In the following $\|.\|$ denote the norm with respect to (M^{2m},φ,g).

Lemma 3. *Let (M^{2m},φ,g) be an anti-paraKähler manifold. Then we have the followings:*

1. ${}^{H}X(g(u,\varphi u))=0$,
2. ${}^{V}X(g(u,\varphi u))=2g(X,\varphi u)$,
3. ${}^{H}X(g(Y,\varphi u))=g(\nabla_X Y,\varphi u)$,
4. ${}^{V}X(g(Y,\varphi u))=g(X,\varphi Y)$,

for any vector fields X,Y on M.

Theorem 4. *[11] Let (M^{2m},φ,g) be an anti-paraKähler manifold and TM be its tangent bundle. The Levi-Civita connection $\widetilde{\nabla}$ of the Berger-type deformed Sasaki metric g^{BS} on TM satisfies the following properties:*

1. $(\widetilde{\nabla}_{{}^{H}X}{}^{H}Y)={}^{H}(\nabla_X Y)-\frac{1}{2}{}^{V}(R(X,Y)u)$,
2. $(\widetilde{\nabla}_{{}^{H}X}{}^{V}Y)=\dfrac{1}{2}{}^{H}(R(u,Y)X)+{}^{V}(\nabla_X Y)$,
3. $(\widetilde{\nabla}_{{}^{V}X}{}^{H}Y)=\dfrac{1}{2}{}^{H}(R(u,X)Y)$,
4. $(\widetilde{\nabla}_{{}^{V}X}{}^{V}Y)=\dfrac{\delta^2}{1+\delta^2\alpha}g(X,\varphi Y)^{V}(\varphi u)$,

for any vector fields X,Y on M, where ∇ is the Levi-Civita connection, R is its Riemannian curvature tensor of (M,g) and $\alpha=g(u,u)$.

Lemma 5. *Let (M^{2m},φ,g) be an anti-paraKähler manifold and (TM,g^{BS}) its tangent bundle equipped with the Berger type deformed Sasaki metric, then we have*

1. $\widetilde{\nabla}_{{}^{H}X}{}^{V}(\varphi u)=0$,
2. $\widetilde{\nabla}_{{}^{V}(\varphi u)}{}^{H}X=0$,
3. $\widetilde{\nabla}_{{}^{V}X}{}^{V}(\varphi u)={}^{V}(\varphi X)+\dfrac{\delta^2}{1+\delta^2\alpha}g(X,u)^{V}(\varphi u)$,
4. $\widetilde{\nabla}_{{}^{V}(\varphi u)}{}^{V}X=\dfrac{\delta^2}{1+\delta^2\alpha}g(X,u)^{V}(\varphi u)$,
5. $\widetilde{\nabla}_{{}^{V}(\varphi u)}{}^{V}(\varphi u)={}^{V}u+\dfrac{\delta^2}{1+\delta^2\alpha}g(u,\varphi u)^{V}(\varphi u)$,

for all vector field X on M.

4. Berger-type deformed Sasaki metric and harmonicity

4.1 Harmonicity of a vector field $X:(M,g)\longrightarrow(TM,g^{BS})$

Lemma 6. *[18] Let (M,g) be a Riemannian manifold. If $X,Y\in\Im^1_0(M)$ are vector fields on M and $(x,u)\in TM$ such that $Y_x=u$, then we have:*

$$d_xY(X_x)={}^{H}X_{(x,u)}+{}^{V}(\nabla_X Y)_{(x,u)}.$$

Lemma 7. *Let (M^{2m},φ,g) be an almost anti-paraHermitian manifold and (TM,g^{BS}) its tangent bundle equipped with the Berger-type deformed Sasaki metric. If $X\in\Im^1_0(M)$, then the energy density associated to X is given by:*

$$e(X) \;=\; m+\frac{1}{2}trace_g\big(g(\nabla X,\nabla X)+\delta^2 g(\nabla X,\varphi X)^2\big). \tag{18}$$

Proof. Let $(x,u) \in TM$, $X \in \Im^1_0(M)$, $X_x = u$ and $(E_1,\ldots,E_{2m})$ be a local orthonormal frame on M, then:

$$\begin{aligned} e(X)_x &= \frac{1}{2} trace_g g^{BS}(dX,dX)_{(x,u)} \\ &= \frac{1}{2}\sum_{i=1}^{2m} g^{BS}(dX(E_i),dX(E_i))_{(x,u)}. \end{aligned}$$

Using Lemma 6, we obtain:

$$\begin{aligned} e(X) &= \frac{1}{2}\sum_{i=1}^{2m} g^{BS}({}^HE_i + {}^V(\nabla_{E_i}X), {}^HE_i + {}^V(\nabla_{E_i}X)) \\ &= \frac{1}{2}\sum_{i=1}^{2m} \big(g^{BS}({}^HE_i, {}^HE_i) + g^{BS}({}^V(\nabla_{E_i}X), {}^V(\nabla_{E_i}X))\big) \\ &= \frac{1}{2}\sum_{i=1}^{2m} \big(g(E_i,E_i) + g(\nabla_{E_i}X, \nabla_{E_i}X) + \delta^2 g(\nabla_{E_i}X, \varphi X)^2\big) \\ &= m + \frac{1}{2} trace_g\big(g(\nabla X,\nabla X) + \delta^2 g(\nabla X,\varphi X)^2\big). \end{aligned}$$

∎

Theorem 8. *Let (M^{2m},φ,g) be an anti-paraKähler manifold and (TM,g^{BS}) its tangent bundle equipped with the Berger-type deformed Sasaki metric. If $X \in \Im^1_0(M)$, then the tension field associated to X is given by:*

$$\tau(X) = {}^H\big(trace_g R(X,\nabla X)*\big) + {}^V\Big(trace_g\big(\nabla^2 X + \frac{\delta^2}{1+\delta^2\alpha} g(\nabla X, \varphi(\nabla X))\varphi X\big)\Big), \tag{19}$$

where $\alpha = g(X,X) = \|X\|^2$.

Proof. Let $(x,u) \in TM$, $X \in \Im^1_0(M)$, $X_x = u$ and $\{E_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on M such that $(\nabla^M_{E_i}E_i)_x = 0$, then

$$\begin{aligned} \tau(X)_x &= \sum_{i=1}^{2m}\{(\nabla^X_{E_i} dX(E_i))_x - dX(\nabla^M_{E_i}E_i)_x\} \\ &= \sum_{i=1}^{2m}\{\widetilde{\nabla}_{dX(E_i)} dX(E_i)\}_{(x,u)} \\ &= \sum_{i=1}^{2m}\{\widetilde{\nabla}_{({}^HE_i + {}^V(\nabla_{E_i}X))}({}^HE_i + {}^V(\nabla_{E_i}X))\}_{(x,u)} \\ &= \sum_{i=1}^{2m}\{\widetilde{\nabla}_{{}^HE_i}{}^HE_i + \widetilde{\nabla}_{{}^HE_i}{}^V(\nabla_{E_i}X) + \widetilde{\nabla}_{{}^V(\nabla_{E_i}X)}{}^HE_i + \widetilde{\nabla}_{{}^V(\nabla_{E_i}X)}{}^V(\nabla_{E_i}X)\}_{(x,u)}. \end{aligned}$$

Using Theorem 4, we obtain

$$\begin{aligned} \tau(X) &= \sum_{i=1}^{2m}\Big({}^H(\nabla_{E_i}E_i) - \frac{1}{2}{}^V(R(E_i,E_i)X) + \frac{1}{2}{}^H(R(X,\nabla_{E_i}X)E_i) + {}^V(\nabla_{E_i}\nabla_{E_i}X) + \frac{1}{2}{}^H(R(X,\nabla_{E_i}X)E_i) \\ &\quad + \frac{\delta^2}{1+\delta^2\alpha} g(\nabla_{E_i}X, \varphi(\nabla_{E_i}X))^V(\varphi X)\Big) \\ &= \sum_{i=1}^{2m}\Big({}^H(R(X,\nabla_{E_i}X)E_i) + {}^V(\nabla_{E_i}\nabla_{E_i}X) + \frac{\delta^2}{1+\delta^2\alpha} g(\nabla_{E_i}X, \varphi(\nabla_{E_i}X))^V(\varphi X)\Big) \\ &= {}^H\big(trace_g R(X,\nabla X)*\big) + {}^V\Big(trace_g\big(\nabla^2 X + \frac{\delta^2}{1+\delta^2\alpha} g(\nabla X, \varphi(\nabla X))\varphi X\big)\Big). \end{aligned}$$

∎

Theorem 9. *Let (M^{2m},φ,g) be an anti-paraKähler manifold and (TM,g^{BS}) its tangent bundle equipped with the Berger-type deformed Sasaki metric. If $X\in\Im^1_0(M)$, then X is harmonic if and only if the following conditions are verified*

$$trace_g\big(R(X,\nabla X)*\big)=0, \tag{20}$$

and

$$trace_g\big(\nabla^2X+\frac{\delta^2}{1+\delta^2\alpha}g(\nabla X,\varphi(\nabla X))\varphi X\big)=0, \tag{21}$$

where $\alpha=g(X,X)=\|X\|^2$.

Proof. The statement is a direct consequence of Theorem 8. ■

Corollary 10. *Let (M^{2m},φ,g) be an anti-paraKähler manifold and (TM,g^{BS}) its tangent bundle equipped with the Berger-type deformed Sasaki metric. Any parallel vector field is harmonic.*

Example 11. *Let $(]0,+\infty[^2,\varphi,g)$ be an anti-paraKähler manifold such that*

$$g=x^2dx^2+y^2dy^2,$$

and

$$\varphi\partial_x=\frac{x}{y}\partial_y,\qquad \varphi\partial_y=\frac{y}{x}\partial_x,$$

where $\partial_x=\frac{\partial}{\partial x}$ The vector field $X=\frac{1}{x}\partial_x+\frac{1}{y}\partial_y$ is harmonic. Indeed, It is enough to set $u=\frac{1}{2}x^2$ and $v=\frac{1}{2}y^2$ to get the Euclidean metric $g=du^2+dv^2$ and $X=\partial_u+\partial_v$ which is trivially parallel.

Example 12. *Let $\mathbb{R}^2$ be endowed with the structure anti-paraKähler (φ,g) in polar coordinate defined by*

$$g=dr^2+r^2d\theta^2,$$

and

$$\varphi\partial_r=\sin 2\theta\partial_r+\frac{1}{r}\cos 2\theta\partial_\theta\quad,\quad \varphi\partial_\theta=r\cos 2\theta\partial_r-\sin 2\theta\partial_\theta.$$

The vector field $X=(\cos\theta+\sin\theta)\partial_r+\frac{1}{r}(\cos\theta-\sin\theta)\partial_\theta$ is harmonic. Indeed, the non-null Christoffel symbols of the Riemannian connection are:

$$\Gamma^2_{12}=\Gamma^2_{21}=\frac{1}{r},\ \Gamma^1_{22}=-r,$$

and

$$\nabla_{\partial_r}\partial_r=0,\ \nabla_{\partial_r}\partial_\theta=\frac{1}{r}\partial_\theta,\ \nabla_{\partial_\theta}\partial_r=\frac{1}{r}\partial_\theta,\ \nabla_{\partial_\theta}\partial_\theta=-r\partial_r,$$

then we have,

$$\nabla_{\partial_r}X = (\cos\theta+\sin\theta)\nabla_{\partial_r}\partial_r-\frac{1}{r^2}(\cos\theta-\sin\theta)\frac{\partial}{\partial\theta}+\frac{1}{r}(\cos\theta-\sin\theta)\nabla_{\partial_r}\frac{\partial}{\partial\theta}=0,$$

$$\nabla_{\partial_\theta}X = (\cos\theta-\sin\theta)\partial_r+(\cos\theta+\sin\theta)\nabla_{\partial_\theta}\partial_r-\frac{1}{r}(\cos\theta+\sin\theta)\partial_\theta+\frac{1}{r}(\cos\theta-\sin\theta)\nabla_{\partial_\theta}\partial_\theta=0,$$

i.e. $\nabla X=0$.

Theorem 13. *Let (M^{2m},φ,g) be an anti-paraKähler compact manifold and (TM,g^{BS}) its tangent bundle equipped with the Berger-type deformed Sasaki metric. If $X\in\Im^1_0(M)$, then X is harmonic if and only if X is parallel.*

Proof. If X is parallel, from Corollary 10, we deduce that X is harmonic vector field. Conversely, let X_t be a variation of X defined by:

$$\begin{array}{rcl} \mathbb{R}\times M & \longrightarrow & T_xM \\ (t,x) & \longmapsto & X_t(x)=(t+1)X. \end{array}$$

From Lemma 7 we have:

$$e(X_t) = m+\frac{(1+t)^2}{2}trace_g g(\nabla X,\nabla X)+\frac{(1+t)^4}{2}\delta^2 trace_g g(\nabla X,\varphi X)^2$$

$$E(X_t) = mVol(M)+\frac{(1+t)^2}{2}\int_M trace_g g(\nabla X,\nabla X)dv_g+\frac{(1+t)^4}{2}\delta^2\int_M trace_g g(\nabla X,\varphi X)^2 dv_g.$$

If X is a critical point of the energy functional, then we have :

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}E(X_t)|_{t=0} \\ &= \frac{\partial}{\partial t}\Big[mVol(M)+\frac{(1+t)^2}{2}\int_M trace_g g(\nabla X,\nabla X)dv_g\Big]_{t=0}+\frac{\partial}{\partial t}\Big[\frac{(1+t)^4}{2}\delta^2\int_M trace_g g(\nabla X,\varphi X)^2 dv_g\Big]_{t=0} \\ &= \int_M trace_g g(\nabla X,\nabla X)dv_g+2\delta^2\int_M trace_g g(\nabla X,\varphi X)^2 dv_g \\ &= \int_M trace_g\big[g(\nabla X,\nabla X)+2\delta^2 g(\nabla X,\varphi X)^2\big]dv_g \end{aligned}$$

which gives

$$g(\nabla X,\nabla X)+2\delta^2 g(\nabla X,\varphi X)^2=0,$$

hence $\nabla X=0$. ■

Example 14. *(Counterexample) Let $(\mathbb{R}^{2m},\varphi,<,>)$ be an anti-paraKähler real Euclidean space (flat manifold and non compact) and $T^*\mathbb{R}^{2m}$ its tangent bundle equipped with the Berger-type deformed Sasaki metric, such that φ is the canonical paracomplex structure on $\mathbb{R}^{2m}$ [19] it is given by the matrix*

$$\begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}.$$

If $X=(X^1,\ldots,X^{2m})\in\Im_0^1(\mathbb{R}^{2m})$ is a vector field on $\mathbb{R}^{2m}$
For $\delta=0$, we have

$$\tau(X)=trace_g\nabla^2 X=(\sum_{i=1}^{2m}\frac{\partial^2 X^1}{\partial x_i^2},\ldots,\sum_{i=1}^{2m}\frac{\partial^2 X^{2m}}{\partial x_i^2}).$$

1) If X is constant, then X is harmonic.
2) If $X_i=a_ix_i$ and $a_i\neq 0$, then X is harmonic ($\tau(X)=0$) but $\nabla X\neq 0$.
Indeed

$$\nabla X(\frac{\partial}{\partial x_j})=\nabla_{\frac{\partial}{\partial x_j}}X=\sum_i a_i\nabla_{\frac{\partial}{\partial x_j}}(x_i\frac{\partial}{\partial x_i})=\sum_i\delta_i^j a_i\frac{\partial}{\partial x_i}=a_j\frac{\partial}{\partial x_j}\neq 0.$$

Remark 15. *In general , using Corollary 10 and Theorem 13, we can construct many examples for harmonic vector fields.*

Isometric immersion

Proposition 16. *[20] Let (M^{2m},φ,g) be an anti-paraKähler manifold, (TM,g^{BS}) its tangent bundle equipped with the Berger-type deformed Sasaki metric and $X:(M^{2m},\varphi,g)\to(TM,g^{BS})$ is an isometric immersion if and only if $\nabla X=0$.*

Proof. Let Y,Z be vector fields and $X_x=u$. From Lemma 6 we have

$$\begin{aligned}g^{BS}(dX(Y),dX(Z)) &= g^{BS}({}^HY+{}^V(\nabla_YX),{}^HZ+{}^V(\nabla_ZX))\\ &= g^{BS}({}^HY,{}^HZ)+g^{BS}({}^V(\nabla_YX),{}^V(\nabla_ZX))\\ &= g(Y,Z)+g(\nabla_YX,\nabla_ZX)+\delta^2g(\nabla_YX,\varphi X)g(\nabla_ZX,\varphi X),\end{aligned}$$

from which it follows that

$$g^{BS}(dX(X),dX(Y))=g(X,Y).$$

Therefore, X is an isometric immersion if and only if

$$g(\nabla_YX,\nabla_ZX)+\delta^2g(\nabla_YX,\varphi X)g(\nabla_ZX,\varphi X)=0,$$

which is equivalent to $\nabla X=0$. ■

As a direct consequence of Theorem 9 and Proposition 16, we obtain the following theorem.

Theorem 17. *[20] Let (M^{2m},φ,g) be an anti-paraKähler manifold, (TM,g^{BS}) its tangent bundle equipped with the Berger-type deformed Sasaki metric. If $X:(M^{2m},\varphi,g)\to(TM,g^{BS})$ is isometric immersion, then X is totally geodesic.*

Corollary 18. *[20] Let (M^{2m},φ,g) be an anti-paraKähler manifold, (TM,g^{BS}) its tangent bundle equipped with the Berger-type deformed Sasaki metric. If $X:(M^{2m},\varphi,g)\to(TM,g^{BS})$ is isometric immersion, then X is harmonic.*

4.2 Harmonicity of the map $\sigma:(M,g)\longrightarrow(TN,h_{BS})$

Lemma 19. *[21] Let $\phi:(M^m,g)\to(N^n,h)$ be a smooth map between Riemannian manifolds and*

$$\begin{aligned}\sigma:M &\longrightarrow TN\\ x &\longmapsto (Y\circ\phi)(x)=(\phi(x),Y_{\phi(x)})\end{aligned}$$

a smooth map, such that Y be a vector field on N. Then

$$d\sigma(X)={}^H(d\phi(X))+{}^H(\nabla^\phi_X\sigma) \tag{22}$$

for all $X\in\Im^1_0(M)$.

Theorem 20. *Let (M^m,g) be a Riemannian manifold, (N^{2n},φ,h) be an anti-paraKähler manifold and let (TN,h_{BS}) the tangent bundle of N equipped with Berger-type deformed Sasaki metric. Let $\phi:M\to N$ be a smooth map and*

$$\begin{aligned}\sigma:M &\longrightarrow TN\\ x &\longmapsto (Y\circ\phi)(x)=(\phi(x),Y_{\phi(x)})\end{aligned}$$

a smooth map, such that Y be a vector field on N. The tension field of σ is given by

$$\tau(\sigma) = {}^H\big(\tau(\phi)+trace_gR^N(\sigma,\nabla^\phi\sigma)d\phi(*)\big)+{}^V\Big(trace_g\big((\nabla^\phi)^2\sigma+\frac{\delta^2}{1+\delta^2\|\sigma\|^2}h(\nabla^\phi\sigma,\varphi\nabla^\phi\sigma)\varphi\sigma\big)\Big), \tag{23}$$

where $\|\sigma\|^2=h(\sigma,\sigma)$.

Proof. Let $x\in M$, $\{E_i\}_{i=\overline{1,m}}$ be a local orthonormal frame on M such that $(\nabla^M_{E_i}E_i)_x=0$ and $\sigma(x)=(\phi(x),v), v=Y_{\phi(x)}\in T_{\phi(x)}N$, then we have

$$\begin{aligned}\tau(\sigma)_x &= trace_g(\nabla d\sigma)_x\\ &= \sum_{i=1}^m\{(\nabla^\sigma_{E_i}d\sigma(E_i))_x-d\sigma(\nabla^M_{E_i}E_i)_x\}\\ &= \sum_{i=1}^m\{\nabla^{TN}_{d\sigma(E_i)}d\sigma(E_i)\}_{(\phi(x),v)}.\end{aligned}$$

Using Lemma 19, we have

$$\begin{aligned}\tau(\sigma)_x &= \sum_{i=1}^{m}\{\nabla^{TN}_{(^H(d\phi(E_i))+^V(\nabla^{\phi}_{E_i}\sigma))}(^H(d\phi(E_i))+^V(\nabla^{\phi}_{E_i}\sigma))\}_{(\phi(x),v)}\\ &= \sum_{i=1}^{m}\{\nabla^{TN}_{^H(d\phi(E_i))}{}^H(d\phi(E_i))+\nabla^{TN}_{^H(d\phi(E_i))}{}^V(\nabla^{\phi}_{E_i}\sigma)+\nabla^{TN}_{^V(\nabla^{\phi}_{E_i}\sigma)}{}^H(d\phi(E_i))+\nabla^{TN}_{^V(\nabla^{\phi}_{E_i}\sigma)}{}^V(\nabla^{\phi}_{E_i}\sigma)\}_{(\phi(x),v)}.\end{aligned}$$

From the Theorem 4, we obtain

$$\begin{aligned}\tau(\sigma) &= \sum_{i=1}^{m}\Big({}^H(\nabla^N_{d\phi(E_i)}d\phi(E_i))-\frac{1}{2}{}^V(R^N(d\phi(E_i),d\phi(E_i))\sigma)+\frac{1}{2}{}^H(R^N(\sigma,\nabla^{\phi}_{E_i}\sigma)d\phi(E_i))+{}^V(\nabla^N_{d\phi(E_i)}\nabla^{\phi}_{E_i}\sigma)\\ &\quad+\frac{1}{2}{}^H(R^N(\sigma,\nabla^{\phi}_{E_i}\sigma)d\phi(E_i))+\frac{\delta^2}{1+\delta^2\|\sigma\|^2}h(\nabla^{\phi}_{E_i}\sigma,\varphi\nabla^{\phi}_{E_i}\sigma)^V(\varphi\sigma)\Big)\\ &= \sum_{i=1}^{m}\Big({}^H(\nabla^{\phi}_{E_i}d\phi(E_i))+{}^H(R^N(\sigma,\nabla^{\phi}_{E_i}\sigma)d\phi(E_i))+{}^V(\nabla^{\phi}_{E_i}\nabla^{\phi}_{E_i}\sigma)+\frac{\delta^2}{1+\delta^2\|\sigma\|^2}h(\nabla^{\phi}_{E_i}\sigma,\varphi\nabla^{\phi}_{E_i}\sigma)^V(\varphi\sigma)\Big).\end{aligned}$$

Therefore we get

$$\tau(\sigma) = {}^H\big(\tau(\phi)+trace_gR^N(\sigma,\nabla^{\phi}\sigma)d\phi(*)\big)+{}^V\Big(trace_g\big((\nabla^{\phi})^2\sigma+\frac{\delta^2}{1+\delta^2\|\sigma\|^2}h(\nabla^{\phi}\sigma,\varphi\nabla^{\phi}\sigma)\varphi\sigma\big)\Big).$$

■

From Theorem 20, we obtain the following:

Theorem 21. *Let (M^m,g) be a Riemannian manifold, (N^{2n},φ,h) be an anti-paraKähler manifold and let (TN,h_{BS}) the tangent bundle of N equipped with the Berger-type deformed Sasaki metric. Let $\phi: M\to N$ be a smooth map and*

$$\begin{aligned}\sigma: M &\longrightarrow TN\\ x &\longmapsto (Y\circ\phi)(x)=(\phi(x),Y_{\phi(x)})\end{aligned}$$

a smooth map, such that Y be a vector field on N, then σ is a harmonic if and only if the following conditions are verified

$$\tau(\phi) = -trace_gR^N(\sigma,\nabla^{\phi}\sigma)d\phi(*), \tag{24}$$

and

$$trace_g\big((\nabla^{\phi})^2\sigma+\frac{\delta^2}{1+\delta^2\|\sigma\|^2}h(\nabla^{\phi}\sigma,\varphi\nabla^{\phi}\sigma)\varphi\sigma\big)=0. \tag{25}$$

4.3 Harmonicity of the map $\Phi:(TM,g^{BS})\longrightarrow(N,h)$

Lemma 22. *Let (M^{2m},φ,g) be an anti-paraKähler manifold and (TM,g^{BS}) its tangent bundle equipped with the Berger-type deformed Sasaki metric. The canonical projection*

$$\begin{aligned}\pi:(TM,g^{BS}) &\longrightarrow (M,g)\\ (x,u) &\longmapsto x\end{aligned}$$

is harmonic i.e. $\tau(\pi)=0$.

Proof. We put $p=2m$, Let $\{E_i\}_{i=\overline{1,p}}$ be a local orthonormal frame on M and $\{\widetilde{E}_j\}_{j=\overline{1,2p}}$ be a local frame on TM, where

$$\widetilde{E}_j=\begin{cases}{}^HE_j, & 1\le j\le p\\ {}^VE_{j-p}, & p+1\le j\le 2p.\end{cases} \tag{26}$$

The tension field of π is given by

$$\begin{aligned}\tau(\pi) &= trace_{g^{BS}}\nabla d\pi\\ &= \sum_{i,j=1}^{2p}G^{ij}\Big\{\nabla^M_{d\pi(\widetilde{E}_i)}d\pi(\widetilde{E}_j)-d\pi(\nabla^{TM}_{\widetilde{E}_i}\widetilde{E}_j)\Big\},\end{aligned}$$

where (G_{ij}) is the matrix of g^{BS} and its inverse matrix is (G^{ij}) such that:

$$\begin{cases} G_{ij} = \delta_{ij} & 1 \le i,j \le p \\ G_{ij} = 0 & 1 \le i \le p,\ p+1 \le j \le 2p \\ G_{ij} = \delta_{ij} + \delta^2(\varphi u)^{i-p}(\varphi u)^{j-p} & p+1 \le i,\ j \le 2p \end{cases}$$

and

$$\begin{cases} G^{ij} = \delta_{ij}\ ,\ 1 \le i,\ j \le p \\ G^{ij} = 0\ ,\ 1 \le i \le p,\ p+1 \le j \le 2p \\ G^{ij} = \dfrac{1}{1+\delta^2\|\varphi u\|^2}\left[\delta_{ij} + \delta^2(\|\varphi u\|^2\delta_{ij} - (\varphi u)^{i-p}(\varphi u)^{j-p})\right],\ p+1 \le i,\ j \le 2p, \end{cases}$$

where $\varphi u = (\varphi u)^k E_k$, then

$$\begin{aligned} \tau(\pi) &= \sum_{1\le i,j\le p} G^{ij}\left\{\nabla^M_{d\pi(^HE_i)} d\pi(^HE_j) - d\pi(\nabla^{TM}_{^HE_i}{}^HE_j)\right\} \\ &\quad + \sum_{p+1\le i,j\le 2p} G^{ij}\left\{\nabla^M_{d\pi(^VE_{i-m})} d\pi(^VE_{j-m}) - d\pi(\nabla^{TM}_{^VE_{i-m}}{}^VE_{j-m})\right\} \end{aligned}$$

as $d\pi(^VX) = 0$ and $d\pi(^HX) = X\circ\pi$ for any $X \in \Im^1_0(M)$ then:

$$\begin{aligned} \tau(\pi) &= \sum_{1\le i,j\le p} G^{ij}\left\{\nabla^M_{(E_i\circ\pi)}(E_j\circ\pi) - d\pi\big(^H(\nabla^M_{E_i}E_j) - \frac{1}{2}{}^V(R(E_i,E_j)u)\big)\right\} \\ &\quad - \sum_{p+1\le i,j\le 2p} G^{ij}\frac{\delta^2}{1+\delta^2\alpha} g(E_{i-p},\varphi E_{j-p}) d\pi(^V(\varphi u)) \\ &= \sum_{1\le i,j\le p} G^{ij}\left\{(\nabla^M_{E_i}E_j)\circ\pi - (\nabla^M_{E_i}E_j)\circ\pi\right\} \\ &= 0. \end{aligned}$$

■

Theorem 23. *Let (M^{2m},φ,g) be an anti-paraKähler manifold, (N^n,h) be a Riemannian manifold and let (TM,g^{BS}) the tangent bundle of M equipped with the Berger-type deformed Sasaki metric. Let $\phi : (M,g) \longrightarrow (N,h)$ a smooth map. The tension field of the map*

$$\begin{aligned} \Phi : (TM,g^{BS}) &\longrightarrow (N,h) \\ (x,u) &\longmapsto \phi(x) \end{aligned}$$

is given by:

$$\tau(\Phi) = \tau(\phi)\circ\pi. \tag{27}$$

Proof. We put $p = 2m$, let $\{E_i\}_{i=\overline{1,p}}$ be a local orthonormal frame on M and $\{\widetilde{E}_j\}_{j=\overline{1,2p}}$ be a local frame on TM defined by (26), as the Φ is defined by $\Phi = \phi\circ\pi$, we have:

$$\begin{aligned} \tau(\Phi) &= \tau(\phi\circ\pi) \\ &= d\phi(\tau(\pi)) + trace_{g^{BS}}\nabla d\phi(d\pi,d\pi) \end{aligned}$$

$$\begin{aligned} trace_{g^{BS}}\nabla d\phi(d\pi,d\pi) &= \sum_{i=1}^{2p} G^{ij}\left\{\nabla^N_{d\phi(d\pi(\widetilde{E}_i))} d\phi(d\pi(\widetilde{E}_j)) - d\phi(\nabla^M_{d\pi(\widetilde{E}_i)} d\pi(\widetilde{E}_j))\right\} \\ &= \sum_{i,j=1}^{p} \delta_{ij}\big(\nabla^N_{d\phi(d\pi(^HE_i))} d\phi(d\pi(^HE_j)) - d\phi(\nabla^M_{d\pi(^HE_i)} d\pi(^HE_j))\big) \\ &= \sum_{i=1}^{p}\big(\nabla^N_{d\phi(E_i)} d\phi(E_i) - d\phi(\nabla^M_{E_i}E_i)\big)\circ\pi \\ &= \tau(\phi)\circ\pi, \end{aligned}$$

Using Lemma 22, we obtain

$$\tau(\Phi) = \tau(\phi)\circ\pi.$$

■

Theorem 24. *Let (M^{2m},φ,g) be an anti-paraKähler manifold, (N^n,h) be a Riemannian manifold and let (TM,g^{BS}) the tangent bundle of M equipped with Berger-type deformed Sasaki metric. Let $\phi:(M,g)\longrightarrow(N,h)$ a smooth map. The map*

$$\begin{aligned}\Phi:(TM,g^{BS}) &\longrightarrow (N,h)\\ (x,u) &\longmapsto \phi(x)\end{aligned}$$

is a harmonic if and only if ϕ is harmonic.

4.4 Harmonicity of the identity map $I:(TM,g^{BS1})\to(TM,g^{BS2})$

Let (M^{2m},φ,g) be an anti-paraKähler manifold and (TM,g^{BS1}) (resp. (TM,g^{BS2})) its tangent bundle equipped with the Berger-type deformed Sasaki metric g^{BS1} (resp. g^{BS2}), such that

$$g^{BS1}=\begin{pmatrix} g_{ij} & 0\\ 0 & g_{ij}+\delta_1^2 g_{ik}g_{hj}(\varphi u)^k(\varphi u)^h\end{pmatrix},\quad g^{BS2}=\begin{pmatrix} g_{ij} & 0\\ 0 & g_{ij}+\delta_2^2 g_{ik}g_{hj}(\varphi u)^k(\varphi u)^h\end{pmatrix},$$

with respect to the adapted frame $({}^H(\frac{\partial}{\partial x^i}),{}^V(\frac{\partial}{\partial x^i}))_{i=\overline{1,2m}}$, where δ_1,δ_2 are constants and $\delta_1\neq\delta_2$.

Proposition 25. *Let (M^{2m},φ,g) be an anti-paraKähler manifold, then the tension field associated to the identity map $I:(TM,g^{BS1})\to(TM,g^{BS2})$ is given by*

$$\tau(I)=\frac{\Omega(\delta_2^2-\delta_1^2)}{(1+\delta_2^2\alpha)(1+\delta_1^2\alpha)}{}^V(\varphi u),$$

where $\Omega=\sum_{j=1}^{2m-1} g(\varphi E_j,E_j)+\frac{1}{1+\delta_1^2\alpha}g(\varphi E_{2m},E_{2m})$.

Proof. Let $(x,u)\in TM$, $\{E_i\}_{i=\overline{1,m}}$ be a local orthonormal frame on M at x and $\left({}^HE_i,{}^V(\varphi E_j),\frac{1}{\sqrt{1+\delta_1^2\alpha}}{}^V(\varphi E_{2m}),\right)_{i=\overline{1,2m},j=\overline{1,2m-1}}$ be a local orthonormal frame on (TM,g^{BS1}) at (x,u), where $E_{2m}=\frac{u}{\|u\|}$. If $\widetilde{\nabla}$ (resp. $\overline{\nabla}$) denote the Levi-Civita connection of

(TM,g^{BS1}) (resp. (TM,g^{BS2})), then, we have

$$\begin{aligned}
\tau(I) &= trace_{g^{BS1}}(\nabla dI)\\
&= \sum_{i=1}^{2m}\big(\overline{\nabla}_{dI(^HE_i)}dI(^HE_i)-dI(\widetilde{\nabla}_{^HE_i}{}^HE_i)\big)+\sum_{j=1}^{2m-1}\big(\overline{\nabla}_{dI(^V(\varphi E_j))}dI(^V(\varphi E_j))-dI(\widetilde{\nabla}_{^V(\varphi E_j)}{}^V(\varphi E_j))\big)\\
&\quad+\overline{\nabla}_{dI(\frac{1}{\sqrt{1+\delta_1^2\alpha}}{}^V(\varphi E_{2m}))}dI(\frac{1}{\sqrt{1+\delta_1^2\alpha}}{}^V(\varphi E_{2m}))-dI(\widetilde{\nabla}_{\frac{1}{\sqrt{1+\delta_1^2\alpha}}{}^V(\varphi E_{2m})}\frac{1}{\sqrt{1+\delta_1^2\alpha}}{}^V(\varphi E_{2m}))\\
&= \sum_{i=1}^{2m}\big(\overline{\nabla}_{^HE_i}{}^HE_i-\widetilde{\nabla}_{^HE_i}{}^HE_i\big)+\sum_{j=1}^{2m-1}\big(\overline{\nabla}_{^V(\varphi E_j)}{}^V(\varphi E_j)-\widetilde{\nabla}_{^V(\varphi E_j)}{}^V(\varphi E_j)\big)\\
&\quad+\overline{\nabla}_{\frac{1}{\sqrt{1+\delta_1^2\alpha}}{}^V(\varphi E_{2m})}\frac{1}{\sqrt{1+\delta_1^2\alpha}}{}^V(\varphi E_{2m})-\widetilde{\nabla}_{\frac{1}{\sqrt{1+\delta_1^2\alpha}}{}^V(\varphi E_{2m})}\frac{1}{\sqrt{1+\delta_1^2\alpha}}{}^V(\varphi E_{2m}).\\
&= \sum_{j=1}^{2m-1}\big(\overline{\nabla}_{^V(\varphi E_j)}{}^V(\varphi E_j)-\widetilde{\nabla}_{^V(\varphi E_j)}{}^V(\varphi E_j)\big)\\
&\quad+\frac{1}{1+\delta_1^2\alpha}\big(\overline{\nabla}_{^V(\varphi E_{2m})}{}^V(\varphi E_{2m})-\widetilde{\nabla}_{^V(\varphi E_{2m})}{}^V(\varphi E_{2m})\big)\\
&= \sum_{j=1}^{2m-1}\Big(\frac{\delta_2^2}{1+\delta_2^2\alpha}g(\varphi E_j,E_j){}^V(\varphi u)-\frac{\delta_1^2}{1+\delta_1^2\alpha}g(\varphi E_j,E_j){}^V(\varphi u)\Big)\\
&\quad+\frac{1}{1+\delta_1^2\alpha}\Big(\frac{\delta_2^2}{1+\delta_2^2\alpha}g(\varphi E_{2m},E_{2m}){}^V(\varphi u)-\frac{\delta_1^2}{1+\delta_1^2\alpha}g(\varphi E_{2m},E_{2m}){}^V(\varphi u)\Big)\\
&= \sum_{j=1}^{2m-1}g(\varphi E_j,E_j)\Big(\frac{\delta_2^2}{1+\delta_2^2\alpha}-\frac{\delta_1^2}{1+\delta_1^2\alpha}\Big){}^V(\varphi u)\\
&\quad+\frac{1}{1+\delta_1^2\alpha}g(\varphi E_{2m},E_{2m})\Big(\frac{\delta_2^2}{1+\delta_2^2\alpha}-\frac{\delta_1^2}{1+\delta_1^2\alpha}\Big){}^V(\varphi u)\\
&= \Big(\sum_{j=1}^{2m-1}g(\varphi E_j,E_j)+\frac{1}{1+\delta_1^2\alpha}g(\varphi E_{2m},E_{2m})\Big)\frac{\delta_2^2-\delta_1^2}{(1+\delta_2^2\alpha)(1+\delta_1^2\alpha)}{}^V(\varphi u)\\
&= \frac{\Omega(\delta_2^2-\delta_1^2)}{(1+\delta_2^2\alpha)(1+\delta_1^2\alpha)}{}^V(\varphi u),
\end{aligned}$$

where $\Omega=\sum_{j=1}^{2m-1}g(\varphi E_j,E_j)+\frac{1}{1+\delta_1^2\alpha}g(\varphi E_1,E_1).$ ■

Theorem 26. *Let (M^{2m},φ,g) be an anti-paraKähler manifold, then identity map $I:(TM,g^{BS1})\to(TM,g^{BS2})$ is harmonic if and only if $\delta_1=\delta_2$ or $\Omega=0$.*

5. Berger-type deformed Sasaki metric on φ-unit tangent bundle $T_1^{\varphi}M$

The φ-unit tangent (sphere) bundle over an anti-paraKähler manifold (M^{2m},φ,g), is the hypersurface

$$T_1^{\varphi}M = \big\{(x,u)\in TM,\ g(u,\varphi u)=1\big\}.$$

If we set

$$\begin{aligned}
F:TM &\to \mathbb{R}\\
(x,u) &\mapsto F(x,u)=g(u,\varphi u)-1,
\end{aligned}$$

then the hypersurface $T_1^{\varphi}M$ is given by

$$T_1^{\varphi}M=\{(x,u)\in TM,\quad F(x,u)=0\},$$

and $\widetilde{grad}F$ (the gradient of F with respect to g^{BS}) is a normal vector field to $T_1^{\varphi}M$. From the Lemma 3, for any vector field X on M, we get

$$g^{BS}({}^{H}X,\widetilde{grad}F) = {}^{H}X(F) = {}^{H}X(g(u,\varphi u)-1) = 0,$$
$$g^{BS}({}^{V}X,\widetilde{grad}F) = {}^{V}X(F) = {}^{V}X(g(u,\varphi u)-1) = 2g(X,\varphi u) = \frac{2}{1+\delta^2\alpha}g^{BS}({}^{V}X,{}^{V}(\varphi u)).$$

So

$$\widetilde{grad}F = \frac{2}{1+\delta^2\alpha}{}^{V}(\varphi u).$$

Then the unit normal vector field to $T_1^{\varphi}M$ is given by

$$\mathcal{N} = \frac{\widetilde{grad}F}{\sqrt{g^{BS}(\widetilde{grad}F,\widetilde{grad}F)}} = \frac{{}^{V}(\varphi u)}{\sqrt{g^{BS}({}^{V}(\varphi u),{}^{V}(\varphi u))}} = \sqrt{\frac{1}{\alpha(1+\delta^2\alpha)}}{}^{V}(\varphi u).$$

The tangential lift ${}^{T}X$ with respect to g^{BS} of a vector $X\in T_xM$ to $(x,u)\in T_1^{\varphi}M$ is the tangential projection of the vertical lift of X to (x,u) with respect to $\mathcal{N}$, that is

$${}^{T}X = {}^{V}X - g^{BS}_{(x,u)}({}^{V}X,\mathcal{N}_{(x,u)})\mathcal{N}_{(x,u)} = {}^{V}X - \frac{1}{\alpha}g_x(X,\varphi u)^{V}(\varphi u)_{(x,u)}.$$

For the sake of notational clarity, we will use $\bar{X} = X - \frac{1}{\alpha}g(X,\varphi u)\varphi u$, then ${}^{T}X = {}^{V}\bar{X}$.

From the above, we get the direct sum decomposition

$$T_{(x,u)}TM = T_{(x,u)}T_1^{\varphi}M \oplus span\{\mathcal{N}_{(x,u)}\} = T_{(x,u)}T_1^{\varphi}M \oplus span\{{}^{V}(\varphi u)_{(x,u)}\},$$

where $(x,u)\in T_1^{\varphi}M$.

Indeed, if $W\in T_{(x,u)}TM$, then they exist $X,Y\in T_xM$, such that

$$\begin{aligned} W &= {}^{H}X + {}^{V}Y \\ &= {}^{H}X + {}^{T}Y + g^{BS}_{(x,u)}({}^{V}Y,\mathcal{N}_{(x,u)})\mathcal{N}_{(x,u)} \\ &= {}^{H}X + {}^{T}Y + \frac{1}{\alpha}g_x(Y,\varphi u)^{V}(\varphi u)_{(x,u)}. \end{aligned} \tag{28}$$

From the (28) we can say that the tangent space $T_{(x,u)}T_1^{\varphi}M$ of $T_1^{\varphi}M$ at (x,u) is given by

$$T_{(x,u)}T_1^{\varphi}M = \{{}^{H}X + {}^{T}Y \,/\, X,Y\in T_xM, Y\in(\varphi u)^{\perp}\},$$

where $(\varphi u)^{\perp} = \{Y\in T_xM,\, g(Y,\varphi u)=0\}$. Hence $T_{(x,u)}T_1^{\varphi}M$ is spanned by vectors of the form ${}^{H}X$ and ${}^{T}Y$.
Given a vector field X on M, the tangential lift ${}^{T}X$ of X is given by

$${}^{T}X_{(x,u)} = \left({}^{V}X - g^{BS}({}^{V}X,\mathcal{N})\mathcal{N}\right)_{(x,u)} = {}^{V}X_{(x,u)} - \frac{1}{\alpha}g_x(X_x,\varphi u)^{V}(\varphi u)_{(x,u)}.$$

Definition 27. *Let (M^{2m},φ,g) be an anti-paraKähler manifold and (TM,g^{BS}) its tangent bundle equipped with the Berger-type deformed Sasaki metric. The Riemannian metric $\hat{g}^{BS}$ on $T_1^{\varphi}M$, induced by g^{BS}, is completely determined by the identities*

$$\begin{aligned} \hat{g}^{BS}({}^{H}X,{}^{H}Y) &= g(X,Y), \\ \hat{g}^{BS}({}^{T}X,{}^{H}Y) &= \hat{g}^{BS}({}^{H}X,{}^{T}Y) = 0, \\ \hat{g}^{BS}({}^{T}X,{}^{T}Y) &= g(X,Y) - \frac{1}{\alpha}g(X,\varphi u)g(Y,\varphi u). \end{aligned}$$

We shall calculate the Levi-Civita connection $\widehat{\nabla}$ of $T_1^{\varphi}M$ with generalized Berger type deformed Sasaki metric $\hat{g}^{BS}$. This connection is characterized by the Gauss formula:

$$\widehat{\nabla}_{\widehat{X}}\widehat{Y} = \widetilde{\nabla}_{\widehat{X}}\widehat{Y} - g^{BS}(\widetilde{\nabla}_{\widehat{X}}\widehat{Y},\mathcal{N})\mathcal{N}, \tag{29}$$

for all vector fields $\widehat{X}$ and $\widehat{Y}$ on $T_1^{\varphi}M$.

Theorem 28. *Let (M^{2m},φ,g) be an anti-paraKähler manifold and $(T_1^{\varphi}M,\hat{g}^{BS})$ its φ-unit tangent bundle equipped with the Berger-type deformed Sasaki metric, then we have the following formulas.*

$$\begin{aligned}
1.\,\widehat{\nabla}_{{}^HX}{}^HY &= {}^H(\nabla_XY)-\frac{1}{2}{}^T(R(X,Y)u),\\
2.\,\widehat{\nabla}_{{}^HX}{}^TY &= \frac{1}{2}{}^H(R(u,Y)X)+{}^T(\nabla_XY),\\
3.\,\widehat{\nabla}_{{}^TX}{}^HY &= \frac{1}{2}{}^H(R(u,X)Y),\\
4.\,\widehat{\nabla}_{{}^TX}{}^TY &= \frac{1}{\alpha^2}g(X,\varphi u)g(Y,\varphi u)^Tu-\frac{1}{\alpha}g(Y,\varphi u)^T(\varphi X),
\end{aligned}$$

for all vector fields X,Y on M, where ∇ is the Levi-Civita connection of (M^{2m},φ,g) and R is its curvature tensor.

Proof. In the proof, we will use Theorem 4, Lemma 5 and formula (29). ∎

5.1 Harmonicity of canonical projection $\pi_1:(T_1^{\varphi}M,\hat{g}^{BS})\longrightarrow(M,g)$

Lemma 29. *Let (M^{2m},φ,g) be an anti-paraKähler manifold and $(T_1^{\varphi}M,\hat{g}^{BS})$ its φ-unit tangent bundle equipped with the Berger-type deformed Sasaki metric. The canonical projection*

$$\begin{aligned}
\pi_1:(T_1^{\varphi}M,\hat{g}^{BS}) &\longrightarrow (M,g)\\
(x,u) &\longmapsto x
\end{aligned}$$

is harmonic i.e. $\tau(\pi_1)=0$.

Proof. Let $(x,u)\in T_1^{\varphi}M$, $\{E_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on M at x, where $E_{2m}=\frac{u}{\|u\|}$ and $\left({}^HE_i,{}^T(\varphi E_j)\right)_{i=\overline{1,2m},j=\overline{1,2m-1}}$ be a local orthonormal frame on $(T_1^{\varphi}M,\hat{g}^{BS})$ at (x,u), then, the tension field of π_1 is given by

$$\begin{aligned}
\tau(\pi_1) &= trace_{\hat{g}^{BS}}\nabla d\pi_1\\
&= \sum_{i=1}^{2m}\left(\nabla^M_{d\pi_1({}^HE_i)}d\pi_1({}^HE_i)-d\pi_1(\widehat{\nabla}_{{}^HE_i}{}^HE_i)\right)+\sum_{j=1}^{2m-1}\left(\nabla^M_{d\pi_1({}^T(\varphi E_j))}d\pi_1({}^T(\varphi E_j))-d\pi_1(\widehat{\nabla}_{{}^T(\varphi E_j)}{}^T(\varphi E_j))\right).
\end{aligned}$$

With $d\pi_1({}^TX)=d\pi({}^TX)=d\pi({}^V\bar{X})=0$ and $d\pi_1({}^HX)=d\pi({}^HX)=X\circ\pi=X\circ\pi_1$ for any $X\in\Im^1_0(M)$, then we find

$$\begin{aligned}
\tau(\pi_1) &= \sum_{i=1}^{2m}\left(\nabla^M_{(E_i\circ\pi_1)}(E_i\circ\pi_1)-d\pi_1\left({}^H(\nabla^M_{E_i}E_i)-\frac{1}{2}{}^T(R(E_i,E_i)u)\right)\right)\\
&\quad-\sum_{j=1}^{2m-1}d\pi_1\left(\frac{1}{\alpha^2}g(E_j,u)^{2T}u-\frac{1}{\alpha}g(E_j,u)^TE_j\right)\\
&= \sum_{j=1}^{2m}\left((\nabla^M_{E_i}E_i)\circ\pi_1-(\nabla^M_{E_i}E_i)\circ\pi_1\right)\\
&= 0.
\end{aligned}$$

∎

Theorem 30. *Let (M^{2m},φ,g) be an anti-paraKähler manifold, (N^n,h) be a Riemannian manifold and let $(T_1^{\varphi}M,\hat{g}^{BS})$ the φ-unit tangent bundle of M equipped with the Berger-type deformed Sasaki metric. Let $\phi:(M,g)\longrightarrow(N,h)$ a smooth map. The tension field of the map*

$$\begin{aligned}
\Phi_1:(T_1^{\varphi}M,\hat{g}^{BS}) &\longrightarrow (N,h)\\
(x,u) &\longmapsto \phi(x)
\end{aligned}$$

is given by:

$$\tau(\Phi_1) = \tau(\phi)\circ\pi_1. \tag{30}$$

Proof. Let $(x,u) \in T_1^{\varphi}M$, $\{E_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on M at x, where $E_{2m} = \frac{u}{\|u\|}$ and $\left({}^{H}E_i, {}^{T}(\varphi E_j)\right)_{i=\overline{1,2m}, j=\overline{1,2m-1}}$ be a local orthonormal frame on $(T_1^{\varphi}M, \hat{g}^{BS})$ at (x,u), as the Φ_1 is defined by $\Phi_1 = \phi \circ \pi_1$, we have:

$$\begin{aligned} \tau(\Phi_1) &= \tau(\phi \circ \pi_1) \\ &= d\phi(\tau(\pi_1)) + trace_{\hat{g}^{BS}} \nabla d\phi(d\pi_1, d\pi_1) \end{aligned}$$

$$\begin{aligned} trace_{\hat{g}^{BS}} \nabla d\phi(d\pi_1, d\pi_1) &= \sum_{i=1}^{2m} \left(\nabla^N_{d\phi(d\pi_1({}^{H}E_i))} d\phi(d\pi_1({}^{H}E_i)) - d\phi(\nabla^M_{d\pi_1({}^{H}E_i)} d\pi_1({}^{H}E_i))\right) \\ &\quad + \sum_{j=1}^{2m-1} \left(\nabla^N_{d\phi(d\pi_1({}^{T}(\varphi E_j)))} d\phi(d\pi_1({}^{T}(\varphi E_j))) - d\phi(\nabla^M_{d\pi_1({}^{T}(\varphi E_j))} d\pi_1({}^{T}(\varphi E_j)))\right) \\ &= \sum_{i=1}^{2m} \left(\nabla^N_{d\phi(E_i)} d\phi(E_i) - d\phi(\nabla^M_{E_i} E_i)\right) \circ \pi_1 \\ &= \tau(\phi) \circ \pi_1. \end{aligned}$$

Using Lemma 29, we obtain

$$\tau(\Phi) = \tau(\phi) \circ \pi_1.$$

■

Theorem 31. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold, (N^n, h) be a Riemannian manifold and let $(T_1^{\varphi}M, \hat{g}^{BS})$ the φ-unit tangent bundle of M equipped with Berger-type deformed Sasaki metric. Let $\phi : (M,g) \longrightarrow (N,h)$ a smooth map. The map*

$$\begin{aligned} \Phi_1 : (T_1^{\varphi}M, \hat{g}^{BS}) &\longrightarrow (N,h) \\ (x,u) &\longmapsto \phi(x) \end{aligned}$$

is a harmonic if and only if ϕ is harmonic.

6. Conclusions

In this work, firstly, we studied the harmonicity of a tangent bundle with the Berger-type deformed Sasaki and we gave the necessary and sufficient conditions when a vector field is harmonic with respect to this metric. Secondly, we searched the harmonicity of the maps between a Riemannian manifold and the tangent bundle over another anti-paraKähler manifold or vice versa. In the last section, we introduce the φ-unit tangent bundle $T_1^{\varphi}M$ equipped with Berger-type deformed Sasaki metric $\hat{g}^{BS}$, and we study the harmonicity of this metric. Also, we can study the harmonicity of an another metrics on the tangent bundle by deformation in the vertical bundle or in the horizontal bundle.

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