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# Properties of Certain Volterra type ABC Fractional Integral Equations

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**Abstract**

In this paper we study existence, uniqueness and other properties of solutions of Volterra type ABC fractional integral equations. We have used Banach fixed point theorem with Bielecki type norm and Gronwall inequality in the frame of ABC fractional integral for proving our results.

*Keywords:* Volterra Equation ABC fractional Existence fractional Gronwall inequality.
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## 1. Introduction

During last few decades many authors have studied the fractional differential and integral equations and its properties by using various techniques. Due to wide applications of fractional calculus in various fields its study has become very interesting [4, 21]. In this paper we consider the Volterra fractional integral equation of the form

$$x(t) = \mathfrak{f}(t, x(t), \ {}_{a}^{AB}I^{\alpha} k\left(t, \tau, x\left(\tau\right)\right), \tag{1.1}$$

for $0 < \alpha < 1$, where $I = [a, b]$, $k \in C(I \times I \times \mathbb{R}, \mathbb{R})$ and $\mathfrak{f} \in C(I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.
Gronwall inequality plays very important role in studying the various properties such as estimates of solution, continuous dependence and others of differential equation. Recently in [6, 25, 27] the authors have obtained the fractional Gronwall inequality using various fractional definition. Fractional calculus is found to be very interesting in modeling the real world situations. Due to the application of fractional calculus various

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definitions of fractional derivative and integral are obtained in the literature such as Generelized fractional derivative and integral with respect to another function that is $\psi$ fractional, Hadmard, Katungampola, Caputo-Fabrizio, Hilfer and others [5, 9, 11, 15, 17]. Recently Atangana and Balaneu has introduced the new definitions of fractional integral and derivative with Mittag-Lefler function [2] . Using these definitions results on Taylors theorem existence, uniqueness and numerical solution have been obtained in [1, 3, 7, 8, 13, 14, 16, 20, 23]. In [12, 18, 19, 24, 26] authors have given various application of fractional calculus using these definitions.

. Motivated by the above mentioned research work the main objective of this paper is to study the fundamental properties of solutions of (1.1). The well known Banach fixed point theorem with Bielecki type norm and Gronwall type inequality given in [10] is used for presenting the results.

## 2. Preliminaries

Now in this we give some basic definitions and lemmas used in our discussions The left Riemann-Liouville fractional integral of order $\alpha$ for $\alpha > 0$ is defined as [15]:

$$\left({}_aI^{\alpha}x\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} x(s)\, ds.$$

The Atagana-Baleanu fractional derivative and integral is defined as follows:

***Definition 2.1.*** [2] Let $x \in H^1(a,b)$, $a < b$ and $\alpha$ in $[0,1]$. The Caputo Atangana-Baleanu(ABC) fractional derivative of $x$ of order $\alpha$ is defined by

$$\left({}_a^{ABC}D^{\alpha}x\right)(t) = \frac{B(\alpha)}{1-\alpha} \int_a^t x'(s) E_{\alpha}\left[-\alpha \frac{(t-s)^{\alpha}}{1-\alpha}\right] ds,$$

where $E_{\alpha}$ is the Mittag-Leffler function defined by $E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)}$ and $B(\alpha)$ is a normalizing positive function satisfying $B(0) = B(1) = 1$.

The Riemann Atangana-Baleanu fractional derivative of $x$ of order $\alpha$ is defined by

$$\left({}_a^{ABR}D^{\alpha}x\right)(t) = \frac{B(\alpha)}{1-\alpha} \int_a^t x(s) E_{\alpha}\left[-\alpha \frac{(t-s)^{\alpha}}{1-\alpha}\right] ds.$$

The associated fractional integral is defined by:

$$\left({}_a^{AB}I^{\alpha}x\right)(t) = \frac{1-\alpha}{B(\alpha)} x(t) + \frac{\alpha}{B(\alpha)} \left(\left({}_aI^{\alpha}x\right)(t)\right).$$

where ${}_aI^{\alpha}$ is the left Riemann-Liouville fractional integral.

Now we construct the appropriate metric space. Let $\xi > 0$ be a constant and consider the space of all continuous function $C(I,\mathbb{R})$ where $I = [a,b]$. We denote this special space by $C_{\xi,\alpha}(I,\mathbb{R})$

$$d_{\xi,\alpha}(u,v) = \underset{t \in I}{Sup} \frac{|u(t) - v(t)|}{E_{\alpha}\left(\xi (t-a)^{\alpha}\right)},$$

with norm defined by

$$|u|_{\xi,\alpha} = \underset{t \in I}{Sup} \frac{|u(t)|}{E_{\alpha}\left(\xi (t-a)^{\alpha}\right)},$$

where $E_{\alpha} : \mathbb{R} \to \mathbb{R}$ is a one parameter Mittag-Leffler function. The above definitions $d_{\xi,\alpha}$ and $|.|_{\xi,\alpha}$ are the variants of Bielecki's metric and norm.

The Gronwall inequality in the frame of fractional integrals associated with the Atangana-Baleanu fractional derivative is given in [10] as follows:

***Theorem 2.1.***. [10] Suppose that $\alpha > 0$, $c(t)\left(1-\frac{1-\alpha}{B(\alpha)}d\left(t\right)\right)^{-1}$ is nonnegative, nondecreasing and locally integrable function on $[a,b)$, $\frac{\alpha d(t)}{B(\alpha)}\left(1-\frac{1-\alpha}{B(\alpha)}d\left(t\right)\right)^{-1}$ is nonnegative and bounded on $[a,b)$ and $x(t)$ is nonnegative and locally integrable on $[a,b)$ with

$$x\left(t\right)\leq c\left(t\right)+d\left(t\right)\left({}_{a}^{AB}I^{\alpha}x\right)\left(t\right).$$

Then

$$x\left(t\right)\leq\frac{c\left(t\right)B\left(\alpha\right)}{B\left(\alpha\right)-\left(1-\alpha\right)d\left(t\right)}E_{\alpha}\left(\frac{\alpha d\left(t\right)\left(t-a\right)^{\alpha}}{B\left(\alpha\right)-\left(1-\alpha\right)d\left(t\right)}\right).$$

## 3. Existence of Solution

Now we give our result on the existence of solution of (1.1)

***Theorem 3.1***. Let $P > 0, Q \geq 0,\ \xi > 1$ be constants. Suppose that the functions $\mathfrak{f}, k$ in (1.1) satisfy the conditions

$$\left|\mathfrak{f}\left(t,u,v\right)-\mathfrak{f}\left(t,\overline{u},\overline{v}\right)\right|\leq Q\left[\left|u-\overline{u}\right|+\left|v-\overline{v}\right|\right], \tag{3.1}$$

$$\left|k\left(t,s,u\right)-k\left(t,s,\overline{u}\right)\right|\leq P\left|u-\overline{u}\right| \tag{3.2}$$

and

$$m_{1}=\sup_{t\in I}\frac{1}{E_{\alpha}\left(\xi\left(t-a\right)^{\alpha}\right)}\left|\mathfrak{f}\left(t,0,{}_{a}^{AB}I^{\alpha}k\left(t,s,0\right)\right)\right|<\infty. \tag{3.3}$$

If $Q\left(1+\frac{P}{\xi}\right)<1$ then the integral equation (1.1) has a unique solution $x\in C_{\xi,\alpha}\left(I,\mathbb{R}\right)$

***Proof.***. Consider the equivalent formulation of (1.1) we have

$$\begin{aligned}x\left(t\right)=\;&\mathfrak{f}\left(t,x(t),{}_{a}^{AB}I^{\alpha}k\left(t,\tau,x(\tau)\right)\right)-\mathfrak{f}\left(t,0,{}_{a}^{AB}I^{\alpha}k\left(t,\tau,0\right)\right)\\&+\mathfrak{f}\left(t,0,{}_{a}^{AB}I^{\alpha}k\left(t,\tau,0\right)\right),\end{aligned} \tag{3.4}$$

for $t\in I$. We shall show that (3.4) has unique solution and thus (1.1) must also have unique solution. Let $x\in C_{\xi,\alpha}\left(I,\mathbb{R}\right)$ and define the operator $\mathfrak{T}$ by

$$\begin{aligned}\left(\mathfrak{T}x\right)\left(t\right)=\;&\mathfrak{f}\left(t,x(t),{}_{a}^{AB}I^{\alpha}k\left(t,\tau,x(\tau)\right)\right)-\mathfrak{f}\left(t,0,{}_{a}^{AB}I^{\alpha}k\left(t,\tau,0\right)\right)\\&+\mathfrak{f}\left(t,0,{}_{a}^{AB}I^{\alpha}k\left(t,\tau,0\right)\right).\end{aligned} \tag{3.5}$$

Now we show that $\mathfrak{T}$ maps $C_{\xi,\alpha}\left(I,\mathbb{R}\right)$ into itself. We have

$$\begin{aligned}\left|\mathfrak{T}x\right|_{\xi,\alpha}&=\underset{t\in I}{Sup}\frac{\left|\left(\mathfrak{T}x\right)\left(t\right)\right|}{E_{\alpha}\left(\xi\left(t-a\right)^{\alpha}\right)}\\&\leq\sup_{t\in I}\frac{1}{E_{\alpha}\left(\xi\left(t-a\right)^{\alpha}\right)}\left|\mathfrak{f}\left(t,x\left(t\right),{}_{a}^{AB}I^{\alpha}k\left(t,\tau,x(\tau)\right)\right)\right.\\&\left.-\mathfrak{f}\left(t,0,{}_{a}^{AB}I^{\alpha}k\left(t,\tau,0\right)\right)\right|+\sup_{t\in I}\frac{1}{E_{\alpha}\left(\xi\left(t-a\right)^{\alpha}\right)}\left|\mathfrak{f}\left(t,0,{}_{a}^{AB}I^{\alpha}k\left(t,\tau,0\right)\right)\right|\\&\leq m_{1}+\sup_{t\in I}\frac{1}{E_{\alpha}\left(\xi\left(t-a\right)^{\alpha}\right)}Q\left[\left|x\left(t\right)\right|+{}_{a}^{AB}I^{\alpha}P\left|x\left(\tau\right)\right|\right]\\&=m_{1}+Q\left[\sup_{t\in I}\frac{\left|x\left(t\right)\right|}{E_{\alpha}\left(\xi\left(t-a\right)^{\alpha}\right)}\right.\\&\left.+P\sup_{t\in I}\frac{1}{E_{\alpha}\left(\xi\left(t-a\right)^{\alpha}\right)}{}_{a}^{AB}I^{\alpha}E_{\alpha}\left(\xi\left(t-a\right)^{\alpha}\right)\frac{\left|x\left(\tau\right)\right|}{E_{\alpha}\left(\xi\left(t-a\right)^{\alpha}\right)}\right]\end{aligned}$$

$$\leq m_1 + Q\left[|x|_{\xi,\alpha} + P\,|x|_{\xi,\alpha} \sup_{t\in I} \frac{1}{E_\alpha\left(\xi\left(t-a\right)^\alpha\right)} {}_a^{AB}I^\alpha E_\alpha\left(\xi\left(t-a\right)^\alpha\right)\right]$$

$$\leq m_1 + Q\left[|x|_{\xi,\alpha} + P\,|x|_{\xi,\alpha} \sup_{t\in I} \frac{1}{E_\alpha\left(\xi\left(t-a\right)^\alpha\right)}\right.$$
$$\left.\left(\frac{(1-\alpha)}{B(\alpha)} E_\alpha\left(\xi\left(t-a\right)^\alpha\right) + \frac{\alpha}{B(\alpha)} {}_aI^\alpha E_\alpha\left(\xi\left(t-a\right)^\alpha\right)\right)\right]$$

$$\leq m_1 + Q\left[|x|_{\xi,\alpha} + P\,|x|_{\xi,\alpha} \sup_{t\in I} \frac{1}{E_\alpha\left(\xi\left(t-a\right)^\alpha\right)}\right.$$
$$\left.\left(\frac{(1-\alpha)}{B(\alpha)} E_\alpha\left(\xi\left(t-a\right)^\alpha\right) + \frac{\alpha}{B(\alpha)} \frac{E_\alpha\left(\xi\left(t-a\right)^\alpha\right)-1}{\xi}\right)\right]$$

$$\leq m_1 + Q\,|x|_{\xi,\alpha}\left[1 + P\frac{1}{B(\alpha)}\left((1-\alpha) + \frac{\alpha}{\xi}\right)\right]. \tag{3.6}$$

Now we show that the operator $\mathfrak{T}$ is a contraction map. Let $v, w \in C_{\xi,\alpha}(I,\mathbb{R})$, from the hypotheses we have

$$d_{\xi,\alpha}(\mathfrak{T}v, \mathfrak{T}w) = \sup_{t\in I} \frac{|(\mathfrak{T}v)(t) - (\mathfrak{T}w)(t)|}{E_\alpha\left(\xi\left(t-a\right)^\alpha\right)}$$
$$= \sup_{t\in I} \frac{1}{E_\alpha\left(\xi\left(t-a\right)^\alpha\right)} \left|f\left(t, v(t), {}_a^{AB}I^\alpha k(t,\tau,v(\tau))\right)\right.$$
$$\left.-f\left(t, w(t), {}_a^{AB}I^\alpha k(t,\tau,w(\tau))\right)\right]$$
$$\leq \sup_{t\in I} \frac{1}{E_\alpha\left(\xi\left(t-a\right)^\alpha\right)} Q\left[|v(t)-w(t)| + {}_a^{AB}I^\alpha P\,|v(\tau)-w(\tau)|\right]$$
$$\leq Q\left[\sup_{t\in I} \frac{|v(t)-w(t)|}{E_\alpha\left(\xi\left(t-a\right)^\alpha\right)} + \sup_{t\in I} \frac{1}{E_\alpha\left(\xi\left(t-a\right)^\alpha\right)}\right.$$
$$\left. P{}_a^{AB}I^\alpha E_\alpha\left(\xi\left(t-a\right)^\alpha\right) \frac{|v(\tau)-w(\tau)|}{E_\alpha\left(\xi\left(t-a\right)^\alpha\right)}\right]$$
$$\leq Q\left[d_{\xi,\alpha}(v,w) + P d_{\xi,\alpha}(v,w) \sup_{t\in I} \frac{1}{E_\alpha\left(\xi\left(t-a\right)^\alpha\right)} {}_a^{AB}I^\alpha E_\alpha\left(\xi\left(t-a\right)^\alpha\right)\right]$$
$$= Q d_{\xi,\alpha}(v,w)$$
$$\left[1 + p d_{\xi,\alpha}(v,w) \sup_{t\in I} \frac{1}{E_\alpha\left(\xi\left(t-a\right)^\alpha\right)}\left(\frac{E_\alpha\left(\xi\left(t-a\right)^\alpha\right)-1}{\xi}\right)\right]$$
$$= Q\,|x|\,d_{\xi,\alpha}(v,w)\,\frac{1}{B(\alpha)}\left[(1-\alpha) + \frac{\alpha}{\xi}\right]. \tag{3.7}$$

It follows from Banach fixed point theorem $\mathfrak{T}$ has a unique fixed point.

## 4. Estimates of Solution

Now we obtain estimates on the solutions of equation (1.1) with some suitable assumptions

***Theorem 4.1.*** Suppose that the functions $\mathfrak{f}, k$ in (1.1) are continuous and satisfy the conditions

$$|\mathfrak{f}(t,u,v) - \mathfrak{f}(t,\overline{u},\overline{v})| \leq G\left[|u-\overline{u}| + |v-\overline{u}|\right], \tag{4.1}$$

$$|k(t,\tau,u) - k(t,\tau,v)| \leq h(t)\,|u-v|, \tag{4.2}$$

where $0 \leq G < 1$ is a constant

$$m_2 = \underset{t\in I}{Sup}\left|\mathfrak{f}\left(t, 0, {}_a^{AB}I^\alpha k(t,s,0)\right)\right| < \infty. \tag{4.3}$$

If $x(t)$ is any solution of (1.1) and $H(t) = \underset{t\in I}{Sup}\, h(t)$ then

$$|x(t)| \leq \frac{m_2}{(1-G)} \frac{B(\alpha)}{\left(B(\alpha) - (1-\alpha)\left(\frac{G}{1-G}\right)H(t)\right)} E_\alpha\left(\frac{\alpha \frac{G}{1-G} H(t)\,(t-a)^\alpha}{\left(B(\alpha) - (1-\alpha)\left(\frac{G}{1-G}\right)H(t)\right)}\right). \tag{4.4}$$

***Proof***. Since the solution $x(t)$ of equation (1.1) satisfies the equation (3.4) and the hypothesis we have

$$\begin{aligned} |x(t)| &\leq \left|\mathfrak{f}\left(t, 0, {}^{AB}_{a}I^\alpha k\,(t,\tau,0)\right)\right| \\ &+ \left|\mathfrak{f}\left(t, x(t), {}^{AB}_{a}I^\alpha k\,(t,\tau,x(\tau))\right) - \mathfrak{f}\left(t, 0, {}^{AB}_{a}I^\alpha k\,(t,\tau,0)\right)\right| \\ &\leq m_2 + G\left[|x(t)| + {}^{AB}_{a}I^\alpha h\,(\tau)\,|x\,(\tau)|\right]. \end{aligned} \tag{4.5}$$

From (4.5) and hypotheses $0 \leq G < 1$ we have

$$|x(t)| \leq \frac{m_2}{(1-G)} + \frac{G}{1-G} H(t) {}^{AB}_{a}I^\alpha\,|x\,(\tau)|\,. \tag{4.6}$$

Now applying the Gronwall inequality Theorem 2.1 to (4.6) we get (4.4).

## 5. Continuous dependence

Now in this section we study the continuous dependence of (1.1) and the functions involved therein. Now consider the equation (1.1) and the corresponding equation

$$y\,(t) = \bar{\mathfrak{f}}\left(t, y\,(t)\,, {}^{AB}_{a}I^\alpha \overline{k}\,(t,\tau,y\,(\tau))\right), \tag{5.1}$$

for $t \in I$, $\tau \leq t$ where $\overline{k} \in C(I \times I \times \mathbb{R}, \mathbb{R})$ and $\overline{f} \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $y_0$ is a given constant in $\mathbb{R}^n$.

Our next theorem deals with continuous dependence of solution of (1.1) and on functions involved therein.

***Theorem 5.1***. Suppose the functions $\mathfrak{f}, k$ in (1.1) are continuous and satisfy the conditions (4.1) and (4.2). Suppose that

$$\left|\mathfrak{f}\left(t, y\,(t)\,, {}^{AB}_{a}I^\alpha k\,(t,\tau,y\,(\tau))\right) - \bar{\mathfrak{f}}\left(t, y\,(t)\,, {}^{AB}_{a}I^\alpha \overline{k}\,(t,\tau,y\,(\tau))\right)\right| \leq \epsilon, \tag{5.2}$$

where $\mathfrak{f}, k$ and $\bar{\mathfrak{f}}, \overline{k}$ are functions involved in (1.1) and (5.1), $\epsilon > 0$ is small constant and $y(t)$ is given solution of (5.1). Then the solution $x(t)$, $t \in I$ of (1.1) depends continuously on the functions involved on the right hand side of (1.1).

***Proof***. Let $v(t) = |x(t) - y(t)|$ for $t \in I$. Since $x(t)$ and $y(t)$ are solutions of equations (1.1), (5.1) and given conditions we have

$$\begin{aligned} v\,(t) &\leq \left|\mathfrak{f}\left(t, x(t), {}^{AB}_{a}I^\alpha k\,(t,\tau,x\,(\tau))\right) - \bar{\mathfrak{f}}\left(t, y(t), {}^{AB}_{a}I^\alpha \overline{k}\,(t,\tau,y\,(\tau))\right)\right| \\ &+ \left|\mathfrak{f}\left(t, y(t), {}^{AB}_{a}I^\alpha k\,(t,\tau,y\,(\tau))\right) - \bar{\mathfrak{f}}\left(t, y(t), {}^{AB}_{a}I^\alpha \overline{k}\,(t,\tau,y\,(\tau))\right)\right| \\ &\leq \varepsilon + G\left[|v\,(t)| + {}^{AB}_{a}I^\alpha h\,(\tau)\,|v\,(\tau)|\right]. \end{aligned} \tag{5.3}$$

From (5.3) and assumption $0 \leq G < 1$ we have

$$v\,(t) \leq \frac{\varepsilon}{1-G} + \frac{G}{1-G} H\,(t)\, {}^{AB}_{a}I^\alpha\,|v\,(\tau)|\,. \tag{5.4}$$

Now by Gronwall's Inequality Theorem 2.1 we have

$$|x\,(t) - y\,(t)| \leq \frac{\varepsilon}{1-G} \frac{B\,(\alpha)}{\left[B\,(\alpha) - (1-\alpha)\,\frac{G}{1-G} H\,(t)\right]} E_\alpha\left(\frac{\alpha \frac{G}{1-G} H\,(t)}{B\,(\alpha) - (1-\alpha)\,\frac{G}{1-G} H\,(t)}\right). \tag{5.5}$$

From (5.5) it follows that the solution of equation (1.1) depends continuously on the functions involved on the right hand side of equation (1.1).

Consider the system of fractional Volterra integral equation

$$w(t) = g\left(t, w(t), {}^{AB}_{a}I^{\alpha} l(t,\tau,w(\tau)), \mu\right) \tag{5.6}$$

and

$$w(t) = g\left(t, w(t), {}^{AB}_{a}I^{\alpha} l(t,\tau,w(\tau)), \mu_0\right), \tag{5.7}$$

for $t \in I$ where $l \in C(I \times I \times \mathbb{R}, \mathbb{R})$ and $g \in C(I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.

Now in our next theorem we give the dependence of solution of equation (5.6)-(5.7) on parameters.

***Theorem 5.2.*** Suppose that the functions $g, l$ in equation $(5.6) - (5.7)$ satisfy the conditions

$$|g(t, v_1, v_2, \mu) - g(t, \overline{v_1}, \overline{v_2}, \mu)| \le \overline{G}\left[|v_1 - \overline{v_1}| + |v_2 - \overline{v_2}|\right], \tag{5.8}$$

$$|g(t, v_1, v_2, \mu_0) - g(t, \overline{v_1}, \overline{v_2}, \mu_0)| \le d(t)\,|\mu - \mu_0|, \tag{5.9}$$

$$|l(t,s,v) - l(t,s,r)| \le \overline{h}(t)\,|v - r|, \tag{5.10}$$

where $0 \le \overline{G} < 1$ is a constant and $d \in C(I, \mathbb{R})$ such that $d(t) \le D < \infty$, $D$ is a constant and $\overline{h}(t) \in C(I, \mathbb{R})$.

Let $w_1(t)$ and $w_2(t)$ be the solutions of (5.6) and (5.7) respectively then

$$\begin{aligned} |w_1(t) - w_2(t)| \le \frac{D\,|\mu-\mu_0|}{1-\overline{G}} & \frac{B(\alpha)}{\left[B(\alpha) - (1-\alpha)\frac{G}{1-G}\overline{H}(t)\right]} \\ & E_{\alpha}\left(\frac{\alpha\frac{\overline{G}}{1-\overline{G}}\overline{H}(t)(t-a)^{\alpha}}{B(\alpha) - (1-\alpha)\frac{G}{1-G}\overline{H}(t)}\right). \end{aligned} \tag{5.11}$$

***Proof.*** Let $w(t) = |w_1(t) - w_2(t)|$, $t \in I$. Since $w_1(t)$ and $w_2(t)$ are solution of equation (5.6) and (5.7) and given conditions we have

$$\begin{aligned} |w(t)| \le & \left|g\left(t, w_1(t), {}^{AB}_{a}I^{\alpha} l(t,\tau,w_1(\tau)), \mu\right)\right. \\ & \left. - g\left(t, w_2(t), {}^{AB}_{a}I^{\alpha} l(t,\tau,w_2(\tau)), \mu_0\right)\right| \\ & + \left|g\left(t, w_2(t), {}^{AB}_{a}I^{\alpha} l(t,\tau,w_2(\tau)), \mu\right)\right. \\ & \left. - g\left(t, w_2(t), {}^{AB}_{a}I^{\alpha} l(t,\tau,w_2(\tau)), \mu_0\right)\right| \\ \le & \overline{G}\left[w(t) + {}^{AB}_{a}I^{\alpha}\overline{h}(\tau)\,w(\tau)\right]. \end{aligned} \tag{5.12}$$

From (5.12) and using the assumption $0 \le \overline{G} < 1$ we have

$$|w(t)| \le \frac{D\,|\mu-\mu_0|}{1-\overline{G}} + \frac{\overline{G}}{1-\overline{G}} H(t)\, {}^{AB}_{a}I^{\alpha}\,|w(\tau)|. \tag{5.13}$$

Now an application of Gronwall inequality Theorem 2.1 to (5.13) we get (5.11).

***Example.*** In order to ilustrate our results we give the following example:
Consider the Volterra type ABC fractional integral equation

$$x(t) = \frac{1}{10(t+1)}x(t) + \frac{1}{10}{}^{AB}_{a}I^{\alpha} e^{-2t} x(\tau), \tag{5.14}$$

for $t \in [0,1]$
Set

$$f(t, x(\tau), \mathrm{X}x(\tau)) = \frac{1}{10(t+1)}x(t) + \frac{1}{10}{}^{AB}_{a}I^{\alpha} e^{-2t} x(\tau),$$

$$\mathrm{X}x(\tau) = {}_{a}^{AB}I^{\alpha}e^{-2t}x\left(\tau\right).$$

From above it can be easy to see that

$$\left|f(t, u(\tau), \mathrm{X}u(\tau)) - f(t, v(\tau), \mathrm{X}v(\tau))\right| \leq \frac{1}{10}\left[|u-v| + |\mathrm{X}u - \mathrm{X}v|\right],$$

$$|\mathrm{X}u - \mathrm{X}v| \leq \frac{1}{10e^{2t}}{}_{a}^{AB}I^{\alpha}x\left(\tau\right).$$

Thus from above equation the conditions $(3.1)-(3.2)$ holds we have $Q = \frac{1}{10}$ and $P = \frac{1}{10e^{2t}}$ then for $\xi = 2$ we have

$$Q\left(1 + \frac{P}{\xi}\right) \cong 0.10067 < 1.$$

Thus the assumptions of the Theorem 3.1 are satisfied and thus the integral equation (5.14) has a unique solution.

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## References

[1] O.A. Arqub and B. Maayah, Numerical solutions of integrodifferential equations of Fredholm operator type in the sense of the Atangana-Baleanu fractional operator, Chaos Solitions Fractals. 117, (2018), 117–124.

[2] A. Atagana and D. Baleanu, New fractional derivatives with nonlocal and nonsingular kernal: Theory and applications to heat transfer model, Therm sci. 20(2), 2016, 763−769.

[3] A. Atangana, J.F. Gomez-Aguilar, Numerical approximation of Riemann-Liouville definition of fractional derivative: from Riemann-Liouville to Atangana-Baleanu, Numer. Methods Partial Differ Equ. 2018;34(5):1502−23 .

[4] D. Baleanu, A.M. Lopes, Handbook of Fractional Calculus with Applications, Applications in Engineering, Life and Social Sciences, *De Gruyter.*, Vol. 7:2019.

[5] M. Caputo, M. Fabrizio, A new defination of fractional derivative without singular kernal, Progr. Fract. Differ Appl. 1(2015), 73−85.

[6] Ding X.L., Daniel C.L., Nieto J.J., A New Generalized Gronwall Inequality with a Double Singularity and Its Applications to Fractional Stochastic Differential Equations Stochastic Analysis and Applications, 019, 37(6), pp. 1042−1056.

[7] Djida J., Atangana A. and Area I., Numerical computation of a Fractional Derivative with nonlocal and non-singular kernal, Math. Model. Nat. Phenom. 12(3), 2017, 4−13.

[8] A. Fernandez and D. Baleanu, The mean value theorem and Taylors theorem for fractional derivatives with Mittag-Leffler kernel, *Advances in Difference Equations*, 2018: 86.

[9] R. Hilfer, Applications of fractional calculusin physics, World Scientific. 2000.

[10] F. Jarad, T. Abdeljawad and Z. Hammouch, On a class of ordinary diferential equations in the frame of Atangana-Baleanu fractional derivative, Chaos Solitions Fractals. 117(2018), 16−20.

[11] U. Katugampola, New approach to generalized fractional derivatives, Bull. Math. Anal. Appl. (2014), 6(4), 16−20.

[12] I. Koca and A. Atangana, Chaos in a simple nonlinear system with Atangana Baleanu derivatives with fractional order, Chaos Solitons Fractals. (2016),89:447−54.

[13] K.D. Kucche and S.T. Sutar, Analysis of nonlinear fractional differential equations involving Atangana-Baleanu-Caputo derivative, Chaos, Solitons Fractals, Volume 143, 2021, 110556.

[14] A. Kumar and D. Pandey, Existence of mild solutions of Atangana-Baleanu fractional differential equations with noninstaneous impulses and with non-local conditions, Chaos Solitions Fractals, 132 (2020), 109551

[15] A. Kilbas, H. Srivastava and J. Trujillio, Theory and application of fractional differential equations, North Holland Mathematics Studies, 2006.

[16] M. MalikkaArjunan, A. Hamiaz and V. Kavitha, Existence results for Atangana-Baleanu fractional neutral integro-differential systems with infinite delay through sectorial operators, Chaos, Solitons Fractals. 149, 2021, 111042.

[17] S. Mekki, J.J. Nieto, A. Ouahab, Stochastic version of Henry type Gronwalls inequality, Infinite Dimensional Analysis, Quantum Probability and Related Topics. 2021, 24(02), 2150013.

[18] K.M. Owolabi and A. Atangana, On the formulation of Adams-Bashforth scheme with Atangana-Baleanu-Caputo fractional, derivative to model chaotic problems. Chaos Solitions Fractals. 2019;29(2):1−12.

[19] K.M. Owolabi, Modelling and simulation of a dynamical system with the Atangana-Baleanu fractional, derivative, Eur. Phys. J. Plus. 2018;133(1):1−13.

[20] S.K. Panda, T. Abdeljawad and C. Ravichandran, Novel fixed point approach to Atangana-Baleanu fractional and Lp-Fredholm integral equations, Alexandria Engineering Journal. (2020) 59,1959−1970.

[21] I. Petras, Handbook of Fractional Calculus with Applications, Applications in Control, De Gruyter.. Vol. 6:2019.

[22] C. Ravichandran, K. Logeswari and F. Jarad, New results on existence in the framework of Atangana-Baleanu derivative for fractional integro-differential equations, Chaos Solitions Fractals. 125(2019), 194−200.

[23] M. Syam and M. Al-Refai, Fractional differential equations with Atangana-Baleanu fractional drivative: Analysis and Applications, Chaos Solitions Fractals:X. (2019), 100013

[24] S. Ucar, E. Ucar, N. Ozdemir and Z. Hammouch, Mathematical analysis and numerical simulation for a smoking model with Atangana-Baleanu derivative, Chaos Solitons Fractals, (2019), 118:300−6 .

[25] Vanterler da J., C. Sousa and E.C. De Oliveira, A Gronwall inequality and the Cauchy type problem by means of $\psi$-Hilfer Operator, Differ. Equ. Appl., 11(1), (2019), 87−106.

[26] H. Vu, B. Ghanbari and N.V. Hoa, Fuzzy fractional differential equations with the generalized Atangana-Baleanu fractional derivative, Fuzzy Sets and Systems , 2020

[27] H. Ye, J. Gao and Y. Ding, A generalized Gronwall inequality and its application to fractional differential equation, J. Math. Anal. Appl., 328(2), (2007), 1075−1081.