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Quasi Bi-Slant Submanifolds of Kaehler Manifolds

Rajendra Prasad, Mehmet Akif Akyol*, Sandeep Kumar Verma and Sumeet Kumar

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ABSTRACT

In this paper, we introduce a new notion which is called quasi bi-slant submanifolds of almost Hermitian manifolds. Necessary and sufficient conditions for the integrability of distributions which are involved in the definition of such submanifolds of a Kaehler manifold are obtained. We also investigate the necessary and sufficient conditions for these submanifolds of Kaehler manifolds to be totally geodesic and study the geometry of foliations determined by the above distributions. Finally, we obtain the necessary and sufficient conditions for a quasi bi-slant submanifold to be local product Riemannian manifold and also construct some examples of such submanifolds.

Keywords: *Quasi bi-slant submanifold, Kaehler manifold, totally geodesic, distribution.*
AMS Subject Classification (2020): *Primary: 53C15; Secondary: 53C40; 53C42.*

1. Introduction

The theory of submanifolds has several important applications in Mathematics, Physics as well as in Mechanics. In the last two decades, the applications of Kaehler manifold is widely recognized (especially in Physics, for the target spaces of non-linear $\sigma-$ models with supersymmetry), see [10]. Nowadays theory of submanifolds plays a key role in computer designing, image processing, economic modelling. It has origin in the study of the geometry of plane curves initiated by Fermat. The study and research work of slant submanifolds in almost Hermitian manifolds begins with the remarkable work of B. Y. Chen ([8], [9]) as a generalization of complex (holomorphic) and totally real submanifolds. Later on, as a natural generalization of CR-submanifold, slant submanifold, holomorphic submanifold and totally real submanifold in almost Hermitian manifold was defined by N. Papaguice [15], which is known as semi-slant submanifold. The theory of slant submanifolds has been studied by several geometers ([3], [5], [6], [7], [12], [13], [14], [15], [20], [21]). Further the notion of slant submanifold generalized as semi-slant submanifolds, pseudo-slant submanifolds, bi-slant submanifolds, etc.

Bi-slant submanifolds of almost contact metric manifolds were studied by J. L. Cabrerizo et al in [7]. B. Y. Chen et al. in [19] investigated bi-slant submanifolds in Kaehler manifolds. The purpose of the present paper is to introduce the notion of quasi bi-slant submanifolds of almost Hermitian manifolds which includes the classes of slant submanifolds, semi-slant submanifolds, hemi-slant submanifolds and bi-slant submanifolds as its particular cases (see also: [1, 16]). Primarily the hemi-slant submanifolds were known as anti-slant submanifolds. Later, B. Şahin [17] named these submanifolds as hemi-slant submanifolds. Hemi-slant submanifolds are one of the classes of bi-slant submanifolds. In 2011, F.R. Solamy et al. [2] obtained some interesting results on totally umbilical hemi-slant submanifolds of Kaehler manifolds. The paper is organized as follows: In section 2, we mention the basic definitions and properties of almost complex manifolds. In section 3, we define quasi bi-slant submanifolds and some basic properties of submanifolds. Section 4 deals with necessary and sufficient conditions for integrability of distributions. In this section, we also find necessary

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*** Corresponding author***

and sufficient conditions for the submanifolds to be totally geodesic. In the last section, we construct some examples of such submanifolds.

2. Preliminaries

Let N be a Riemannian manifold with an almost complex structure J and Hermitian metric g satisfying

$$J^2 = -I \tag{2.1}$$

and

$$g(JX, JY) = g(X, Y), \tag{2.2}$$

for any $X, Y \in \Gamma(TN)$, where $\Gamma(TN)$ is the Lie algebra of vector fields in N, then N is called an almost Hermitian manifold. If the almost complex structure J also satisfies

$$(\bar{\nabla}_X J)Y = 0, \tag{2.3}$$

for every $X, Y \in \Gamma(TN)$, where $\bar{\nabla}$ is the Levi-Civita connection on N, then N is said to be a Kaehler manifold [11, 22].

The covariant derivative of the complex structure J is defined as

$$(\bar{\nabla}_X J)Y = \bar{\nabla}_X JY - J\bar{\nabla}_X Y,$$

using (2.3) , we have

$$J\bar{\nabla}_X Y = \bar{\nabla}_X JY. \tag{2.4}$$

Throughout this paper, A and h denote the shape operator and second fundamental form of submanifold M into N respectively. If ∇ is the induced Riemannian connection on M, then the Gauss and Weingarten formulae are given by [8]

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.5}$$

and

$$\bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{2.6}$$

for all vector fields $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where $\nabla^{\perp}$ denotes the connection on the normal bundle $(T^{\perp}M)$ of M. The shape operator and the second fundamental form are related by

$$g(A_V X, Y) = g(h(X, Y), V). \tag{2.7}$$

The mean curvature vector is defined by

$$H = \frac{1}{n} trace(h) = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i), \tag{2.8}$$

where n denotes the dimension of submanifold M and $\{e_1, e_2, ..., e_n\}$ is the local orthonormal basis of tangent space at each point of M.

A submanifold M of Kaehler manifold N is said to be totally umbilical if

$$h(X, Y) = g(X, Y)H, \tag{2.9}$$

where H is the mean curvature vector. If $h(X, Y) = 0$ for every $X, Y \in \Gamma(TM)$, then M is said to be totally geodesic and if $H = 0$, then M is said to be a minimal submanifold.

For any $X \in \Gamma(TM)$, we can write

$$JX = \phi X + \omega X, \tag{2.10}$$

where ϕX and ωX are the tangential and normal components of JX on M respectively. Similarly for any $V \in \Gamma(T^{\perp}M)$, we have

$$JV = BV + CV, \tag{2.11}$$

where BV and CV are the tangential and normal components of JV on M respectively.

The covariant derivative of projection morphisms in (2.10) and (2.11) are defined as

$$(\bar{\nabla}_X \phi)Y = \nabla_X \phi Y - \phi \nabla_X Y, \tag{2.12}$$

$$(\bar{\nabla}_X \omega)Y = \nabla^{\perp}_X \omega Y - \omega \nabla_X Y, \tag{2.13}$$

$$(\bar{\nabla}_X B)V = \nabla_X BV - B\nabla^{\perp}_X V \tag{2.14}$$

and

$$(\bar{\nabla}_X C)V = \nabla^{\perp}_X CV - C\nabla^{\perp}_X V \tag{2.15}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

Now we recall the following definitions for later use:

Definition 2.1. Let M be a Riemannian manifold isometrically immersed in an almost Hermitian manifold N. A submanifold M of an almost Hermitian manifold N is said to be invariant (holomorphic or complex) [5] if $J\,(T_xM) \subseteq T_xM$, for every point $x \in M$.

Definition 2.2. A submanifold M of an almost Hermitian manifold N is said to be anti-invariant (totally really) [13] if $J\,(T_xM) \subseteq T_x^{\perp}M$, for every point $x \in M$.

Definition 2.3. A submanifold M of an almost Hermitian manifold N is said to be slant [14], if for each non-zero vector X tangent to M at $x \in M$, the angle $\theta(X)$ between JX and T_xM is constant, i.e., it does not depends on the choice of the point $x \in M$ and $X \in T_xM$. In this case, the angle θ is called the slant angle of the submanifold. A slant submanifold M is called proper slant submanifold if neither $\theta = 0$ nor $\theta = \frac{\pi}{2}$.

We note that on a slant submanifold M if $\theta = 0$, then it is an invariant submanifold and if $\theta = \frac{\pi}{2}$, then it is an anti-invariant submanifold. This means slant submanifold is a generalization of invariant and anti-invariant submanifolds.

Definition 2.4. A submanifold M of an almost Hermitian manifold N is said to be semi-invariant [4], if there exist two orthogonal complementary distributions D and $D^{\perp}$ on M such that

$$TM = D \oplus D^{\perp},$$

where D is invariant and $D^{\perp}$ is anti-invariant.

Definition 2.5. A submanifold M of an almost Hermitian manifold N is said to be semi-slant [15], if there exist two orthogonal complementary distributions D and D_θ on M such that

$$TM = D \oplus D_\theta,$$

where D is invariant and D_θ is slant with slant angle θ. In this case, the angle θ is called semi-slant angle.

Definition 2.6. A submanifold M of an almost Hermitian manifold N is said to be hemi-slant [18, 2], if there exist two orthogonal complementary distributions D_θ and $D^{\perp}$ on M such that

$$TM = D_\theta \oplus D^{\perp},$$

where D_θ is slant with slant angle θ and $D^{\perp}$ is anti-invariant. In this case, the angle θ is called hemi-slant angle.

Definition 2.7. A submanifold M of an almost Hermitian manifold N is said to be bi-slant [19], if there exist two orthogonal complementary distributions D_{θ_1} and D_{θ_2} on M such that

$$TM = D_{\theta_1} \oplus D_{\theta_2},$$

where D_{θ_1} and D_{θ_2} are slants with slant angles θ_1 and θ_2 respectively.

Now, we shall introduce the notion of quasi bi-slant submanifolds of almost Hermitian manifolds:

3. Quasi Bi-Slant Submanifolds

In the present section of the paper, we introduce quasi bi-slant submanifolds of almost Hermitian manifolds and obtain the necessary and sufficient conditions for the distributions involved in the definition of such submanifolds to be integrable.

Definition 3.1. A submanifold M of an almost Hermitian manifold N is called a quasi bi-slant submanifold if there exist distributions D, D_1 and D_2 such that:

(i) TM admits the orthogonal direct decomposition as

$$TM = D \oplus D_1 \oplus D_2,$$

(ii) $J(D) = D$ i.e., D is invariant,

(iii) $J(D_1) \perp D_2$,

(iv) For any non-zero vector field $X \in (D_1)_p$, $p \in M$, the angle θ_1 between JX and $(D_1)_p$ is constant and independent of the choice of point p and X in $(D_1)_p$,

(v) For any non-zero vector field $Z \in (D_2)_q$, $q \in M$, the angle θ_2 between JZ and $(D_2)_q$ is constant and independent of the choice of point q and Z in $(D_2)_q$,

These angles θ_1 and θ_2 are called slant angles of the submanifold.
We easily observe that
(a) If $\dim D \neq 0, \dim D_1 = 0$ and $\dim D_2 = 0$, then M is an invariant submanifold.
(b) If $\dim D \neq 0, \dim D_1 \neq 0, 0 < \theta_1 < \frac{\pi}{2}$ and $\dim D_2 = 0$, then M is proper semi-slant submanifold.
(c) If $\dim D = 0$, $\dim D_1 \neq 0, 0 < \theta_1 < \frac{\pi}{2}$ and $\dim D_2 = 0$, then M is slant submanifold with slant angle θ_1.
(d) If $\dim D = 0, \dim D_1 = 0$ and $\dim D_2 \neq 0$, $0 < \theta_2 < \frac{\pi}{2}$, then M is slant submanifold with slant angle θ_2.
(e) If $\dim D = 0, \dim D_1 \neq 0$, $\theta_1 = \frac{\pi}{2}$ and $\dim D_2 = 0$, then M is an anti-invariant submanifold.
(f) If $\dim D \neq 0, \dim D_1 \neq 0$, $\theta_1 = \frac{\pi}{2}$ and $\dim D_2 = 0$, then M is semi-invariant submanifold.
(g) If $\dim D = 0, \dim D_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $\dim D_2 \neq 0$, $\theta_2 = \frac{\pi}{2}$, then M is hemi-slant submanifold.
(h) If $\dim D = 0, \dim D_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $\dim D_2 \neq 0$, $0 < \theta_2 < \frac{\pi}{2}$, then M is bi-slant submanifold.
(i) If $\dim D \neq 0, \dim D_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $\dim D_2 \neq 0$, $\theta_2 = \frac{\pi}{2}$, then we may call M is quasi hemi-slant submanifold.
(j) If $\dim D \neq 0$, and $0 < \theta_1 = \theta_2 < \frac{\pi}{2}$, then M is proper semi-slant submanifold.
(k) If $\dim D \neq 0, \dim D_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $\dim D_2 \neq 0$, $0 < \theta_2 < \frac{\pi}{2}$, then M is proper quasi bi-slant submanifold.
This means notion of quasi bi-slant submanifold is a generalization of invariant, anti-invariant, slant, hemi-slant and semi-slant submanifolds.

Remark 3.1. Above definition can be generalized by taking $TM = D \oplus D_{\theta_1} \oplus D_{\theta_2} ... \oplus D_{\theta_k}$. Hence we can define multi-slant submanifolds, quasi multi-slant submanifolds etc.

Let M be a quasi bi-slant submanifold of an almost Hermitian manifold N. We denote the projections of $X \in \Gamma(TM)$ on the distributions D, D_1 and D_2 by P, Q and R, respectively. Then we can write, for any $X \in \Gamma(TM)$

$$X = PX + QX + RX. \tag{3.1}$$

Now, put

$$JX = \phi X + \omega X, \tag{3.2}$$

where ϕX and ωX are tangential and normal components of JX on M, respectively.
Using (3.1) and (3.2), we obtain

$$\begin{aligned} JX &= JPX + JQX + JRX \\ &= \phi PX + \omega PX + \phi QX + \omega QX + \phi RX + \omega RX. \end{aligned} \tag{3.3}$$

Since $JD = D$, we have $\omega PX = 0$. Therefore, we get

$$JX = \phi PX + \phi QX + \omega QX + \phi RX + \omega RX. \tag{3.4}$$

This means, for any $X \in \Gamma(TM)$, we have

$$\begin{aligned} \phi X &= \phi PX + \phi QX + \phi RX \text{ and} \\ \omega X &= \omega QX + \omega RX. \end{aligned}$$

Thus, we have the following decomposition

$$J(TM) \subset D \oplus \phi D_1 \oplus \omega D_1 \oplus \phi D_2 \oplus \omega D_2. \tag{3.5}$$

Since $\omega D_1 \subset (T^\perp M)$ and $\omega D_2 \subset (T^\perp M)$, we have

$$T^\perp M = \omega D_1 \oplus \omega D_2 \oplus \mu, \tag{3.6}$$

where μ is the orthogonal complement of $\omega D_1 \oplus \omega D_2$ in $(T^\perp M)$ and it is invariant with respect to J.
For any $Z \in \Gamma(T^\perp M)$, we put

$$JZ = BZ + CZ, \tag{3.7}$$

where $BZ \in \Gamma(TM)$ and $CZ \in \Gamma(T^\perp M)$.
Followings are some easy observations which we write for later use:

$$(a)\ \phi D \subseteq D, \quad (b)\ \omega D = \{0\}, \quad (c)\ \phi D_i \subseteq D_i \text{ for } i = 1, 2, \quad (d)\ B\left(T^\perp M\right) \subset D_1 \oplus D_2.$$

There are three other important classes of submanifolds of an almost Hermitian manifold determined by the behavior of the tangent bundle of the submanifold under the action of almost complex structure of the ambient manifold. A distribution D on a manifold $\bar{M}$ is called auto parallel if $\bar{\nabla}_X Y \in D$ for any $X, Y \in D$ and is called parallel if $\bar{\nabla}_U X \in D$ for any $X \in D$ and $U \in T\bar{M}$. If a distribution D on $\bar{M}$ is auto parallel, then it is clearly integrable and by Gauss formula D is totally geodesic in $\bar{M}$. If D is parallel then the orthogonal complementary distribution $D^\perp$ is also parallel which implies that D is parallel if and only if $D^\perp$ is parallel. In this case $\bar{M}$ is locally product of the leaves of D and $D^\perp$. Let M be a submanifold of $\bar{M}$. For two distributions D_1 and D_2 on M, we say that M is $(D1; D2)$ mixed totally geodesic if for all $X \in D_1$ and $Y \in D_2$, we have $h(X, Y) = 0$, where h is the second fundamental form of M.

Lemma 3.1. *Let M be a quasi bi-slant submanifold of an almost Hermitian manifold N. Then the endomorphism ϕ and projection morphisms ω, B and C on the tangent bundle of M, satisfy the following identities:*

(i) $\phi^2 + B\omega = -I$ *on* TM,

(ii) $\omega\phi + C\omega = 0$ *on* TM,

(iii) $\omega B + C^2 = -I$ *on* $(T^\perp M)$,

(iv) $\phi B + BC = 0$ *on* $(T^\perp M)$,

where I is the identity operator.

Proof. From equations (3.2) , (3.7) and using the fact that $J^2 = -I$, then on comparing tangential and normal components, one can easily get these assertions. □

Lemma 3.2. *Let M be a quasi bi-slant submanifold of an almost Hermitian manifold N. Then*

(i) $\phi^2 X = -(\cos^2\theta_1)X$,

(ii) $g(\phi X, \phi Y) = (\cos^2\theta_1)g(X, Y)$,

(iii) $g(\omega X, \omega Y) = (\sin^2\theta_1)g(X, Y)$

for any $X, Y \in \Gamma(D_1)$, where θ_1 denotes the slant angle of D_1.

Proof. (i) For any non-zero $X \in \Gamma(D_1)$, we have

$$\cos\theta_1 = \frac{g(JX, \phi X)}{\|JX\| \cdot \|\phi X\|} = \frac{-g(X, \phi^2 X)}{\|X\| \cdot \|\phi X\|}$$

and $\cos\theta_1 = \frac{\|\phi X\|}{\|JX\|}$, we have

$$\cos^2\theta_1 = \frac{-g(X,\phi^2 X)}{\|X\|^2}$$

which implies

$$\cos^2\theta_1 g(X,X) = -g(X,\phi^2 X).$$

By polarization,

$$\phi^2 X = -(\cos^2\theta_1)X, \text{ for } X \in \Gamma(D_1).$$

(ii) For any $X, Y \in \Gamma(D_1)$, using equations (2.2), (3.2) and Lemma 2 (i), we have

$$\begin{aligned} g(\phi X, \phi Y) &= g(JX - \omega X, \phi Y) \\ &= -g(X, \phi^2 Y) \\ &= (\cos^2\theta_1) g(X,Y). \end{aligned}$$

(iii) Using equations (2.2), (3.2) and Lemma 2 (ii), we have Lemma 2(iii). □

In a similar way as in above, we obtain the following Lemma:

Lemma 3.3. *Let M be a quasi bi-slant submanifold of an almost Hermitian manifold N. Then*

(i) $\phi^2 Z = -(\cos^2\theta_2)Z$,

(ii) $g(\phi Z, \phi W) = (\cos^2\theta_2) g(Z,W)$,

(iii) $g(\omega Z, \omega W) = (\sin^2\theta_2) g(Z,W)$

for any $Z, W \in \Gamma(D_2)$, where θ_2 denotes the slant angle of D_2.

Using equations (2.3), (2.5), (2.6), (2.10) and (2.11), and then on comparing tangential and normal components, we have following:

Lemma 3.4. *Let M be a submanifold of a Kaehler manifold N, then for any $X, Y \in \Gamma(TM)$, we have*

$$\nabla_X \phi Y - A_{\omega Y} X - \phi \nabla_X Y - Bh(X,Y) = 0$$

and

$$h(X, \phi Y) + \nabla^\perp_X \omega Y - \omega(\nabla_X Y) - Ch(X,Y) = 0.$$

Using equations (2.12) and (2.13) in above Lemma, we have the following:

Lemma 3.5. *Let M be a quasi bi-slant submanifold of a Kaehler manifold N. Then*

$$(\bar\nabla_X \phi)Y = A_{\omega Y} X + Bh(X,Y),$$

$$(\bar\nabla_X \omega)Y = Ch(X,Y) - h(X, \phi Y)$$

for any $X, Y \in \Gamma(TM)$.

4. Integrability of distributions and decomposition theorems

In this section we investigate the integrability conditions for the distributions involved in the definition of quasi bi-slant submanifolds.

Theorem 4.1. *Let M be a proper quasi bi-slant submanifold of a Kaehler manifold N. Then the invariant distribution D is integrable if and only if*

$$g(\nabla_Z \phi W - \nabla_W \phi Z, \phi QX + \phi RX) = g(h(W, \phi Z) - h(Z, \phi W), \omega QX + \omega RX), \tag{4.1}$$

for any $Z, W \in \Gamma(D)$ and $X \in \Gamma(D_1 \oplus D_2)$.

Proof. For any $Z, W \in \Gamma(D)$ and $X = QX + RX \in \Gamma(D_1 \oplus D_2)$, using (2.2) , (2.4) , (2.5) and (3.2), we have

$$\begin{aligned} g([Z,W],X) &= g(\bar{\nabla}_Z\, \phi W, JX) - g(\bar{\nabla}_W\, \phi Z, JX), \\ &= g(\nabla_Z\, \phi W - \nabla_W\, \phi Z, \phi QX + \phi RX) \\ &\quad + g(h(Z,\phi W) - h(W,\phi Z), \omega QX + \omega RX). \end{aligned}$$

This completes the proof. □

Theorem 4.2. *Let M be a proper quasi bi-slant submanifold of a Kaehler manifold N. Then the slant distribution D_1 is integrable if and only if*

$$g(A_{\omega W}Z - A_{\omega Z}W, \phi X) = g(A_{\omega\phi W}Z - A_{\omega\phi Z}W, X) + g(\nabla^{\perp}_Z \omega W - \nabla^{\perp}_W \omega Z, \omega RX), \tag{4.2}$$

for any $Z, W \in \Gamma(D_1)$ and $X \in \Gamma(D \oplus D_2)$.

Proof. For any $Z, W \in \Gamma(D_1)$ and $X = PX + RX \in \Gamma(D \oplus D_2)$, using (2.2), (2.4) and (3.2), we obtain

$$\begin{aligned} g([Z,W],X) &= g(\bar{\nabla}_Z \omega W, JX) - g(\bar{\nabla}_Z J\phi W, X) \\ &\quad - g(\bar{\nabla}_W \omega Z, JX) + g(\bar{\nabla}_W J\phi Z, X). \end{aligned}$$

Then from (2.5), (2.6) and (3.4) and using Lemma 2, we have

$$\begin{aligned} g([Z,W],X) &= -g(A_{\omega W}Z - A_{\omega Z}W, JX) + \cos^2\theta_1 g([Z,W],X) \\ &\quad + g(A_{\omega\phi W}Z - A_{\omega\phi Z}W, X) + g(\nabla^{\perp}_Z \omega W - \nabla^{\perp}_W \omega Z, JX), \end{aligned}$$

which leads to

$$\begin{aligned} \sin^2\theta_1 g([Z,W],X) &= g(A_{\omega\phi W}Z - A_{\omega\phi Z}W, X) + g(\nabla^{\perp}_Z \omega W - \nabla^{\perp}_W \omega Z, \omega RX) \\ &\quad - g(A_{\omega W}Z - A_{\omega Z}W, \phi PX + \phi RX). \end{aligned}$$

Thus the proof follows. □

From the above Theorem, we have the following sufficient conditions for the slant distribution D_1 to be integrable:

Corollary 4.1. *Let M be a proper quasi bi-slant submanifold of a Kaehler manifold N. If*

$$\begin{aligned} \nabla^{\perp}_Z \omega W - \nabla^{\perp}_W \omega Z &\in \omega D_1 \oplus \mu, \\ A_{\omega\phi W}Z - A_{\omega\phi Z}W &\in D_1 \quad \textit{and} \\ A_{\omega W}Z - A_{\omega Z}W &\in D_1, \end{aligned} \tag{4.3}$$

for any $Z, W \in \Gamma(D_1)$, then the slant distribution D_1 is integrable.

In a similar way to Theorem 2, we can conclude the following:

Theorem 4.3. *Let M be a proper quasi bi-slant submanifold of a Kaehler manifold N. Then the slant distribution D_2 is integrable if and only if*

$$g(A_{\omega W}Z - A_{\omega Z}W, \phi X) = g(A_{\omega\phi W}Z - A_{\omega\phi Z}W, X) + g(\nabla^{\perp}_Z \omega W - \nabla^{\perp}_W \omega Z, \omega QX),$$

for any $Z, W \in \Gamma(D_2)$ and $X \in \Gamma(D \oplus D_1)$.

From the above Theorem, we have the following sufficient conditions for the slant distribution D_2 to be integrable:

Corollary 4.2. *Let M be a proper quasi bi-slant submanifold of a Kaehler manifold N if*

$$\begin{aligned} \nabla^{\perp}_Z \omega W - \nabla^{\perp}_W \omega Z &\in \omega D_2 \oplus \mu, \\ A_{\omega\phi W}Z - A_{\omega\phi Z}W &\in D_2 \quad \textit{and} \\ A_{\omega W}Z - A_{\omega Z}W &\in D_2, \end{aligned} \tag{4.4}$$

for any $Z, W \in \Gamma(D_2)$, then the slant distribution D_2 is integrable.

Now, we obtain a necessary and sufficient condition for a quasi bi-slant submanifold to be totally geodesic.

Theorem 4.4. *Let M be a proper quasi bi-slant submanifold of a Kaehler manifold N. Then M is totally geodesic if and only if*

$$\begin{aligned} g(h(X,PY)+\cos^2\theta_1 h(X,QY)+\cos^2\theta_2 h(X,RY),U) &= g(\nabla^{\perp}_X\omega\phi QY+\nabla^{\perp}_X\omega\phi RY,U)\\ &+g(A_{\omega Y}X,BU)-g(\nabla^{\perp}_X\omega Y,CU) \end{aligned} \tag{4.5}$$

for any $X,Y\in\Gamma(TM)$ and $U\in\Gamma(T^{\perp}M)$.

Proof. For any $X,Y\in\Gamma(TM)$ and $U\in\Gamma(T^{\perp}M)$, using (2.2), (2.4) , (3.1) and (3.2), we obtain

$$\begin{aligned} g(\bar\nabla_X Y,U) &= g(\bar\nabla_X PY,U)+g(\bar\nabla_X QY,U)+g(\bar\nabla_X RY,U)\\ &= g(\bar\nabla_X JPY,JU)+g(\bar\nabla_X\phi QY,JU)+g(\bar\nabla_X\omega QY,JU)\\ &\quad +g(\bar\nabla_X\phi RY,JU)+g(\bar\nabla_X\omega RY,JU). \end{aligned}$$

Using (2.2), (2.5), (2.6), Lemma 2 and Lemma 3, we have

$$\begin{aligned} g(\bar\nabla_X Y,U) &= g(\bar\nabla_X PY,U)-g(\bar\nabla_X\phi^2 QY,U)-g(\bar\nabla_X\omega\phi QY,U)\\ &\quad +g(\bar\nabla_X\omega QY,JU)-g(\bar\nabla_X\phi^2 RY,U)\\ &\quad -g(\bar\nabla_X\omega\phi RY,U)+g(\bar\nabla_X\omega RY,JU)\\ &= g(h(X,PY),U)+\cos^2\theta_1 g(h(X,QY),U)+\cos^2\theta_2 g(h(X,RY),U)\\ &\quad -g(\nabla^{\perp}_X\omega\phi QY,U)-g(\nabla^{\perp}_X\omega\phi RY,U)\\ &\quad +g(-A_{\omega QY}X+\nabla^{\perp}_X\omega QY,JU)+g(-A_{\omega RY}X+\nabla^{\perp}_X\omega RY,JU). \end{aligned}$$

Since $\omega Y=\omega QY+\omega RY$, we have

$$\begin{aligned} g(\bar\nabla_X Y,U) &= g(h(X,PY)+\cos^2\theta_1 h(X,QY)+\cos^2\theta_2 h(X,RY),U)\\ &\quad -g(\nabla^{\perp}_X\omega\phi QY,U)-g(\nabla^{\perp}_X\omega\phi RY,U)\\ &\quad -g(A_{\omega Y}X,BU)+g(\nabla^{\perp}_X\omega Y,CU). \end{aligned} \tag{4.6}$$

Hence the proof follows. □

Now, we investigate the geometry of leaves of foliations determined by above distributions.

Theorem 4.5. *Let M be a proper quasi bi-slant submanifold of a Kaehler manifold N. Then the invariant distribution D defines a totally geodesic foliation on M if and only if*

$$\begin{aligned} g(\nabla_X\phi Y,\phi Z) &= -g(h(X,\phi Y),\omega Z)\\ g(\nabla_X\phi Y,B\xi) &= -g(h(X,\phi Y),C\xi), \end{aligned} \tag{4.7}$$

for any $X,Y\in\Gamma(D)$, $Z\in\Gamma(D_1\oplus D_2)$ and $\xi\in\Gamma(T^{\perp}M)$.

Proof. For any $X,Y\in\Gamma(D)$, $Z=QZ+RZ\in\Gamma(D_1\oplus D_2)$ and using (2.2), (2.4) , (3.2) and $\omega Y=0$, we have

$$\begin{aligned} g(\bar\nabla_X Y,Z) &= g(\bar\nabla_X\phi Y,JZ),\\ &= g(\nabla_X\phi Y,\phi Z)+g(h(X,\phi Y),\omega Z). \end{aligned}$$

Now, for any $\xi\in\Gamma(T^{\perp}M)$ and $X,Y\in\Gamma(D)$, we have

$$\begin{aligned} g(\bar\nabla_X Y,\xi) &= g(\bar\nabla_X\phi Y,J\xi)\\ &= g(\nabla_X\phi Y,B\xi)+g(h(X,\phi Y),C\xi). \end{aligned}$$

Hence the proof follows. □

Theorem 4.6. *Let M be a proper quasi bi-slant submanifold of a Kaehler manifold N. Then the slant distribution D_1 defines a totally geodesic foliation on M if and only if*

$$g(\nabla^{\perp}_X \omega Y, \omega RZ) = g(A_{\omega Y}X, \phi Z) - g(A_{\omega\phi Y}X, Z) \tag{4.8}$$
$$g(A_{\omega Y}X, B\xi) = g(\nabla^{\perp}_X \omega Y, C\xi) - g(\nabla^{\perp}_X \omega\phi Y, \xi), \tag{4.9}$$

for any $X, Y \in \Gamma(D_1)$, $Z \in \Gamma(D \oplus D_2)$ and $\xi \in \Gamma(T^{\perp}M)$.

Proof. For any $X, Y \in \Gamma(D_1)$, $Z = PZ + RZ \in \Gamma(D \oplus D_2)$ and using (2.2), (2.4) and (3.2), we have

$$\begin{aligned} g(\bar\nabla_X Y, Z) &= g(\bar\nabla_X JY, JZ) = g(\bar\nabla_X \phi Y, JZ) + g(\bar\nabla_X \omega Y, JZ) \\ &= -g(\bar\nabla_X \phi^2 Y, Z) - g(\bar\nabla_X \omega\phi Y, Z) + g(\bar\nabla_X \omega Y, \phi PZ + \phi RZ + \omega RZ). \end{aligned}$$

Then using (3.1), (2.6), lemma 2 and the fact that $\phi PZ + \phi RZ = \phi Z$, $\omega PZ = 0$, we have

$$\begin{aligned} g(\bar\nabla_X Y, Z) &= \cos^2\theta_1 g(\bar\nabla_X Y, Z) + g(A_{\omega\phi Y}X, Z) \\ &\quad - g(A_{\omega Y}X, \phi PZ + \phi RZ) + g(\nabla^{\perp}_X \omega Y, \omega RZ), \\ \sin^2\theta_1 g(\bar\nabla_X Y, Z) &= g(A_{\omega\phi Y}X, Z) + g(\nabla^{\perp}_X \omega Y, \omega RZ) - g(A_{\omega Y}X, \phi Z). \end{aligned} \tag{4.10}$$

Similarly, we get

$$\sin^2\theta_1 g(\bar\nabla_X Y, \xi) = -g(\nabla^{\perp}_X \omega\phi Y, \xi) - g(A_{\omega Y}X, B\xi) + g(\nabla^{\perp}_X \omega Y, C\xi). \tag{4.11}$$

Thus from (4.10) and (4.11), we have the assertions. □

In a similar way to the above theorem, we can conclude the following:

Theorem 4.7. *Let M be a proper quasi bi-slant submanifold of a Kaehler manifold N. Then the slant distribution D_2 defines a totally geodesic foliation on M if and only if*

$$g(\nabla^{\perp}_X \omega Y, \omega QZ) = g(A_{\omega Y}X, \phi Z) - g(A_{\omega\phi Y}X, Z) \tag{4.12}$$
$$g(A_{\omega Y}X, B\xi) = g(\nabla^{\perp}_X \omega Y, C\xi) - g(\nabla^{\perp}_X \omega\phi Y, \xi),$$

for any X, $Y \in \Gamma(D_2)$, $Z \in \Gamma(D \oplus D_1)$ and $\xi \in \Gamma(T^{\perp}M)$.

5. Examples

Example 5.1. Consider a $14-$dimensional differentiable manifold $\overline{M} = \mathbb{R}^{14}$

$$\overline{M} = \{(x_i, y_i) = (x_1, x_2, ..., x_7, y_1, y_2, ..., y_7) \in \mathbb{R}^{14};\ \ i = 1, 2, ..., 7.\}.$$

We choose the vector fields

$$E_i = \frac{\partial}{\partial y_i},\ \ E_{7+i} = \frac{\partial}{\partial x_i},\ \ \text{for}\ \ i = 1, 2, ..., 7.$$

Let g be a Hermitian metric defined by

$$g = (dx_1)^2 + (dx_2)^2 + ... + (dx_7)^2 + (dy_1)^2 + (dy_2)^2 + ... + (dy_7)^2.$$

Here $\{E_1, E_2, ..., E_{14}\}$ forms an orthonormal basis. We define $(1,1)$-tensor field J as

$$J\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i},\ \ J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j},\ \ \ \forall\, i, j = 1, 2, ..., 7.$$

By using linearity of J and g, we have

$$J^2 = -I,$$
$$g(JX, JY) = g(X, Y),\ \ \text{for any}\ X, Y \in \Gamma(T\overline{M})$$

We can easily show that $(\overline{M}, J, g)$ is a Kaehler manifold of dimension 14.

Now, we consider a submanifold M of $\overline{M}$ defined by immersion f as follows:

$$f(u, v, w, r, s, t) = (u, w, 0, s, 0, 0, 0, v, r\cos\theta_1, r\sin\theta_1, t\cos\theta_2, 0, 0, t\sin\theta_2)$$

with $\{\theta_1, \theta_2\} \subset (0, \frac{\pi}{2})$.

By direct computation, it is easy to check that the tangent bundle of M is spanned by a linearly independent set $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6\}$, where

$$Z_1 = \frac{\partial}{\partial x_1}, \;\; Z_2 = \frac{\partial}{\partial y_1}, \;\; Z_3 = \frac{\partial}{\partial x_2},$$
$$Z_4 = \cos\theta_1 \frac{\partial}{\partial y_2} + \sin\theta_1 \frac{\partial}{\partial y_3}, \;\; Z_5 = \frac{\partial}{\partial x_4},$$
$$Z_6 = \cos\theta_2 \frac{\partial}{\partial y_4} + \sin\theta_2 \frac{\partial}{\partial y_7}.$$

Then using almost complex structure of $\overline{M}$, we have

$$JZ_1 = \frac{\partial}{\partial y_1}, \;\; JZ_2 = -\frac{\partial}{\partial x_1}, \;\; JZ_3 = \frac{\partial}{\partial y_2},$$
$$JZ_4 = -\left(\cos\theta_1 \frac{\partial}{\partial x_2} + \sin\theta_1 \frac{\partial}{\partial x_3}\right), \;\; JZ_5 = \frac{\partial}{\partial y_4},$$
$$JZ_6 = -\left(\cos\theta_2 \frac{\partial}{\partial x_4} + \sin\theta_2 \frac{\partial}{\partial x_7}\right).$$

Now, let the distributions $D = Span\{Z_1, Z_2\}$, $D_1 = Span\{Z_3, Z_4\}$, $D_2 = Span\{Z_5, Z_6\}$.

Then it is easy to see that D is invariant, D_1 and D_2 are slant distributions with slant angles θ_1 and θ_2 respectively.

Hence f defines a proper 6-dimensional quasi bi-slant submanifold M in $\overline{M}$.

Example 5.2. Consider a submanifold N of Kaehler manifold $\overline{M}$ (see example 1) defined by immersion ψ as follows:

$$\psi(u, v, w, r, s, t) = \left(\frac{u}{\sqrt{2}}, w, 0, \sqrt{3}s, 0, 0, \frac{u}{\sqrt{2}}, \frac{v}{\sqrt{2}}, r, r, t, s, 0, \frac{v}{\sqrt{2}}\right)$$

By direct computation, it is easy to check that the tangent bundle of N is spanned by a linearly independent set $\{X_1, X_2, X_3, X_4, X_5, X_6\}$, where

$$X_1 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_7}\right), \;\; X_2 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_7}\right),$$
$$X_3 = \frac{\partial}{\partial x_2}, \;\; X_4 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_3}\right),$$
$$X_5 = \frac{\sqrt{3}}{2}\frac{\partial}{\partial x_4} + \frac{1}{2}\frac{\partial}{\partial y_5}, \;\; X_6 = \frac{\partial}{\partial y_4}.$$

Then using almost complex structure of $\overline{M}$, we have

$$JX_1 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_7}\right), \;\; JX_2 = -\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_7}\right),$$
$$JX_3 = \frac{\partial}{\partial y_2}, \;\; JX_4 = -\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}\right),$$
$$JX_5 = \frac{\sqrt{3}}{2}\frac{\partial}{\partial y_4} - \frac{1}{2}\frac{\partial}{\partial x_5}, \;\; JX_6 = -\frac{\partial}{\partial x_4}.$$

Now, let the distributions $D = Span\{X_1, X_2\}$, $D_1 = Span\{X_3, X_4\}$, $D_2 = Span\{X_5, X_6\}$.

Then it is easy to see that D is invariant, D_1 and D_2 are slant distributions with slant angles $\frac{\pi}{4}$ and $\frac{\pi}{6}$ respectively.

Hence ψ defines a proper 6-dimensional quasi bi-slant submanifold N of $\overline{M}$.

Example 5.3. Consider $\mathbb{R}^{2n}$ with standard coordinates $(x_1, x_2, x_3, x_4, ..., x_{2n-1}, x_{2n})$. We can canonically choose an almost complex structure J on $\mathbb{R}^{2n}$ as follows :

$$\begin{aligned} & J(a_1\frac{\partial}{\partial x_1} + a_2\frac{\partial}{\partial x_2} + a_3\frac{\partial}{\partial x_3} + a_4\frac{\partial}{\partial x_4} + ... + a_{2n-1}\frac{\partial}{\partial x_{2n-1}} + a_{2n}\frac{\partial}{\partial x_{2n}}) \\ = & (a_1\frac{\partial}{\partial x_2} - a_2\frac{\partial}{\partial x_1} + a_3\frac{\partial}{\partial x_4} - a_4\frac{\partial}{\partial x_3} + ... + a_{2n-1}\frac{\partial}{\partial x_{2n}} - a_{2n}\frac{\partial}{\partial x_{2n-1}}), \end{aligned}$$

where $a_1, a_2, a_3, ...a_{2n}$ are C^∞ functions defined on $\mathbb{R}^{2n}$.

Consider a submanifold M of $\mathbb{R}^{10}$ defined by

$$\begin{aligned} f(x_1, x_2, x_3, x_4, x_5, x_6) &= (\frac{x_1 + x_2}{\sqrt{2}}, \frac{x_1 - x_2}{\sqrt{2}}, x_3, x_4\cos\theta_1, x_5, x_6\cos\theta_2, 0, \\ & \quad x_4\sin\theta_1, 0, x_6\sin\theta_2) \end{aligned}$$

with $\{\theta_1, \theta_2\} \subset (0, \frac{\pi}{2})$.

By direct computation, it is easy to check that the tangent space at each point of M is spanned by a linearly independent set $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6\}$, where

$$\begin{aligned} Z_1 &= \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right), \quad Z_2 = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}), \\ Z_3 &= \frac{\partial}{\partial x_3}, \quad Z_4 = \cos\theta_1\frac{\partial}{\partial x_4} + \sin\theta_1\frac{\partial}{\partial x_8}, \\ Z_5 &= \frac{\partial}{\partial x_5}, \quad Z_6 = \cos\theta_2\frac{\partial}{\partial x_6} + \sin\theta_2\frac{\partial}{\partial x_{10}}. \end{aligned}$$

Let g be a Hermitian metric on $\mathbb{R}^{10}$ such that

$$g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i}\right) = 1 \text{ ; for } 1 \leq i \leq 10,$$

and

$$g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = 0 \quad i \neq j, \text{ for } 1 \leq i, j \leq 10,$$

Then, using the canonical Hermitian structure of $\mathbb{R}^{10}$, we have

$$\begin{aligned} JZ_1 &= \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1}\right), \quad JZ_2 = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}), \quad JZ_3 = \frac{\partial}{\partial x_4}, \\ JZ_4 &= -\cos\theta_1\frac{\partial}{\partial x_3} - \sin\theta_1\frac{\partial}{\partial x_7}, \quad JZ_5 = \frac{\partial}{\partial x_6}, \\ JZ_6 &= -\cos\theta_2\frac{\partial}{\partial x_5} - \sin\theta_2\frac{\partial}{\partial x_9}. \end{aligned}$$

Let $D = Span\{Z_1, Z_2\}$, $D_1 = Span\{Z_3, Z_4\}$ and $D_2 = Span\{Z_5, Z_6\}$. Then it is easy to see that D is invariant and D_1, D_2 are slant distributions with slant angles θ_1 and θ_2 respectively.

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Affiliations

RAJENDRA PRASAD
ADDRESS: Lucknow University, Dept. of Mathematics and Astronomy, 226007, Lucknow (U.P.), India.
E-MAIL: rp.manpur@rediffmail.com
ORCID ID: 0000-0002-7502-0239

MEHMET AKIF AKYOL
ADDRESS: Bingöl University, Dept. of Mathematics, 12000, Bingöl, Turkey.
E-MAIL: mehmetakifakyol@bingol.edu.tr
ORCID ID: 0000-0003-2334-6955

SANDEEP KUMAR VERMA
ADDRESS: Lucknow University, Dept. of Mathematics and Astronomy, 226007, Lucknow (U.P.), India.
E-MAIL: skverma1208@gmail.com
ORCID ID: 0000-0003-2793-0120

SUMEET KUMAR
ADDRESS: Lucknow University, Dept. of Mathematics and Astronomy, 226007, Lucknow (U.P.), India.
E-MAIL: itssumeetkumar@gmail.com
ORCID ID: 0000-0003-1214-5701