

# Surface Family with a Common Natural Asymptotic Lift of a Timelike Curve in Minkowski 3-space

Ergin Bayram<sup>1\*</sup>, Evren Ergün<sup>2</sup>

<sup>1</sup>Ondokuz Mayıs University, Faculty of Science and Letters, Department of Mathematics, Samsun, +90 362 3121919, erginbayram@yahoo.com

<sup>2</sup>Ondokuz Mayıs University, Çarşamba Ticaret Borsası Vocational High School, Çarşamba, Samsun, +90 362 3121919

\*Corresponding author / İletişimden sorumlu yazar

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## Özet

Bu çalışmada, 3-boyutlu Minkowski uzayında verilen bir timelike eğrinin tabii liftini asimptotik eğri kabul eden yüzey ailesi elde edildi. Verilen eğrinin tabii liftinin yüzey üzerinde asimptotik eğri olması için gerekli ve yeterli koşullar verildi. Son olarak, bazı örnekler ile sunulan metod desteklendi

**Anahtar Kelimeler** – Yüzey ailesi, asimptotik eğri, tabii lift

## 3-Boyutlu Minkowski Uzayındaki Bir Timelike Eğrinin Tabii Liftini Asimptotik Eğri Kabul Eden Yüzey Ailesi

### Abstract

In the present paper, we find a surface family possessing the natural lift of a given timelike curve as an asymptotic curve in Minkowski 3-space. We express necessary and sufficient conditions for a given curve such that its natural lift is an asymptotic curve on any member of the surface family. Finally, we illustrate the method with some examples.

**Keywords** – Surface family, asymptotic curve, natural lift

## 1 Introduction

The problem of finding surfaces with a given common curve as a special curve was firstly handled by Wang et.al. [1]. They constructed surfaces with a common geodesic. Li et.al. [2] obtained necessary and sufficient condition for a given curve to be a line of curvature on a surface pencil. Bayram et.al. [3] studied surface pencil with a common asymptotic curve. In 2014, Ergün et.al. [4] presented a constraint for surfaces with a common line of curvature in Minkowski 3-space. In [5 - 11] authors found necessary and sufficient conditions for a given curve to be a common special curve on a surface family in Euclidean and Minkowski spaces.

Inspired with the above studies, we find necessary and sufficient condition for a surface family possessing the natural lift of a given timelike curve as a common asymptotic curve. First, we start with fundamentals required for the paper. Minkowski 3-space  $\mathbb{R}_1^3$  is the vector space  $\mathbb{R}^3$  equipped with the Lorentzian inner product  $g$  given by

$$g(\mathbf{X}, \mathbf{X}) = -x_1^2 + x_2^2 + x_3^2$$

where  $\mathbf{X} = (x_1, x_2, x_3) \in \mathbb{R}^3$ . A vector  $\mathbf{X} = (x_1, x_2, x_3) \in \mathbb{R}^3$  is said to be timelike if  $g(\mathbf{X}, \mathbf{X}) < 0$ , spacelike if  $g(\mathbf{X}, \mathbf{X}) > 0$  or  $\mathbf{X} = 0$

and lightlike (or null) if  $g(X, X) = 0$  and  $X \neq 0$ .

Similarly, an arbitrary curve  $\alpha = \alpha(s)$  in  $\mathbb{R}^3_1$  can locally be timelike, spacelike or null (lightlike), if all of its velocity vectors  $\alpha'(s)$  are respectively timelike, spacelike or null (lightlike), for every  $s \in I \subset \mathbb{R}$ . A lightlike vector  $X$  is said to be positive (resp. negative) if and only if  $x_1 > 0$  (resp.  $x_1 < 0$ ) and a timelike vector  $X$  is said to be positive (resp. negative) if and only if  $x_1 > 0$  (resp.  $x_1 < 0$ ). The norm of a vector  $X$  is defined by  $\|X\|_{LL} = \sqrt{|g(X, X)|}$  [12].

The vectors  $X = (x_1, x_2, x_3)$ ,  $Y = (y_1, y_2, y_3) \in \mathbb{R}^3_1$  are orthogonal if and only if  $g(X, Y) = 0$  [13].

Now let  $X$  and  $Y$  be two vectors in  $\mathbb{R}^3_1$ , then the Lorentzian cross product is given by [14]

$$X \times Y = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_2y_3 - x_3y_2, x_1y_3 - x_3y_1, x_2y_1 - x_1y_2).$$

**Lemma 1 :** Let  $X$  and  $Y$  be nonzero Lorentz orthogonal vectors in  $\mathbb{R}^3_1$ . If  $X$  is timelike, then  $Y$  is spacelike [13].

**Lemma 2 :** Let  $X$  and  $Y$  be positive (negative) timelike vectors in  $\mathbb{R}^3_1$ . Then

$$g(X, Y) \leq \|X\| \|Y\|$$

with equality if and only if  $X$  and  $Y$  are linearly dependent [13].

**Lemma 3 :**

i) Let  $X$  and  $Y$  be positive (negative) timelike vectors in  $\mathbb{R}^3_1$ . By Lemma 2, there is a unique nonnegative real number  $\phi(X, Y)$  such that

$$g(X, Y) = \|X\| \|Y\| \cosh \phi(X, Y).$$

The Lorentzian timelike angle between  $X$  and  $Y$  is defined to be  $\phi(X, Y)$  [13].

ii) Let  $X$  and  $Y$  be spacelike vectors in  $\mathbb{R}^3_1$  that span a spacelike vector subspace. Then we have

$$|g(X, Y)| \leq \|X\| \|Y\|.$$

Hence, there is a unique real number  $\phi(X, Y)$  between 0 and  $\pi$  such that

$$g(X, Y) = \|X\| \|Y\| \cos \phi(X, Y).$$

$\phi(X, Y)$  is defined to be the Lorentzian spacelike angle between  $X$  and  $Y$  [13].

iii) Let  $X$  and  $Y$  be spacelike vectors in  $\mathbb{R}^3_1$  that span a timelike vector subspace. Then, we have

$$g(X, Y) > \|X\| \|Y\|.$$

Hence, there is a unique positive real number  $\phi(X, Y)$  between 0 and  $\pi$  such that

$$|g(X, Y)| = \|X\| \|Y\| \cosh \phi(X, Y).$$

$\phi(X, Y)$  is defined to be the Lorentzian timelike angle between  $X$  and  $Y$  [13].

iv) Let  $X$  be a spacelike vector and  $Y$  be a positive timelike vector in  $\mathbb{R}^3_1$ . Then there is a unique nonnegative real number  $\phi(X, Y)$  such that

$$|g(X, Y)| = \|X\| \|Y\| \sinh \phi(X, Y).$$

$\phi(X, Y)$  is defined to be the Lorentzian timelike angle between  $X$  and  $Y$  [13].

We denote by  $\{T(s), N(s), B(s)\}$  the moving Frenet frame along the curve  $\alpha$ , where  $T, N$  and  $B$  are the tangent, the principal normal and the binormal vectors of the curve  $\alpha$ , respectively.

Let  $\alpha$  be a unit speed timelike curve with curvature  $\kappa$  and torsion  $\tau$ . So,  $T$  is a timelike vector field,  $N$  and  $B$  are spacelike vector fields. For these vectors, we can write

$$T \times N = -B, \quad N \times B = T, \quad B \times T = -N,$$

where  $\times$  is the Lorentzian cross product in  $\mathbb{R}^3_1$  [15].

The binormal vector field  $B(s)$  is the unique spacelike unit vector field perpendicular to the timelike plane  $\{T(s), N(s)\}$  at every point  $\alpha(s)$  of  $\alpha$ , such that

$\{T, N, B\}$  has the same orientation as  $\mathbb{R}^3_1$ . Then,

Frenet formulas are given by [15]

$$T' = \kappa N, \quad N' = \kappa T + \tau B, \quad B' = -\tau N.$$

Let  $\alpha$  be a unit speed spacelike curve with spacelike binormal. Now,  $T$  and  $B$  are spacelike vector fields and  $N$  is a timelike vector field. In this situation,

$$T \times N = -B, \quad N \times B = -T, \quad B \times T = N.$$

The binormal vector field  $B(s)$  is the unique spacelike unit vector field perpendicular to the timelike plane  $\{T(s), N(s)\}$  at every point  $\alpha(s)$  of  $\alpha$ , such that  $\{T, N, B\}$  has the same orientation as  $\square_1^3$ . Then, Frenet formulas are given by [15]

$$T' = \kappa N, \quad N' = \kappa T + \tau B, \quad B' = \tau N.$$

Let  $\alpha$  be a unit speed spacelike curve with timelike binormal. Now,  $T$  and  $N$  are spacelike vector fields and  $B$  is a timelike vector field. In this case,

$$T \times N = B, \quad N \times B = -T, \quad B \times T = -N,$$

The binormal vector field  $B(s)$  is the unique timelike unit vector field perpendicular to the spacelike plane  $\{T(s), N(s)\}$  at every point  $\alpha(s)$  of  $\alpha$ , such that  $\{T, N, B\}$  has the same orientation as  $\square_1^3$ . Then, Frenet formulas are given by [15]

$$T' = \kappa N, \quad N' = -\kappa T + \tau B, \quad B' = \tau N.$$

Let  $P$  be a surface in  $\square_1^3$  and let  $\alpha : I \rightarrow P$  be a parametrized curve.  $\alpha$  is called an integral curve of  $X$  if

$$\frac{d}{ds}(\alpha(s)) = X(\alpha(s)), \quad \forall t \in I,$$

where  $X$  is a smooth tangent vector field on  $P$  [12]. We have

$$TP = \bigcup_{p \in P} T_p P = \chi(P),$$

where  $T_p P$  is the tangent space of  $P$  at  $p$  and  $\chi(P)$  is the space of tangent vector fields on  $P$ .

For any parametrized curve  $\alpha : I \rightarrow P$ ,  $\bar{\alpha} : I \rightarrow TP$  is given by

$$\bar{\alpha}(s) = (\alpha(s), \alpha'(s)) = \alpha'(s)|_{\alpha(s)}$$

is called the natural lift of  $\alpha$  on  $TP$  [16].

Let  $\alpha(s)$ ,  $L_1 \leq s \leq L_2$ , be a timelike curve with arclength parameter  $s$ . Then, the natural lift  $\bar{\alpha}$  of  $\alpha$  is a spacelike curve with timelike or spacelike binormal. We have following relations between the Frenet frame  $\{T(s), N(s), B(s)\}$  of  $\alpha$  and the Frenet frame  $\{\bar{T}(s), \bar{N}(s), \bar{B}(s)\}$  of  $\bar{\alpha}$ . **a)** Let the natural lift  $\bar{\alpha}$  of  $\alpha$  is a spacelike curve with timelike binormal.

**i)** If the Darboux vector  $W$  of the curve  $\alpha$  is a spacelike vector, then we have

$$\begin{pmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \cosh \theta & 0 & \sinh \theta \\ \sinh \theta & 0 & \cosh \theta \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}. \quad (1)$$

**ii)** If  $W$  is a timelike vector, then we have

$$\begin{pmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \sinh \theta & 0 & \cosh \theta \\ \cosh \theta & 0 & \sinh \theta \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}. \quad (2)$$

**b)** Let the natural lift  $\bar{\alpha}$  of  $\alpha$  is a spacelike curve with spacelike binormal.

**i)** If  $W$  is a spacelike vector, then we have

$$\begin{pmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \cosh \theta & 0 & \sinh \theta \\ -\sinh \theta & 0 & -\cosh \theta \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}. \quad (3)$$

**ii)** If  $W$  is a timelike vector, then we have

$$\begin{pmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \sinh \theta & 0 & \cosh \theta \\ -\cosh \theta & 0 & -\sinh \theta \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}. \quad (4)$$

## 2 Surface family with a common natural asymptotic lift of a timelike curve in Minkowski 3-space

Suppose we are given a 3-dimensional timelike curve  $\alpha(s)$ ,  $L_1 \leq s \leq L_2$ , in which  $s$  is the arc length and  $\|\alpha''(s)\| \neq 0$ ,  $L_1 \leq s \leq L_2$ . Let  $\bar{\alpha}(s)$ ,  $L_1 \leq s \leq L_2$ , be the natural lift of the given curve  $\alpha(s)$ . Now,  $\bar{\alpha}$  is a spacelike curve with timelike or spacelike binormal.

Surface family that interpolates  $\bar{\alpha}(s)$  as a common curve is given in the parametric form as

$$P(s, t) = \bar{\alpha}(s) + u(s, t)\bar{T}(s) + v(s, t)\bar{N}(s) + w(s, t)\bar{B}(s), \quad (5)$$

where  $u(s, t)$ ,  $v(s, t)$  and  $w(s, t)$  are  $C^1$  functions, called *marching-scale functions*, and  $\{\bar{T}(s), \bar{N}(s), \bar{B}(s)\}$  is the Frenet frame of the curve  $\bar{\alpha}$ .

**Remark 4 :** Observe that choosing different marching-scale functions yields different surfaces possessing

$\bar{\alpha}(s)$  as a common curve.

Our goal is to find the necessary and sufficient conditions for which the curve  $\bar{\alpha}(s)$  is isoparametric and asymptotic curve on the surface  $P(s, t)$ . Firstly, as  $\bar{\alpha}(s)$  is an isoparametric curve on the surface  $P(s, t)$ , there exists a parameter  $t_0 \in [T_1, T_2]$  such that

$$u(s, t_0) = v(s, t_0) = w(s, t_0) \equiv 0, L_1 \leq s \leq L_2. \quad (6)$$

Secondly the curve  $\bar{\alpha}$  is an asymptotic curve on the surface  $P(s, t)$  if and only if along the curve the normal vector field  $n(s, t_0)$  of the surface is parallel to the binormal vector field  $\bar{B}$  of the curve  $\bar{\alpha}$ . The normal vector of  $P(s, t)$  can be written as

$$n(s, t) = \frac{\partial P(s, t)}{\partial s} \times \frac{\partial P(s, t)}{\partial t}.$$

Along the curve  $\bar{\alpha}$ , one can obtain the normal vector  $n(s, t_0)$  using Eqns. (5, 6) with an appropriate equation in Eqns. (1-4). It has one of the following forms:

i) if  $\bar{\alpha}$  is a spacelike curve with timelike binormal and Darboux vector  $W$  is spacelike or timelike, then

$$n(s, t_0) = \kappa \left[ \frac{\partial w}{\partial t}(s, t_0) \bar{N}(s) + \frac{\partial v}{\partial t}(s, t_0) \bar{B}(s) \right], \quad (7)$$

ii) if  $\bar{\alpha}$  is a spacelike curve with spacelike binormal and Darboux vector  $W$  is spacelike, then

$$n(s, t_0) = -\kappa \left[ \frac{\partial w}{\partial t}(s, t_0) \bar{N}(s) + \frac{\partial v}{\partial t}(s, t_0) \bar{B}(s) \right], \quad (8)$$

iii) if  $\bar{\alpha}$  is a spacelike curve with spacelike binormal and Darboux vector  $W$  timelike, then

$$n(s, t_0) = \kappa \left[ \frac{\partial w}{\partial t}(s, t_0) \bar{N}(s) - \frac{\partial v}{\partial t}(s, t_0) \bar{B}(s) \right], \quad (9)$$

where  $\kappa$  is the curvature of the curve  $\alpha$ .

Since  $\kappa(s) \neq 0, L_1 \leq s \leq L_2$ , the curve  $\bar{\alpha}$  is an asymptotic curve on the surface  $P(s, t)$  if and only if

$$\frac{\partial w}{\partial t}(s, t_0) \equiv 0, \frac{\partial v}{\partial t}(s, t_0) \neq 0.$$

So, we can present :

**Theorem 5 :** Let  $\alpha(s), L_1 \leq s \leq L_2$ , be a unit speed timelike curve with nonvanishing curvature and  $\bar{\alpha}(s), L_1 \leq s \leq L_2$ , be its natural lift.  $\bar{\alpha}$  is an asymptotic curve on the surface (5) if and only if

$$\begin{cases} u(s, t_0) = v(s, t_0) = w(s, t_0) = \frac{\partial w}{\partial t}(s, t_0) \equiv 0, \\ \frac{\partial v}{\partial t}(s, t_0) \neq 0, \end{cases} \quad (10)$$

where  $L_1 \leq s \leq L_2, T_1 \leq t, t_0 \leq T_2$  ( $t_0$  fixed).

**Corollary 6 :** Let  $\alpha(s), L_1 \leq s \leq L_2$ , be a unit speed timelike curve with nonvanishing curvature and  $\bar{\alpha}(s), L_1 \leq s \leq L_2$ , be its natural lift. If

$$\begin{cases} u(s, t) = v(s, t) = t - t_0, w(s, t) \equiv 0 \\ \text{or} \\ u(s, t) = w(s, t) \equiv 0, v(s, t) = t - t_0, \end{cases} \quad (11)$$

where  $L_1 \leq s \leq L_2, T_1 \leq t, t_0 \leq T_2$  ( $t_0$  fixed) then (5) is a ruled surface possessing  $\bar{\alpha}$  as an asymptotic curve.

**Proof :** By taking marching scale functions as  $u(s, t) = v(s, t) = t - t_0, w(s, t) \equiv 0$  or  $u(s, t) = w(s, t) \equiv 0, v(s, t) = t - t_0$ , the surface Eqn. (5) takes the form

$$P(s, t) = \bar{\alpha}(s) + (t - t_0) [\bar{T}(s) + \bar{N}(s)]$$

or

$$P(s, t) = \bar{\alpha}(s) + (t - t_0) \bar{N}(s),$$

which is a ruled surface satisfying Eqn. (10).

### 3 Examples

**Example 1 :** Let  $\alpha(s) = (\sinh s, 0, \cosh s)$  be a timelike curve. It is easy to show that

$$T(s) = (\cosh s, 0, \sinh s),$$

$$N(s) = (\sinh s, 0, \cosh s),$$

$$B(s) = (0, -1, 0).$$

The natural lift of the curve  $\alpha$  is

$\bar{\alpha}(s) = (\cosh s, 0, \sinh s)$  and its Frenet vectors

$$\bar{T}(s) = (\sinh s, 0, \cosh s),$$

$$\bar{N}(s) = (\cosh s, 0, \sinh s),$$

$$\bar{B}(s) = (0, 1, 0).$$

Choosing marching scale functions as  $u(s, t) = t$ ,  $v(s, t) = \sinh t$ ,  $w(s, t) \equiv 0$  Eqn. (10) is satisfied and we obtain the surface

$$P_1(s, t) = (\cosh s + t \sinh s + (\sinh t) \cosh s, \sinh t, t \cosh s + \sinh s + (\sinh t) \sinh s).$$

$-1 \leq s \leq 1$ ,  $-1 \leq t \leq 0$ , possessing  $\bar{\alpha}$  as a common natural asymptotic lift (Fig. 1).

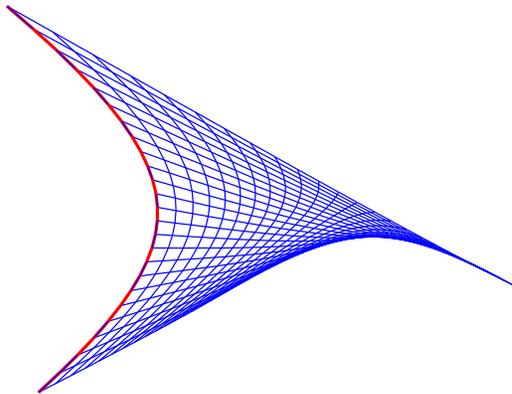


Fig. 1.  $P_1(s,t)$  as a member of the surface family with a common natural asymptotic lift  $\bar{\alpha}$ .

For the same curve, if we choose  $u(s, t) \equiv 0$ ,  $v(s, t) = (\sinh s) \sinh t$ ,  $w(s, t) = t - \sinh t$  we get the surface

$$P_2(s, t) = ((\cosh s)(1 + (\sinh s) \sinh t), t - \sinh t, (\sinh s)(1 + (\sinh s) \sinh t)),$$

$0 < s \leq 1$ ,  $-1 \leq t \leq 1$ , satisfying Eqn. (10) and accepting  $\bar{\alpha}$  as a common natural asymptotic lift (Fig. 2).

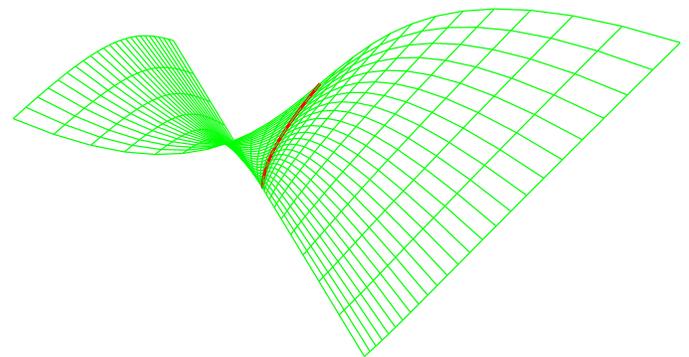


Fig. 2.  $P_2(s,t)$  as a member of the surface family with a common natural asymptotic lift  $\bar{\alpha}$ .

**Example 2 :** Given the arclength timelike curve  $\alpha(s) = (\frac{5}{3}s, \frac{4}{9} \cos 3s, \frac{4}{9} \sin 3s)$  its Frenet apparatus are

$$T(s) = \left( \frac{5}{3}, -\frac{4}{3} \sin 3s, \frac{4}{3} \cos 3s \right),$$

$$N(s) = (0, -\cos 3s, -\sin 3s),$$

$$B(s) = \left( -\frac{4}{3}, \frac{5}{3} \sin 3s, -\frac{5}{3} \cos 3s \right).$$

The natural lift of the curve  $\alpha$  is  $\bar{\alpha}(s) = (\frac{5}{3}, -\frac{4}{3} \sin 3s, \frac{4}{3} \cos 3s)$  and its Frenet vectors

$$\bar{T}(s) = (0, -\cos 3s, -\sin 3s),$$

$$\bar{N}(s) = (0, \sin 3s, -\cos 3s),$$

$$\bar{B}(s) = (-1, 0, 0).$$

If we let marching scale functions as  $u(s, t) = w(s, t) \equiv 0$ ,  $v(s, t) = t$ , we get the ruled surface

$$P_3(s, t) = \left( \frac{5}{3}, \left( t - \frac{4}{3} \right) \sin 3s, \left( \frac{4}{3} - t \right) \cos 3s \right),$$

$-1.1 \leq s \leq 1$ ,  $-1 \leq t \leq 1$ , satisfying Eqn. (11) and passing through  $\bar{\alpha}$  as a common natural asymptotic lift (Fig. 3).

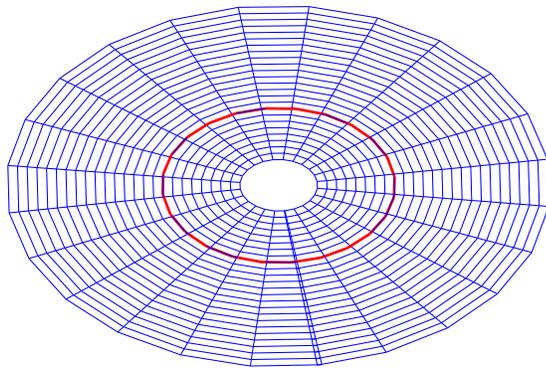


Fig. 3. Ruled surface  $P_3(s,t)$  with a common natural asymptotic lift  $\bar{\alpha}$ .

For the same curve, if we choose  $u(s, t) \equiv 0, v(s, t) = t \ln s, w(s, t) = t^2 e^s$  we obtain the surface

$$P_4(s, t) = \left( \frac{5}{3} - t^2 e^s, \left( t \ln s - \frac{4}{3} \right) \sin(3s), \left( \frac{4}{3} - t \ln s \right) \cos(3s) \right),$$

$1 < s \leq 2, 0 \leq t \leq 1$ , satisfying Eqn. (10) and possessing  $\bar{\alpha}$  as a common natural asymptotic lift (Fig. 4).

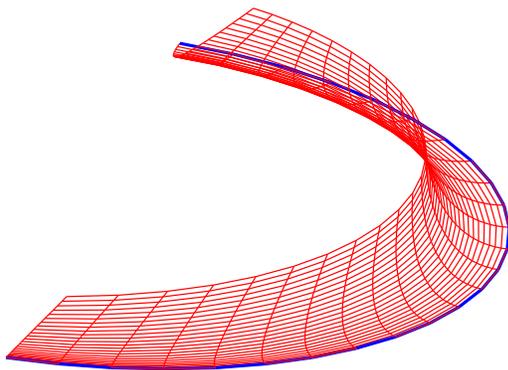


Fig. 4.  $P_4(s,t)$  as a member of the surface family with a common natural asymptotic lift  $\bar{\alpha}$ .

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### 6 References

- [1] Wang, G.J.; Tang, K.; Tai, C.L. Parametric Representation of a Surface Pencil with a Common Spatial Geodesic. *Comput. Aided Des.*, 36, 447-459 (2004).
- [2] Li, C.Y.; Wang, R.H.; Zhu, C.G. Parametric Representation of a Surface Pencil with a Common Line of Curvature. *Comput. Aided Des.* 2011; 43, 1110-1117.
- [3] Bayram, E.; Kasap, E. Hypersurface Family with a Common Isoasymptotic Curve. *Geometry.* 2014; no. 2014.
- [4] Ergün, E.; Bayram, E.; Kasap, E. Surface Pencil with a Common Line of Curvature in Minkowski 3-space. *Acta Mathematica Sinica-English Series.* 2014; 30, 12, 2103-2118.
- [5] Ergün, E.; Bayram, E.; Kasap, E. Surface Family with a Common Natural Line of Curvature Lift. *Journal of Science and Arts.* 2015; 4(31), 321-328.
- [6] Bayram, E.; Ergün, E.; Kasap, E. Surface Family with a Common Natural Asymptotic Lift. *Journal of Science and Arts.* 2015; 2(31), 117-124.
- [7] Bayram, E.; Bilici, M. Surface Family with a Common Involute Asymptotic Curve. *International Geometric Methods in Modern Physics.* 2016; 13, 5, 1650062.
- [8] Ergün, E.; Bayram, E. Surface Family with a Common Natural Geodesic Lift. *Int. J. Math. Combin.* 2016; 1, 34-41.
- [9] Şaffak Atalay, G.; Kasap, E. Surfaces with Common Null Asymptotic. *Applied Mathematics and Computation.* 2015; 260, 135-139.
- [10] Kasap, E.; Akyıldız, F.T. Surfaces with Common Geodesic in Minkowski 3-space. *Appl. Math. Comput.* 2008; 177, 260-270.
- [11] Şaffak, G.; Kasap, E. Family of Surface with a Common Null Geodesic. *Int. J. Phys. Sci.* 2009; 4 8, 428-433.
- [12] O'Neill, B. *Semi-Riemannian Geometry with Applications to Relativity*; Academic Press: New York, 1983.
- [13] Ratcliffe, J.G. *Foundations of Hyperbolic Manifolds*; Springer-Verlag: New York, 1994.
- [14] Önder, M.; Uğurlu, H.H. Frenet Frames and Invariants of Timelike Ruled Surfaces. *Ain Shams Engineering Journal.* 2013; 4, 502-513.
- [15] Walrave, J. *Curves and Surfaces in Minkowski Space*, PhD. Thesis, K. U. Leuven Faculteit Der Wetenschappen. 1995.
- [16] Ergün, E.; Çalışkan, M. On Geodesic Sprays in Minkowski 3-space. *Int. J. Contem. Math. Sci.* 2011; 6, 39, 1929-1933.