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Research Article

Powers of Dirichlet kernels and approximation by discrete linear operators I: direct results

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ABSTRACT. The second and third powers of the Dirichlet kernel are used to construct discrete linear operators for the approximation of continuous periodic functions. An estimate of the rate of convergence is given. Approximation of non-periodic functions are also considered.

Keywords: Discrete linear operators, rate of convergence, direct results, Dirichlet kernel.

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1. INTRODUCTION

Let $\mathbb{T}_n$ be the family of all trigonometric polynomial of degree non greater than n and $C_{2\pi}$ the space of 2π-periodic continuous functions f with the norm $\|f\| = \sup\{|f(x)| : x \in [-\pi, \pi]\}$. We denote by $C^r_{2\pi}$ the space of r-times continuously differentiable functions. For $f \in C^r_{2\pi}$ we set $D^r f = f^{(r)}$.

For $f \in C_{2\pi}$, $r \in \mathbb{N}$ and $t > 0$, the modulus of smoothness of order r is defined by

$$\omega_r(f,t) = \sup_{0<h\leq t} \|\Delta^r_h f\|, \quad \text{where} \quad \Delta^r_h f(x) = \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} f(x+kh).$$

For the approximation of continuous periodic functions several convolution operators have been used. From the computational point of view, it is more useful to work with operators defined discretely (they are given in terms of a finite family of values of the functions). Some authors have employed Riemann sums to replace the integrals in the convolution by discrete sums (see [1]).

For $r \in \mathbb{N}$ and $k \in \mathbb{Z}$, throughout the paper we set

$$x_{r,k} = \frac{2k\pi}{(r+1)}.$$

The Dirichlet kernel is given by (see [3, p. 42])

$$D_n(x) = 1 + 2\sum_{k=1}^{n} \cos(kx) = \frac{\sin((2n+1)x/2)}{\sin(x/2)}, \qquad x \neq 2j\pi, j \in \mathbb{Z}, \tag{1.1}$$

and $D_n(x) = 2n+1$, $x = 2j\pi$, $j \in \mathbb{Z}$. We also set

$$\mathcal{D}_n(x) = \frac{1}{2n+1} D_n(x)$$

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for the normalized Dirichlet kernel. It follows from (1.1) that $|D_n(x)| \leq 2n+1$ and equality holds if $x = 0$. That is the reason why we prefer the normalization given by $\mathcal{D}_n(x)$.

For $f \in C_{2\pi}$ the interpolating polynomial of degree n at the equidistant points $x_{2n,k}$ can be written as

$$L_n(f,x) = \sum_{k=0}^{2n} \mathcal{D}_n(x - x_{2n,k}) f(x_{2n,k}). \tag{1.2}$$

The operator L_n is a Riemann sum approximation of the partial sum of the Fourier series of f given by

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) D_n(x-t)dx.$$

Notice that for $0 \leq j < k \leq 2n$

$$\mathcal{D}_n(x_{2n,j} - x_{2n,k}) = \frac{1}{2n+1}\frac{\sin((j-k)\pi)}{\sin((j-k)\pi/(2n+1))} = 0.$$

Since for every $i \in \mathbb{N}$, $\mathcal{D}_n^i(0) = 1$, each operator

$$L_{n,i}(f,x) = \sum_{k=0}^{2n} \mathcal{D}_n^i(x - x_{2n,k}) f(x_{2n,k}),$$

interpolates the function f at the points $x_{2n,k}$. It is clear that the new polynomials are of degree non greater than ni. Moreover, if the real numbers $a_1, a_2, \ldots, a_m$ satisfy $\sum_{i=1}^{m} a_i = 1$, then the linear combination

$$\mathcal{M}_{nm}(f,x) = \sum_{i=1}^{m} a_i L_{n,i}(f,x) \tag{1.3}$$

provides an interpolation process. The operators $\mathcal{M}_{nm}$ are useful when we want to approximate properties better than the one provided by $L_{n,1}$.

For instance, Kis and Vértesi studied in [9] the operators

$$K_{4n}(f,x) = 4L_{2n,3}(f,x) - 3L_{2n,4}(f,x),$$

while the arguments given by Saxena and Srivastava in [7] can be used to consider the operators

$$S_{6n}(f,x) = \frac{25}{3}L_{2n,4}(f,x) - \frac{32}{3}L_{2n,5}(f,x) + \frac{10}{3}L_{2n,6}(f,x).$$

In [7] only a modification to non-periodic was included. Notice that, in both cases, the sum of the coefficients is one. Thus, they are interpolating operators of the form (1.3).

It was proved in [9] that there exists a constant C such that, for each $f \in C_{2\pi}$ and $n \in \mathbb{N}$,

$$\|f - K_{4n}(f)\| \leq C\omega\Big(f, \frac{1}{n}\Big). \tag{1.4}$$

Another approach to improve the rate of convergence of a linear approximation process considers iterative combinations. For instance, for a linear operator $L : C_{2\pi} \to \mathbb{T}_n$, we construct the new operator

$$\widetilde{L}(f) = 2L(f) - L^2(f),$$

where $L^2(f) = L(L(f))$. But, for linear interpolation operators this approach is not useful. In particular, if L_n is given by (1.2), then $L_n^2(f) = L_n(f)$. We can avoid this inconvenience by using other Riemann sums in the discretization of a convolution operator.

For $n, m \in \mathbb{N}$ and $f \in C_{2\pi}$, in this paper we study the polynomial operators defined by

$$M_{mn,2}(f,x) = \frac{1}{(2n+1)(mn+1)} \sum_{k=0}^{mn} f(x_{mn,k}) D_n^2(x - x_{mn,k}), \tag{1.5}$$

$$M_{mn,3}(f,x) = \frac{1}{(3n^2+3n+1)} \frac{1}{(mn+1)} \sum_{k=0}^{mn} f(x_{mn,k}) D_n^3(x - x_{mn,k}), \tag{1.6}$$

and

$$Q_{3n}(f,x) = C_n \sum_{k=0}^{4n} f(x_{4n,k}) \Big(\mathcal{D}_n^2(x - x_{4n,k}) + \mathcal{D}_n^3(x - x_{4n,k}) \Big), \tag{1.7}$$

where

$$C_n = \frac{(2n+1)^3}{(7n^2+7n+2)(4n+1)}.$$

We will prove in Section 5 that

$$\|Q_{3n}^2(f) - 2Q_{3n}(f) + f\| \le 14\omega_2\Big(f, \frac{2\pi}{n+1}\Big). \tag{1.8}$$

There are some differences between (1.4) and (1.8). Our polynomials are of a lower degree and the rate of convergence is given in terms of the second order modulus of smoothness, but we need more nodes.

Since, for $m \in \mathbb{N}$, D_n^m is an even trigonometric polynomial of degree nm, there are unique real numbers $\varrho_{n,m}(i)$, $0 \le i \le mn$, such that

$$D_n^m(x) = \sum_{i=0}^{mn} \varrho_{n,m}(i) \cos(ix). \tag{1.9}$$

In particular, for $1 \le i \le mn$,

$$\varrho_{n,m}(i) = \frac{1}{\pi} \int_{-\pi}^{\pi} D_n^m(x) \cos(ix) dx. \tag{1.10}$$

For our approach we need explicit expressions of the coefficients $\varrho_{n,2}(i)$ and $\varrho_{n,3}(i)$, but only for $0 \le i \le n$. This will be accomplished in Section 3. In Section 4 we study the behavior of the operators (1.5) and (1.6) for polynomials of lower degree. The main results are presented in Section 5. Finally, in the last section we investigate the case of approximation of non-periodic functions.

A strong converse result, as well as the saturation class, will be given in the second part of the paper.

2. Auxiliary results

Recall that the Fejér kernel is defined by (see [3, p. 43])

$$F_n(x) = \frac{1}{n+1} \sum_{k=0}^{n} D_k(x) = 1 + 2 \sum_{k=1}^{n} \Big(1 - \frac{k}{n+1}\Big) \cos(kx).$$

If $\sin(x/2) \ne 0$, then

$$F_n(x) = \frac{1}{(n+1)} \Big(\frac{\sin((n+1)x/2)}{\sin(x/2)} \Big)^2. \tag{2.11}$$

For $f \in C_{2\pi}$ the associated Fejér operator is defined by

$$\sigma_n(f,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t)F_n(t)dt.$$

Lemma 2.1. *If $g \in C^1_{2\pi}$ and $n \in \mathbb{N}$, then $D(\sigma_n(f)) = \sigma_n(Df)$.*

Proof. It is known that (see [3, Proposition 1.1.14]) if $g \in C_{2\pi}$ and $f \in C^1_{2\pi}$, then $f * g \in C^1_{2\pi}$ and $D(f * g) = (g * D(f))$. $\square$

The following quadrature formula is known.

Proposition 2.1. ([5, p. 20]) *If $x \in \mathbb{R}$, $n \in \mathbb{N}$ and $T \in \mathbb{T}_n$, then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} T(t)dt = \frac{1}{n+1}\sum_{k=0}^{n} T\Big(x + \frac{2k\pi}{n+1}\Big).$$

If

$$T(x) = a_0 + \sum_{j=1}^{n}(a_j\cos(jx) + b_j\sin(jx)) = \sum_{j=0}^{n} A_j(T,x), \tag{2.12}$$

the conjugate of T is given by $\widetilde{T}(x) = \sum_{j=1}^{n}(-b_j\cos(jx) + a_j\sin(jx))$. Simple equations related with the conjugate polynomials are presented in Lemma 2.2.

Lemma 2.2. *If $T \in \mathbb{T}_n$ is given by (2.12) and $W = D\widetilde{T}$, then*

$$D\widetilde{T} = \sum_{j=1}^{n} jA_j(T), \qquad D^2T = -\sum_{j=1}^{n} j^2A_j(T),$$

$$D\widetilde{W} = -D^2(T) \qquad \textit{and} \qquad D(\widetilde{D^2T}) = D^3\widetilde{T}.$$

Lemma 2.3. *If $n \in \mathbb{N}$, σ_n is the Fejér operator and $T \in \mathbb{T}_n$, then*

$$(I - \sigma_n)T = \frac{1}{(n+1)}D\widetilde{T} \qquad \textit{and} \qquad D^3\widetilde{T} = (n+1)(I-\sigma_n)(D^2T).$$

Proof. The first equation is well known (for instance see [2]). For the second one we write

$$(I-\sigma_n)(D^2T) = \frac{1}{(n+1)}D(\widetilde{D^2T}) = \frac{1}{(n+1)}D^3\widetilde{T},$$

where we use Lemma 2.2. $\square$

Theorem 2.1 (Stechkin, [8])**.** *If $r, n \in \mathbb{N}$ and $T \in \mathbb{T}_n$, then*

$$\|D^rT\| \le \left(\frac{n}{2\sin(nh/2)}\right)^r \|\Delta_h^r T\| \tag{2.13}$$

for any $h \in (0, 2\pi/n)$.

We will use the Stechkin theorem in a more convenient form for our purposes.

Proposition 2.2. *If $r, n \in \mathbb{N}$, $f \in C_{2\pi}$, and $T \in \mathbb{T}_n$, then*

$$\frac{1}{n^r}\|D^rT\| \le \frac{1}{2^r}\omega_r\Big(f, \frac{\pi}{n}\Big) + \|f - T_n\|. \tag{2.14}$$

Proof. It follows directly from Theorem 2.1 with $h = \pi/n$ and the inequality $\|\Delta_h^r T\| \le 2^r\|f - T\| + \|\Delta_h^r f\|$. $\square$

We will use Proposition 2.2 in the case when T is the polynomial of the best approximation for f in $\mathbb{T}_n$. It is known that, for every $f \in C_{2\pi}$ and $n \in \mathbb{N}_0$, there exists an unique polynomial $T \in \mathbb{T}_n$ (called the polynomial of the best approximation) such that

$$E_n(f) = \inf_{T_n \in \mathbb{T}_n} \|T_n - f\| = \|T - f\|.$$

Proposition 2.3. *If $f \in C_{2\pi}$, $T \in \mathbb{T}_n$ and $E_n(f) = \|T - f\|$, then*

$$\|D^2 T\| \le n^2 \Big(\frac{1}{4} \omega_2 \Big(f, \frac{\pi}{n} \Big) + E_n(f) \Big),$$
$$\|D^4 T\| \le n^4 \Big(\frac{1}{4} \omega_2 \Big(f, \frac{\pi}{n} \Big) + E_n(f) \Big),$$

and

$$\|D^3 \widetilde{T}\| \le 2n^2 (n+1) \Big(\frac{1}{4} \omega_2 \Big(f, \frac{\pi}{n} \Big) + E_n(f) \Big).$$

Proof. It follows from Proposition 2.2 that

$$\|D^2 T\| \le n^2 \Big(\frac{1}{4} \omega_2 \Big(f, \frac{\pi}{n} \Big) + E_n(f) \Big)$$
$$\|D^4 T\| \le n^4 \Big(\frac{1}{2^4} \omega_4 \Big(f, \frac{\pi}{n} \Big) + E_n(f) \Big) \le n^4 \Big(\frac{1}{4} \omega_2 \Big(f, \frac{\pi}{n} \Big) + E_n(f) \Big),$$

because $\omega_4(f,t) \le 4\omega_2(f,t)$. The last inequality is a consequence of Lemma 2.3. In fact

$$\|D^3 \widetilde{T}\| = (n+1) \|(I - \sigma_n)(D^2 T)\| \le 2(n+1) \|D^2 T\|.$$

□

3. Expansion of Dirichlet kernels

Proposition 3.4. *For each $n \in \mathbb{N}$, one has*

$$D_n^2(x) = 2n + 1 + 2 \sum_{k=1}^{2n} (2n + 1 - k) \cos(kx).$$

That is, $\varrho_{n,2}(0) = 2n+1$ and $\varrho_{n,2}(j) = 2(2n+1-j)$, for $1 \le j \le 2n$ (see (1.9)).

Proof. The computation of D_n^2 is simple, because taking into account (1.1) and (2.11) one has (for $\sin(x/2) \ne 0$)

$$\frac{D_n^2(x)}{2n+1} = \frac{\sin^2((2n+1)x/2)}{(2n+1)\sin^2(x/2)} = F_{2n}(x) = 1 + 2 \sum_{k=1}^{n} \Big(1 - \frac{k}{2n+1} \Big) \cos(kx).$$

□

For D_n^3 we need some preparatory computations.

Lemma 3.4. *For each $n, k \in \mathbb{N}$,*

$$\cos(kx) D_n(x) = \begin{cases} \sum\limits_{i=1}^{n+k} \cos(ix) + \sum\limits_{i=0}^{n-k} \cos(ix), & \text{if} \quad 1 \le k \le n \\ \sum\limits_{i=k-n}^{n+k} \cos(ix), & \text{if} \quad k > n. \end{cases}$$

Proof. If $k \leq n$,

$$\begin{aligned}\cos(kx)D_n(x) &= \cos(kx) + 2\sum_{j=1}^{n}\cos(kx)\cos(jx)\\ &= \cos(kx) + \sum_{j=1}^{n}(\cos((k+j)x) + \cos((k-j)x))\\ &= \cos(kx) + \sum_{i=k+1}^{n+k}\cos(ix) + \sum_{i=1}^{k-1}\cos(ix) + \sum_{i=0}^{n-k}\cos(ix)\\ &= \sum_{i=1}^{n+k}\cos(ix) + \sum_{i=0}^{n-k}\cos(ix).\end{aligned}$$

If $k > n$, then

$$\begin{aligned}\cos(kx)D_n(x) &= \sum_{i=k}^{n+k}\cos(ix) + \sum_{j=1}^{n}\cos((k-j)x)\\ &= \sum_{i=k}^{n+k}\cos(ix) + \sum_{i=k-n}^{k-1}\cos(ix) = \sum_{i=k-n}^{n+k}\cos(ix).\end{aligned}$$

□

Proposition 3.5. *If $n \in \mathbb{N}$, $n \geq 3$, and D_n^3 is given as in (1.9), then*

$$\varrho_{n,3}(0) = 3n^2 + 3n + 1,$$

and

$$\varrho_{n,3}(i) = 2(3n^2 + 3n + 1 - i^2), \quad for \quad 1 \leq i \leq n.$$

Proof. Let $\Pi_n : \mathbb{T}_{3n} \to \mathbb{T}_n$ be the projection given by (see (2.12))

$$\Pi_n(T) = \Pi_n\Big(\sum_{j=0}^{3n} A_j(T,x)\Big) = \sum_{j=0}^{n} A_j(T,x).$$

In this proof (for a fixed n) we denote $\varrho(k) = \varrho_{n,2}(k)$ and consider the expansion of D_n^2 given in Proposition 3.4. Hence

$$\begin{aligned}D_n^3(x) &= (D_n^2(x))D_n(x) = \Big(\sum_{k=0}^{2n}\varrho(k)\cos(kx)\Big)D_n(x)\\ &= \varrho(0)D_n(x) + D_n(x)\sum_{k=1}^{n}\varrho(k)\cos(kx) + D_n(x)\sum_{k=n+1}^{2n}\varrho(k)\cos(kx)\\ &= A_1(x) + A_2(x) + A_3(x).\end{aligned}$$

For $A_2(x)$ one has

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}A_2(x)dx &= \frac{1}{2\pi}\sum_{k=1}^{n}\varrho(k)\int_{-\pi}^{\pi}D_n(x)\cos(kx)dx\\ &= \frac{1}{2\pi}\sum_{k=1}^{n}\varrho(k)\int_{-\pi}^{\pi}\Big(\cos(kx) + 2\sum_{i=1}^{n}\cos(ix)\cos(kx)\Big)dx = \sum_{k=1}^{n}\varrho(k),\end{aligned}$$

and, for $1 \le j \le n$, taking into account Lemma 3.4,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} A_2(x)\cos(jx)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\sum_{k=1}^{n}\varrho(k)\Big(\sum_{i=1}^{n+k}\cos(ix)+\sum_{i=0}^{n-k}\cos(ix)\Big)\cos(jx)dx$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\Big(\cos^2(jx)\sum_{k=1}^{n}\varrho(k)+\cos(jx)\sum_{i=0}^{n-1}\cos(ix)\Big(\sum_{k=1}^{n-i}\varrho(k)\Big)dx.$$

Hence

$$\Pi_n(A_2)(x) = \sum_{k=1}^{n}\varrho(k)+\sum_{j=1}^{n-1}\Big(\sum_{k=1}^{n}\varrho(k)+\sum_{k=1}^{n-j}\varrho(k)\Big)\cos(jx)+\cos(nx)\sum_{k=1}^{n}\varrho(k).$$

For $j=0$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}A_3(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}D_n(x)\sum_{k=n+1}^{2n}\varrho(k)\cos(kx)dx = 0,$$

and, for $1 \le j \le n$,

$$\frac{1}{\pi}\int_{-\pi}^{\pi}A_3(x)\cos(jx)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\Big(D_n(x)\sum_{k=n+1}^{2n}\varrho(k)\cos(kx)\Big)\cos(jx)dx$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\Big(\sum_{k=n+1}^{2n}\varrho(k)\Big(\sum_{i=k-n}^{n+k}\cos(ix)\Big)\Big)\cos(jx)dx$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\Big(\sum_{i=1}^{n}\cos(ix)\Big(\sum_{k=n+1}^{n+i}\varrho(k)\Big)\Big)\cos(jx)dx$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\Big(\sum_{k=n+1}^{n+j}\varrho(k)\Big)\cos^2(jx)dx = \sum_{k=n+1}^{n+j}\varrho(k).$$

Hence

$$\Pi_n(A_3)(x) = \sum_{j=1}^{n}\Big(\sum_{k=n+1}^{n+j}\varrho(k)\Big)\cos(jx).$$

Therefore

$$\Pi_n(D_n^3)(x) = \sum_{k=0}^{n}\varrho(k)+\sum_{j=1}^{n-1}\Big(\sum_{k=0}^{n}\varrho(k)+\sum_{k=0}^{n-j}\varrho(k)+\sum_{k=n+1}^{n+j}\varrho(k)\Big)\cos(jx)$$
$$+\Big(2\varrho(0)+\sum_{k=1}^{n}\varrho(k)+\sum_{k=n+1}^{2n}\varrho(k)\Big)\cos(nx)$$
$$= 3n^2+3n+1+2\sum_{j=1}^{n}(3n^2+3n+1-j^2)\cos(jx).$$

□

4. THE OPERATORS $M_{mn,2}$, $M_{mn,3}$ AND POLYNOMIALS OF LOWER DEGREE

In order to proof the estimate announced in (1.8) we follow a method used in [2]. In particular, for $T \in \mathbb{T}_n$, in Proposition 5.9 we will find a representation of $Q_{3n}(T)$ in terms of the some derivatives of the polynomials.

As Proposition 4.6 shows, the operators $M_{mn,2}(f)$ reproduce the constant functions. But, unfortunately, they are not uniformly bounded. Moreover, if we increase the number of points of interpolation the result does not change. That is the reason why we consider only $m = 3$ for the operators Q_{3n}.

Proposition 4.6. *If $m > 2$, $T \in \mathbb{T}_n$ and $M_{mn,2}$ is defined by (1.5), then*

$$M_{mn,2}(T,x) = T(x) - \frac{1}{(2n+1)} D\widetilde{T}(x).$$

Proof. If $T_n \in \mathbb{T}_n$, then $T_n D_n^2 \in \mathbb{T}_{3n}$ and, taking into account Proposition 2.1, one has

$$\sum_{k=0}^{mn} \frac{T(x_{mn,k})}{(mn+1)} D_n^2(x - x_{mn,k}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} T_n(t) D_n^2(x-t)dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} T_n(x+t) D_n^2(t)dt.$$

If T is written as in (2.12), then

$$\begin{aligned}
&\frac{1}{(2n+1)(mn+1)} \sum_{k=0}^{mn} T(x_{mn,k}) D_n^2(x - x_{mn,k}) \\
=&\frac{a_0}{(2n+1)} \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n^2(t)dt + \sum_{j=1}^{n} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{A_j(T,x)}{(2n+1)} \cos(jt) D_n^2(t)dt \\
=&a_0 + \sum_{j=1}^{n} \frac{1}{2\pi} \frac{A_j(T,x)}{(2n+1)} \int_{-\pi}^{\pi} 2(2n+1-j)\cos^2(jt)dt \\
=&a_0 + \frac{1}{(2n+1)} \sum_{j=1}^{n} A_j(T,x)(2n+1-j) \\
=&T(x) - \frac{1}{(2n+1)} \sum_{j=1}^{n} jA_j(T,x) \\
=&T(x) - \frac{1}{(2n+1)} D\widetilde{T}(x),
\end{aligned}$$

where Proposition 3.4 and Lemma 2.2 were used. □

Proposition 4.7. *If $m > 3$, $T \in \mathbb{T}_n$, and $M_{mn,3}$ is defined by (1.6), then*

$$M_{mn,3}(T,x) = T(x) + \frac{1}{(3n^2+3n+1)} D^2T(x).$$

Proof. Set $u(n) = 3n^2 + 3n + 1$. As before, if $T_n \in \mathbb{T}_n$, then $T_n D_n^3 \in \mathbb{T}_{4n}$ and, taking into account Proposition 2.1, one has

$$\frac{1}{(mn+1)} \sum_{k=0}^{mn} T(x_{mn,k}) D_n^3(x - x_{mn,k}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} T_n(x+t) D_n^2(t)dt.$$

If T is written as in (2.12), then

$$\begin{aligned}&\frac{1}{(mn+1)}\sum_{k=0}^{mn}T(x_{mn,k})D_n^3(x-x_{mn,k})\\ =&a_0\frac{1}{2\pi}\int_{-\pi}^{\pi}D_n^3(t)dt+\sum_{j=1}^{n}\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{A_j(T,x)}{(2n+1)}\cos(jt)D_n^3(t)dt.\end{aligned}$$

Taking into account Proposition 3.5

$$\begin{aligned}&\frac{1}{u(n)(mn+1)}\sum_{k=0}^{mn}T(x_{mn,k})D_n^3(x-x_{mn,k})\\ =&a_0+\frac{1}{u(n)}\sum_{j=1}^{n}\frac{A_j(T,x)(3n^2+3n+1-j^2)}{2\pi}\int_{-\pi}^{\pi}2\cos^2(jt)dt\\ =&a_0+\frac{1}{u(n)}\sum_{j=1}^{n}A_j(T,x)(3n^2+3n+1-j^2)\\ =&T(x)-\frac{1}{u(n)}\sum_{j=1}^{n}j^2A_j(T,x)=T(x)+\frac{1}{u(n)}D^2T(x),\end{aligned}$$

here we use Lemma 2.2. □

5. Main results

In the first result of this section we estimate the norms of the operators.

Proposition 5.8. *If $n\in\mathbb{N}$, Q_{3n} is defined by (1.7) and $f\in C_{2\pi}$, then*

$$\|Q_{3n}(f)\|\le\|f\|$$

and

$$\|Q_{3n}^2(f)-2Q_{3n}(f)+f\|\le 4\|f\|.$$

Proof. Since $|\mathcal{D}_n(x)|\le 1$, $1+\mathcal{D}_n(x)\ge 0$. Therefore

$$\mathcal{D}_n^2(x)+\mathcal{D}_n^3(x)=\mathcal{D}_n^2(x)(1+\mathcal{D}_n(x))\ge 0.$$

It is sufficient to verify that Q_{3n} is a positive operator. Moreover

$$\begin{aligned}Q_{3n}(f,x)&=C_n\sum_{k=0}^{4n}f(x_{4n,k})(\mathcal{D}_n^2(x_{4n,k})+\mathcal{D}_n^3(x-x_{4n,k}))\\ &=\frac{(2n+1)}{(7n^2+7n+2)(4n+1)}\sum_{k=0}^{4n}f(x_{4n,k})D_n^2(x-x_{4n,k})\\ &+\frac{1}{(7n^2+7n+2)(4n+1)}\sum_{k=0}^{4n}f(x_{4n,k})D_n^3(x-x_{4n,k}))\\ &=\frac{(2n+1)^2}{(7n^2+7n+2)}M_{4n,2}(f,x)+\frac{(3n^2+3n+1)}{(7n^2+7n+2)}M_{4n,3}(f,x).\end{aligned}$$

It follows from Propositions 4.6 and 4.7 that

$$Q_{3n}(1,x)=\frac{(2n+1)^2}{(7n^2+7n+2)}+\frac{(3n^2+3n+1)}{(7n^2+7n+2)}=1.$$

If $f \in C_{2\pi}$ and $x \in [-\pi, \pi)$, then

$$|Q_{3n}(f,x)| \leq \|f\| Q_{3n}(1,x) = \|f\|.$$

The second assertion is a simple consequence of the first one. $\square$

Proposition 5.9. *If $n \in \mathbb{N}$, Q_{3n} is defined by (1.7) and $T \in \mathbb{T}_n$, then*

$$Q_{3n}^2 T - 2Q_{3n}T + T = \frac{-(2n+1)^2 D^2 T - 2(2n+1) D^3 \widetilde{T} + D^4 T}{(7n^2+7n+2)^2}.$$

Proof. It follows from Propositions 4.6 and 4.7 that (we set $u(n) = 3n^2+3n+1$ $v(n) = 7n^2+7n+2$ and $W = D\widetilde{T}$)

$$\begin{aligned}
Q_{3n}T &= \frac{(2n+1)^2}{v(n)} M_{4n,2}T + \frac{(3n^2+3n+1)}{v(n)} M_{4n,3}T \\
&= \frac{(2n+1)^2}{v(n)} \Big(T - \frac{1}{(2n+1)} D\widetilde{T} \Big) + \frac{(3n^2+3n+1)}{v(n)} \Big(T + \frac{1}{u(n)} D^2 T \Big) \\
&= T + \frac{1}{v(n)} \Big(D^2 T - (2n+1) W \Big). \qquad (5.15)
\end{aligned}$$

Hence

$$\begin{aligned}
Q_{3n}^2 T &= \frac{(2n+1)^2}{v(n)} M_{4n,2} \Big(T + \frac{D^2T - (2n+1)W}{v(n)} \Big) + \frac{u(n)}{v(n)} M_{4n,3} \Big(T + \frac{D^2T - (2n+1)W}{v(n)} \Big) \\
&= Q_{3n}T + \frac{(2n+1)^2}{v^2(n)} M_{4n,2} \Big(D^2T - (2n+1)W \Big) + \frac{u(n)}{v^2(n)} M_{4n,3} \Big(D^2T - (2n+1)W \Big) \\
&= Q_{3n}T + \frac{(2n+1)^2}{v^2(n)} \Big(D^2 T - \frac{D(\widetilde{D^2T})}{(2n+1)} - (2n+1)W + D\widetilde{W} \Big) \\
&+ \frac{u(n)}{v^2(n)} \Big(D^2T + \frac{D^4T}{u(n)} - (2n+1)W - \frac{(2n+1)D^2W}{u(n)} \Big)
\end{aligned}$$

(recall $D(\widetilde{D^2T}) = D^3\widetilde{T}$ and $D\widetilde{W} = -D^2T$)

$$\begin{aligned}
&= Q_{3n}T + \frac{1}{v(n)} D^2T - \frac{(2n+1)}{v^2(n)} D^3\widetilde{T} - \frac{(2n+1)}{v(n)} W - \frac{(2n+1)^2}{v^2(n)} D^2T \\
&+ \frac{1}{v^2(n)} D^4T - \frac{(2n+1)}{v^2(n)} D^3\widetilde{T} \\
&= Q_{3n}T + \frac{D^2T}{v(n)} - \frac{(2n+1)D\widetilde{T}}{v(n)} - \frac{(2n+1)^2 D^2T}{v^2(n)} - \frac{2(2n+1)}{v^2(n)} D^3\widetilde{T} + \frac{D^4T}{v^2(n)}.
\end{aligned}$$

Taking into account (5.15) we conclude that

$$\begin{aligned}
Q_{3n}^2(T) - 2Q_{3n}(T) + T &= T - Q_{3n}(T) + \frac{D^2T}{v(n)} - \frac{(2n+1)}{v(n)} D\widetilde{T} \\
&\quad - \frac{(2n+1)^2 D^2 T}{v^2(n)} - \frac{2(2n+1)}{v^2(n)} D^3\widetilde{T} + \frac{1}{v^2(n)} D^4 T \\
&= -\frac{1}{v(n)}\Big(D^2T - (2n+1)D\widetilde{T}\Big) + \frac{D^2T}{v(n)} - \frac{(2n+1)}{v(n)} D\widetilde{T} \\
&\quad - \frac{(2n+1)^2 D^2 T}{v^2(n)} - \frac{2(2n+1)}{v^2(n)} D^3\widetilde{T} + \frac{1}{v^2(n)} D^4 T \\
&= -\frac{(2n+1)^2 D^2 T}{v^2(n)} - \frac{2(2n+1)}{v^2(n)} D^3\widetilde{T} + \frac{1}{v^2(n)} D^4 T.
\end{aligned}$$

□

Theorem 5.2. *If $n \in \mathbb{N}$ $(n \geq 3)$, Q_{3n} is defined by (1.7), and $f \in C_{2\pi}$, then*

$$\|Q_{3n}^2(f) - 2Q_{3n}(f) + f\| \leq 5E_n(f) + \omega_2\Big(f, \frac{\pi}{n}\Big).$$

Proof. Fix $f \in C_{2\pi}$ and, for each $n \in \mathbb{N}$, let $T_n \in \mathbb{T}_n$ be the polynomial of the best approximation for f in $\mathbb{T}_n$.

If we set $M_n(f) = Q_{3n}^2(f) - 2Q_{3n}(f)$ and $v(n) = 7n^2 + 7n + 2$, taking into account Propositions 5.8, 5.9, and 2.3 one has

$$\begin{aligned}
\|M_n(f) + f\| &= \|M_n(f - T_n) + f - T_n + M_n(T_n) + T_n\| \\
&\leq 4\|f - T_n\| + \|M_n(T_n) + T_n\| \\
&\leq 4E_n(f) + \frac{\|D^4T\| + 2(2n+1)\|D^3\widetilde{T}\| + (2n+1)^2\|D^2T\|}{v^2(n)} \\
&\leq 4E_n(f) + \frac{n^4 + 4n^2(n+1)(2n+1) + n^2(2n+1)^2}{v^2(n)}\Big(\frac{1}{4}\omega_2\Big(f, \frac{\pi}{n}\Big) + E_n(f)\Big) \\
&= 4E_n(f) + \frac{n^2(13n^2 + 16n + 5)}{v^2(n)}\Big(\frac{1}{4}\omega_2\Big(f, \frac{\pi}{n}\Big) + E_n(f)\Big) \\
&\leq 4E_n(f) + \frac{n^2(14n^2 + 14n + 4)}{v^2(n)}\Big(\frac{1}{4}\omega_2\Big(f, \frac{\pi}{n}\Big) + E_n(f)\Big) \\
&= 4E_n(f) + \frac{2n^2}{v(n)}\Big(\frac{1}{4}\omega_2\Big(f, \frac{\pi}{n}\Big) + E_n(f)\Big) \\
&\leq 4E_n(f) + \frac{2}{7}\Big(\frac{1}{4}\omega_2\Big(f, \frac{\pi}{n}\Big) + E_n(f)\Big) \leq 5E_n(f) + \omega_2\Big(f, \frac{\pi}{n}\Big).
\end{aligned}$$

□

Remark 5.1. *The term $E_n(f)$ in Theorem 5.2 can be estimate as (see* [6, Theorem 2.5]*)*

$$E_n(f) \leq \frac{5}{2}\omega_2\Big(f, \frac{2\pi}{n+1}\Big).$$

Therefore

$$\|Q_{3n}^2(f) - 2Q_{3n}(f) + f\| \leq \frac{25}{2}\omega_2\Big(f, \frac{2\pi}{n+1}\Big) + \omega_2\Big(f, \frac{\pi}{n}\Big) \leq 14\omega_2\Big(f, \frac{2\pi}{n+1}\Big). \tag{5.16}$$

6. Approximation of non-periodic functions

Let $C[-1,1]$ the space of continuous functions $f:[-1,1]\to\mathbb{R}$ provided with the sup norm $\|f\|_\infty=\sup\{|f(x)|: x\in[-1,1]\}$. In this section we follow a known procedure to pass from approximation by trigonometric polynomials to approximation by algebraic polynomials (see Proposition 6.10 below).

For $f\in C[-1,1]$ and $x,h\in[-1,1]$ define

$$(\tau_h f)(x)=\frac{1}{2}\Big(f\Big(xh+\sqrt{(1-x^2)(1-h^2)}\Big)+f\Big(xh-\sqrt{(1-x^2)(1-h^2)}\Big)\Big)$$

and

$$\omega^T(f,t)=\sup_{t\le h\le 1}\|f-\tau_h f\|.$$

We also set

$$E_n(f)_\infty=\inf_{P\in\mathbb{P}_n}\|f-P\|_\infty,$$

where $\mathbb{P}_n$ be the family of all algebraic polynomial of degree not greater than n.

We introduce operators similar to Q_{3n} by setting

$$R_{3n}(f,x)=C_n\sum_{k=0}^{4n} f(\cos x_{4n,k})(\mathcal{D}_n^2(\arccos x-x_{4n,k})+\mathcal{D}_n^3(\arccos x-x_{4n,k}))$$

for $f\in C[-1,1]$ and $x\in[-1,1]$. Notice that $D_n(\arccos x-x_{4n,k})$ can be written in terms of the Chebyshev polynomials. Hence $R_{3n}(f,x)$ is an algebraic polynomial of degree not greater than $3n$ (see Proposition 6.10 below).

Theorem 6.3. *If $n\in\mathbb{N}$ $(n\ge 3)$ and $f\in C[-1,1]$, then*

$$\|R_{3n}^2(f)-2R_{3n}(f)+f\|\le 14\omega^T\Big(f,\cos\frac{2\pi}{n+1}\Big).$$

Proof. Fix $f\in C[-1,1]$ and set $F(t)=f(\cos t)$. It is known that (see [4, Lemma 3]), for $t\in[-1,1]$,

$$\omega^T(f,t)=\omega_2(F,\arccos t). \tag{6.17}$$

If $x\in[-1,1]$ and $x=\cos t$ $(0\le t\le\pi)$, it follows from Theorem 5.2 and (6.17) that

$$\begin{aligned}|R_{3n}^2(f,x)-2R_{3n}(f,x)+f(x)| &= |R_{3n}^2(f,\cos t)-2R_{3n}(f,\cos t)+f(\cos t)|\\ &=\big|Q_{3n}^2(F,t)-2Q_{3n}(F,t)+F(t)\big|\\ &\le 14\omega_2\Big(F,\frac{2\pi}{n+1}\Big)\\ &=14\omega^T\Big(f,\cos\frac{2\pi}{n+1}\Big).\end{aligned}$$

□

Remark 6.2. *Here we only consider estimates in norm, pointwise estimates require another approach.*

Remark 6.3. *Let $X^1[-1,1]$ be the family of $f\in C[-1,1]$ for which there exists $g\in C[-1,1]$ such that*

$$\lim_{h\to 1-}\Big\|\frac{\tau_h f-f}{1-h}-g\Big\|_\infty=0.$$

If $f\in X^1[-1,1]$, then $\omega^T(f,t)\le C(1-t)$ (see [4, Lemma 6]*). Hence, for $f\in X^1[-1,1]$,*

$$\|R_{3n}^2(f)-2R_{3n}(f)+f\|\le C\Big(1-\cos\frac{2\pi}{n+1}\Big)\le\frac{2C\pi^2}{(n+1)^2}.$$

The following result is known, but we include a proof for the benefit of the reader.

Proposition 6.10. *For each $n, m \in \mathbb{N}$, $f \in C[-1,1]$ and $x \in [-1,1]$, the function*

$$\sum_{k=0}^{4n} f(\cos x_{4n,k}) D_n^m(\arccos x - x_{4n,k})$$

is an algebraic polynomial of degree not greater than mn.

Proof. For $k \in \mathbb{N}_0$, let $T_k(x) = \cos(k \arccos x)$ be the Chebyshev polynomial of degree k.

Since

$$D_n(\arccos x) = 1 + 2\sum_{k=1}^{n} \cos(k \arccos x) = 1 + 2\sum_{k=1}^{n} T_k(x),$$

one has $f(1)D_n^m(\arccos x)$ is an algebraic polynomial.

For $1 \le j, k \le 2n$, we consider the trigonometric identities

$$\cos(jx_{4n,4n+1-k}) = \cos\frac{2j(4n+1-k)\pi}{4n+1} = \cos\frac{2kj\pi}{4n+1} = \cos(x_{4n,jk}),$$
$$\sin(jx_{4n,4n+1-k}) = -\sin\frac{2jk\pi}{4n+1} = -\sin x_{4n,jk}$$

and

$$\begin{aligned}
&\cos j(\arccos x - x_{4n,k}) + \cos j(\arccos x - x_{4n,4n+1-k}) \\
=&T_j(x)\Big(\cos(jx_{4n,k}) + \cos(jx_{4n+1-k,k})\Big) + \sin(j \arccos x)\Big(\sin(jx_{4n,k}) + \sin(jx_{4n+1-k,k})\Big) \\
=&2\cos(jx_{4n,k})\, T_j(x),
\end{aligned}$$

to obtain

$$\begin{aligned}
&\sum_{k=1}^{4n} f(\cos x_{4n,k}) D_n^m(\arccos x - x_{4n,k}) \\
=&\sum_{k=1}^{2n} f(\cos x_{4n,k})\Big(D_n^m(\arccos x - x_{4n,k}) + D_n^m(\arccos x - x_{4n,4n+1-k})\Big) \\
=&\sum_{k=1}^{2n} f(\cos x_{4n,k}) \sum_{j=0}^{mn} \varrho_{n,m}(j)\Big(\cos(j(\arccos x - x_{4n,k})) \\
&+ \cos(j(\arccos x - x_{4n,4n+1-k}))\Big) \\
=&2\sum_{k=1}^{2n} f(\cos x_{4n,k}) \sum_{j=0}^{mn} \varrho_{n,m}(j) \cos(jx_{4n,k})\, T_j(x).
\end{aligned}$$

$\square$

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