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Determine When a Parametric Surface is a Surface of Revolution

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ABSTRACT

A surface of revolution is a surface that can be generated by rotating a planar curve, the directrix, around a straight line, the axis, in the same plane. Using the mathematics of quaternions, we provide a parametric equation of a surface of revolution generated by rotating a directrix about an axis by quaternion multiplication of the parametric representations of the directrix curve and the line of axis. Then, we describe an algorithm to determine whether a parametric surface is a surface of revolution, and identify the axis and the directrix. Examples are provided to illustrate our algorithm.

Keywords: *Surface of revolution, quaternion multiplication, rotation axis, directrix.*
AMS Subject Classification (2020): *Primary: 51-08; Secondary: 51N20.*

1. Introduction

Quaternions were discovered by Sir William Rowan Hamilton [6] as an extension of the complex numbers in 1843. An important property of quaternions is that every unit quaternion represents a rotation and this plays a special role in the study of rotations in three dimensional spaces. Quaternions are used in both theoretical and applied mathematics, especially in the areas involving calculations of three dimensional rotations, such as in three dimensional computer graphics, computer vision, animations, and aerospace applications.

A *surface of revolution* is a surface that can be generated by rotating a planar curve (the *directrix*) around a straight line (the *axis*) in the same plane. Surfaces of revolution are everywhere in our daily life. For example, tubes, wheels, vases, footballs, doorknobs and light bulbs are all surfaces of revolution. They are common in computer-aided geometric design (CAGD) and computer graphics, and can be used to create 3D digital surfaces for automotive and industrial design [12]. In particular, the use of surfaces of revolution is essential for digital designs for the applications in the fields of physics and engineering, since the areas of these surfaces can be determined without measuring the length and radius of the objects being designed. Because of it's abundant properties and applications, the surface of revolution is a fundamental topic extensively studied in calculus and analytic geometry [11], differential geometry [9], engineering mathematics [10], and physics [8].

A parametric surface is a surface in the Euclidean 3-space which is defined by a parametric equation with two parameters. Parametric representations are used to study the properties of the surfaces. For instance, a surface is given in parametric form to prove Stokes' theorem and divergence theorem - two main theorems in the vector calculus [11]; and to compute the geometric invariants such as first and second fundamental forms, Gaussian curvature, mean curvature, and principal curvature in differential geometry [9]. Parametric surfaces are widely used in CAGD projects since it is easy to describe the points of the surface by means of the parameters values. To assure the simplicity of the design for CAGD, it is essential to describe parametric surfaces in a geometrically intuitive and easy to understand manner, although the underlying mathematics may be quite sophisticated.

In this paper, we first formulate the parametric representation of a surface of revolution from space curves using quaternion multiplication. Then, we provide an algorithm for determining whether or not a parametric

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surface is a surface of revolution, and identify the rotation axis and directrix. Examples are provided to illustrate our algorithm.

Our approach here is to employ quaternion operations to study surfaces of revolution. To be specific, we internally consider the rotation axis and the directrix curve of a surface of revolution as quaternions, and provide a parametric representation of this surface by quaternion multiplication of these two curves. Our parametric description of surfaces of revolution makes it simple to construct and intuitive for design. This type of construction is initiated by [5], but their ultimate goal is to find the implicit equations of the parametric surfaces generated by two curves by quaternion products. In this paper, we emphasize on determine whether a parametric surface is a surface of revolution by using the properties of this quaternion construction of the surfaces of revolution, which to our knowledge has not been studied in the literature.

In Section 2 of this paper, we start with a brief review of some basic facts about quaternions, and then show that a surface of revolution is in fact a surface generated from two curves by quaternion multiplication. Using this insight, in Section 3 we provide a theorem for determining whether or not a parametric surface is a surface of revolution, and identify the rotation axis and directrix. Examples are provided to illustrate our results.

2. Quaternions and Surfaces of Revolution

2.1. Quaternions.

Quaternions can be used to represent and compute rotations in 3-dimensions. Therefore quaternions have applications in many contemporary areas of computational science and engineering, including computer graphics, computer vision, classical mechanics, and robotics [1], [2], [3], [4], [7]. We are going to generalize the easy to understand idea of using a line and a planar curve to represent a classical surface of revolution by taking advantage of the fact that quaternions give a natural way to represent rotations.

2.1.1. Quaternion Algebra. An arbitrary *quaternion* has the form $q = s_q + v_q$, where s_q is a scalar and $v_q = v_1 i + v_2 j + v_3 k$ is a vector in $\mathbb{R}^3$. The *conjugate* of q is denoted by $q^* = s_q - v_q = s_q - v_1 i - v_2 j - v_3 k$. Addition of one quaternion $q_1 = s_{q_1} + v_{q_1}$ to another quaternion $q_2 = s_{q_2} + v_{q_2}$ is defined by:

$$q_1 + q_2 = (s_{q_1} + s_{q_2}) + (v_{q_1} + v_{q_2}).$$

Multiplication of two quaternions is carried out as follows:

$$q_1\, q_2 = (s_{q_1} + v_{q_1})\,(s_{q_2} + v_{q_2}) = s_{q_1} s_{q_2} - v_{q_1} \cdot v_{q_2} + s_{q_1}\, v_{q_2} + s_{q_2}\, v_{q_1} + v_{q_1} \times v_{q_2},$$

where $\cdot$ and $\times$ are the usual dot and cross product of vectors in $\mathbb{R}^3$. Notice that $qq^* = s_q^2 + v_1^2 + v_2^2 + v_3^2 = |q|^2$ is a non-negative scalar; $|q|$ is called the *norm* of q; $qq^* = 0$ if and only if q is a zero quaternion. If $|q| = 1$, then q is called a *unit quaternion*.

2.1.2. Quaternion Geometry. A *pure quaternion* is a quaternion q whose scalar part $s_q = 0$. A pure quaternion $v_q = v_1 i + v_2 j + v_3 k$ is interpreted geometrically as the vector from the origin to the point located at (v_1, v_2, v_3) in $\mathbb{R}^3$. Similarly, if $s_q \neq 0$, the quaternion $q = s_q + v_q = s_q + v_1 i + v_2 j + v_3 k$ is interpreted geometrically as the mass-point with mass s_q located at $(v_1/s_q, v_2/s_q, v_3/s_q)$ in $\mathbb{R}^3$. With this interpretation the quaternion $q = s_q + v_1 i + v_2 j + v_3 k$ is akin to the homogeneous coordinate representation (s_q, v_1, v_2, v_3) for points and vectors in 3-dimensions. Notice, in particular, that with this interpretation the quaternion $q = 1$ represents the origin in $\mathbb{R}^3$, and the quaternion $q = 1 + v_q$ represents the point in affine space located at the tip of the vector v_q. Quaternions are widely used in computer graphics when a 3-dimensional character rotation is involved [4]. Theorem 2.1 (below) recalls a well-known quaternion formula for rotation in 3-dimensions [4].

Theorem 2.1. *(Quaternion Rotation Theorem) Let $v_q = v_1 i + v_2 j + v_3 k$ be a pure unit quaternion and set*

$$R = \cos(\theta/2) + \sin(\theta/2) v_q.$$

Then R is a unit quaternion and the sandwiching map $x \to RxR^*$ rotates points and vectors in $\mathbb{R}^3$ by the angle θ around the line through the origin in the direction of the vector v_q in $\mathbb{R}^3$.

Remark 2.1. It is known [4] that if $q = s_q + q_1 i + q_2 j + q_3 k$ is a unit quaternion and $p = (p_1, p_2, p_3)$ a point in $\mathbb{R}^3$, then a quaternion rotation

$$qpq^* = \mathbf{R}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 1-2(q_2^2+q_3^2) & 2(q_1q_2 - q_3s_q) & 2(q_1q_3+q_2s_q) \\ 2(q_1q_2+q_3s_q) & 1-2(q_1^2+q_3^2) & 2(q_2q_3-q_1s_q) \\ 2(q_1q_3-q_2s_q) & 2(q_2q_3+q_1s_q) & 1-2(q_1^2+q_2^2) \end{bmatrix} \text{ is the rotation matrix.}$$

Furthermore,

$$\mathbf{R}^T\mathbf{R} = \mathbf{R}\mathbf{R}^T = \mathbf{I}, \text{ where } \mathbf{I} \text{ is the identity matrix.}$$

2.2. *Parametric Representation of Surfaces of Revolution by Quaternion Multiplication*

We shall formulate the parametric representation of a surface of revolution from curves by taking advantage of the fact that quaternions provide a natural way to represent rotations in 3-dimensions.

Theorem 2.2. *Let a line* $\mathbf{f}$ *and a curve* $\mathbf{g}$ *be given as the following parametric representations:*

$$\mathbf{f}(s) = (f_1(s), f_2(s), f_3(s)) = (a_1 s + b_1, a_2 s + b_2, a_3 s + b_3), \qquad \mathbf{g}(t) = (g_1(t), g_2(t), g_3(t)).$$

Then, the surface of revolution generated by rotating the directrix curve $\mathbf{g}(t)$ *around the axis line* $\mathbf{f}(s)$ *has a parametric representation with parameters* t, θ *in the matrix form:*

$$\mathbf{h}(\theta, t) = \mathbf{b} + R(\theta)(\mathbf{g}(t) - \mathbf{b})R^*(\theta) = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + \mathbf{R}(\theta)\begin{bmatrix} g_1(t) - b_1 \\ g_2(t) - b_2 \\ g_3(t) - b_3 \end{bmatrix}, \textit{ where} \tag{2.1}$$

$$R(\theta) = \cos(\theta/2) + \sin(\theta/2)\frac{a_1 i + a_2 j + a_3 k}{\sqrt{a_1^2 + a_2^2 + a_3^2}}, \textit{ and}$$

$$\mathbf{R}(\theta) = \begin{bmatrix} \frac{a_1^2+(a_2^2+a_3^2)\cos(\theta)}{a_1^2+a_2^2+a_3^2} & \frac{-a_3\sin(\theta)}{\sqrt{a_1^2+a_2^2+a_3^2}} + \frac{2a_1a_2\sin^2(\theta/2)}{a_1^2+a_2^2+a_3^2} & \frac{a_2\sin(\theta)}{\sqrt{a_1^2+a_2^2+a_3^2}} + \frac{2a_1a_3\sin^2(\theta/2)}{a_1^2+a_2^2+a_3^2} \\ \frac{a_3\sin(\theta)}{\sqrt{a_1^2+a_2^2+a_3^2}} + \frac{2a_1a_2\sin^2(\theta/2)}{a_1^2+a_2^2+a_3^2} & \frac{a_2^2+(a_1^2+a_3^2)\cos(\theta)}{a_1^2+a_2^2+a_3^2} & \frac{-a_1\sin(\theta)}{\sqrt{a_1^2+a_2^2+a_3^2}} + \frac{2a_2a_3\sin^2(\theta/2)}{a_1^2+a_2^2+a_3^2} \\ \frac{-a_2\sin(\theta)}{\sqrt{a_1^2+a_2^2+a_3^2}} + \frac{2a_1a_3\sin^2(\theta/2)}{a_1^2+a_2^2+a_3^2} & \frac{a_1\sin(\theta)}{\sqrt{a_1^2+a_2^2+a_3^2}} + \frac{2a_2a_3\sin^2(\theta/2)}{a_1^2+a_2^2+a_3^2} & \frac{a_3^2+(a_1^2+a_2^2)\cos(\theta)}{a_1^2+a_2^2+a_3^2} \end{bmatrix}. \tag{2.2}$$

Proof. Since the parametric representation of the axis line $\mathbf{f}(s)$ is:

$$\mathbf{f}(s) = (f_1, f_2, f_3) = (a_1 s + b_1, a_2 s + b_2, a_3 s + b_3), \text{ where } a_i, b_i \in \mathbb{R},\ 1 \le i \le 3$$

the line $\mathbf{f}(s)$ has the direction vector $\langle a_1, a_2, a_3 \rangle$, and passes through the point $\mathbf{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$.

First, perform a linear change of coordinates $(X, Y, Z) = (x - b_1, y - b_2, z - b_3)$, which transforms the point $\mathbf{b} = (b_1, b_2, b_3)$ in the original xyz-coordinate system to the origin $(0, 0, 0)$ in the new XYZ-coordinate system. In this new XYZ-coordinate system, the image of the line $\mathbf{f}(s)$ is a line through the origin with direction vector $a_1 i + a_2 j + a_3 k$, and the image of the directrix curve $\mathbf{g}(t)$ is parametrized as $\mathbf{g}(t) - \mathbf{b}$.

Now, let $v_q = \frac{a_1 i + a_2 j + a_3 k}{\sqrt{a_1^2+a_2^2+a_3^2}}$ be a pure unit quaternion, and set $R(\theta) = \cos(\theta/2) + \sin(\theta/2)v_q$. By quaternion rotation Theorem 2.1, $R(\theta)(\mathbf{g}(t) - \mathbf{b})R^*(\theta)$ rotates points $\mathbf{g}(t) - \mathbf{b}$ by the angle θ around the vector v_q through the origin. Therefore, in the new XYZ-coordinate system, the quaternion product $R(\theta)(\mathbf{g}(t) - \mathbf{b})R^*(\theta)$ is the image of rotating the shifted directrix curve $\mathbf{g}(t)$ by angle θ about the shifted axis line $\mathbf{f}(s)$.

Finally, in the original xyz-coordinate system, the surface generated by rotating the curve $\mathbf{g}(t)$ by angle θ about the axis $\mathbf{f}(s)$ is the composition of a rotation in the new coordinate, i.e., $R(\theta)(\mathbf{g}(t) - \mathbf{b})R^*(\theta)$ followed by a translation $(x, y, z) = (X, Y, Z) + \mathbf{b}$. Thus, the parametrization of the surface of revolution generated by rotating the curve $\mathbf{g}(t)$ about the axis $\mathbf{f}(s)$ can be written as:

$$\mathbf{h}(\theta, t) = \mathbf{b} + R(\theta)(\mathbf{g}(t) - \mathbf{b})R^*(\theta), \text{ where } R(\theta) = \cos(\theta/2) + \sin(\theta/2)\frac{a_1 i + a_2 j + a_3 k}{\sqrt{a_1^2 + a_2^2 + a_3^2}}.$$

Since $R(\theta) = \cos(\theta/2) + \sin(\theta/2)\frac{a_1 i + a_2 j + a_3 k}{\sqrt{a_1^2+a_2^2+a_3^2}}$, by setting

$$q_s = \cos(\theta/2), \quad q_1 = \frac{a_1\sin(\theta/2)}{\sqrt{a_1^2+a_2^2+a_3^2}}, \quad q_2 = \frac{a_2\sin(\theta/2)}{\sqrt{a_1^2+a_2^2+a_3^2}}, \quad q_3 = \frac{a_3\sin(\theta/2)}{\sqrt{a_1^2+a_2^2+a_3^2}},$$

the rotation matrix $\mathbf{R}$ in Remark 2.1 can be expressed as

$$\mathbf{R}(\theta)=\begin{bmatrix} \frac{a_1^2+(a_2^2+a_3^2)\cos(\theta)}{a_1^2+a_2^2+a_3^2} & \frac{-a_3\sin(\theta)}{\sqrt{a_1^2+a_2^2+a_3^2}}+\frac{2a_1a_2\sin^2(\theta/2)}{a_1^2+a_2^2+a_3^2} & \frac{a_2\sin(\theta)}{\sqrt{a_1^2+a_2^2+a_3^2}}+\frac{2a_1a_3\sin^2(\theta/2)}{a_1^2+a_2^2+a_3^2} \\ \frac{a_3\sin(\theta)}{\sqrt{a_1^2+a_2^2+a_3^2}}+\frac{2a_1a_2\sin^2(\theta/2)}{a_1^2+a_2^2+a_3^2} & \frac{a_2^2+(a_1^2+a_3^2)\cos(\theta)}{a_1^2+a_2^2+a_3^2} & \frac{-a_1\sin(\theta)}{\sqrt{a_1^2+a_2^2+a_3^2}}+\frac{2a_2a_3\sin^2(\theta/2)}{a_1^2+a_2^2+a_3^2} \\ \frac{-a_2\sin(\theta)}{\sqrt{a_1^2+a_2^2+a_3^2}}+\frac{2a_1a_3\sin^2(\theta/2)}{a_1^2+a_2^2+a_3^2} & \frac{a_1\sin(\theta)}{\sqrt{a_1^2+a_2^2+a_3^2}}+\frac{2a_2a_3\sin^2(\theta/2)}{a_1^2+a_2^2+a_3^2} & \frac{a_3^2+(a_1^2+a_2^2)\cos(\theta)}{a_1^2+a_2^2+a_3^2} \end{bmatrix}.$$

Hence, by Remark 2.1, the surface of revolution generated by rotating the directrix curve $\mathbf{g}(t)$ along the axis $\mathbf{f}(s)$ has a parametric representation with parameters t,θ in the matrix form:

$$\mathbf{h}(\theta,t)=\mathbf{b}+R(\theta)(\mathbf{g}(t)-\mathbf{b})R^*(\theta)=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}+\mathbf{R}(\theta)\begin{bmatrix}g_1(t)-b_1\\g_2(t)-b_2\\g_3(t)-b_3\end{bmatrix}.$$

□

We will use following example to illustrate Theorem 2.2.

Example 2.1. In this example, we provide a parametric representation for the surface of revolution generated by rotating the unit circle $(x-4)^2+y^2=1$ around the line $x=-1$. First, we parametrize the axis $\mathbf{f}(s)$ and directrix $\mathbf{g}(t)$

$$\begin{aligned}\mathbf{f}(s) &= (-1,s,0), \quad \text{where } a_1=0,\ a_2=1,\ a_3=0,\ b_1=-1,\ b_2=0,\ b_3=0;\\ \mathbf{g}(t) &= \left(\frac{3t^2+5}{t^2+1},\frac{2t}{t^2+1},0\right).\end{aligned}$$

Hence, the parametric equation for this surface of revolution is

$$\mathbf{h}(\theta,t)=\mathbf{b}+R(\theta)(\mathbf{g}(t)-\mathbf{b})R^*(\theta)=\begin{bmatrix}-1\\0\\0\end{bmatrix}+\begin{bmatrix}\cos\theta&0&\sin\theta\\0&1&0\\-\sin\theta&0&\cos\theta\end{bmatrix}\begin{bmatrix}\frac{3t^2+5}{t^2+1}+1\\\frac{2t}{t^2+1}\\0\end{bmatrix}=\begin{bmatrix}\frac{\cos\theta(4t^2+6)}{t^2+1}-1\\\frac{2t}{t^2+1}\\\frac{-\sin\theta(4t^2+6)}{t^2+1}\end{bmatrix}.$$

In fact, the surface of revolution given in Example 2.1 is a torus. A torus is a surface of revolution generated by a circle of radius r_2 rotating about an axis that is at a distance r_1 away from the center of the circle. If the axis is z-axis, then the parametric equation of a torus can be written in terms of r_1,r_2 and angle α and β as

$$(x(\alpha,\beta),y(\alpha,\beta),z(\alpha,\beta))=((r_1+r_2\cos\alpha)\cos\beta,(r_1+r_2\cos\alpha)\sin\beta,r_1\sin\alpha).$$

That is, this torus is generated by rotating a circle $(x-r_1)^2+z^2=r_2^2$ in the xz-plane around the z-axis. This parametric representation of a torus requires that z-axis is the axis of rotation, and cannot be applied to a torus with other rotation axis. The parametrization of surfaces of revolution via quaternion multiplications proposed in this section has the advantage, since the parametric representation of the surface of revolution depends only on the rotation axis and the directrix curve. Therefore, given the rotation axis and the directrix of a surface of revolution, using quaternion multiplication is a much intuitive and direct approach to provide a parametric representation of a surface of revolution.

3. Determine when a parametric surface is a surface of revolution

In this section, we will discuss how to determine whether or not a given parametric surface is a surface of revolution, and how to identify the rotation axis and the directrix. We will first discuss some properties of the surfaces of revolution.

Proposition 3.1. *Given a surface of revolution expressed in Equation* (2.1)

$$\mathbf{h}(\theta,t)=(h_1,h_2,h_3)=\mathbf{b}+R(\theta)(\mathbf{g}(t)-\mathbf{b})R^*(\theta)=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}+\mathbf{R}(\theta)\begin{bmatrix}g_1(t)-b_1\\g_2(t)-b_2\\g_3(t)-b_3\end{bmatrix},$$

then

$$\begin{aligned} |\mathbf{h}_t(\theta,t)|^2 &= \sum_{i=1}^{3}\left(\frac{\partial h_i(\theta,t)}{\partial t}\right)^2 = G_1(t), \text{ for some single variable function } G_1(t);\\ |\mathbf{h}_\theta(\theta,t)|^2 &= \sum_{i=1}^{3}\left(\frac{\partial h_i(\theta,t)}{\partial \theta}\right)^2 = G_2(t), \text{ for some single variable function } G_2(t). \end{aligned}$$

Proof. Since

$$\mathbf{h}_t(\theta,t) = \mathbf{R}(\theta)\mathbf{g}'(t), \text{ where } \mathbf{g}'(t) \text{ is the derivative of } \mathbf{g}(t) \text{ with respect to } t,$$

$$\begin{aligned} |\mathbf{h}_t(\theta,t)|^2 &= \mathbf{h}_t(\theta,t)\cdot\mathbf{h}_t(\theta,t) = (\mathbf{R}(\theta)\mathbf{g}'(t))^T(\mathbf{R}(\theta)\mathbf{g}'(t)) = (\mathbf{g}'(t))^T(\mathbf{R}^T(\theta)\mathbf{R}(\theta))\mathbf{g}'(t)\\ &= (\mathbf{g}'(t))^T\mathbf{I}\mathbf{g}'(t) \quad \text{this equality holds since by Remark 2.1, } \mathbf{R}^T\mathbf{R} = \mathbf{I},\\ &= |\mathbf{g}'(t)|^2 = G_1(t), \quad \text{for some function } G_1(t). \end{aligned}$$

Similarly, since

$$\mathbf{h}_\theta(\theta,t) = \mathbf{R}'(\theta)(\mathbf{g}(t)-\mathbf{b}), \quad \text{where } \mathbf{R}'(\theta) \text{ is the matrix resulted by taking each entry of } \mathbf{R}(\theta) \text{ with respect to } \theta,$$

$$\mathbf{R}'(\theta) = \begin{bmatrix} \frac{-(a_2^2+a_3^2)\sin(\theta)}{a_1^2+a_2^2+a_3^2} & \frac{a_1a_2\sin(\theta)}{a_1^2+a_2^2+a_3^2} - \frac{a_3\cos(\theta)}{\sqrt{a_1^2+a_2^2+a_3^2}} & \frac{a_1a_3\sin(\theta)}{a_1^2+a_2^2+a_3^2} + \frac{a_2\cos(\theta)}{\sqrt{a_1^2+a_2^2+a_3^2}}\\ \frac{a_1a_2\sin(\theta)}{a_1^2+a_2^2+a_3^2} + \frac{a_3\cos(\theta)}{\sqrt{a_1^2+a_2^2+a_3^2}} & \frac{-(a_1^2+a_3^2)\sin(\theta)}{a_1^2+a_2^2+a_3^2} & \frac{a_2a_3\sin(\theta)}{a_1^2+a_2^2+a_3^2} - \frac{a_1\cos(\theta)}{\sqrt{a_1^2+a_2^2+a_3^2}}\\ \frac{a_1a_3\sin(\theta)}{a_1^2+a_2^2+a_3^2} - \frac{a_2\cos(\theta)}{\sqrt{a_1^2+a_2^2+a_3^2}} & \frac{a_2a_3\sin(\theta)}{a_1^2+a_2^2+a_3^2} + \frac{a_1\cos(\theta)}{\sqrt{a_1^2+a_2^2+a_3^2}} & \frac{-(a_1^2+a_2^2)\sin(\theta)}{a_1^2+a_2^2+a_3^2} \end{bmatrix},$$

$$\begin{aligned} |\mathbf{h}_\theta(\theta,t)|^2 &= \mathbf{h}_\theta(\theta,t)\cdot\mathbf{h}_\theta(\theta,t) = (\mathbf{R}'(\theta)(\mathbf{g}(t)-\mathbf{b}))^T(\mathbf{R}(\theta)(\mathbf{g}(t)-\mathbf{b})) = (\mathbf{g}(t)-\mathbf{b})^T(\mathbf{R}'^T(\theta)\mathbf{R}'(\theta))(\mathbf{g}(t)-\mathbf{b})\\ &= (\mathbf{g}(t)-\mathbf{b})^T \begin{bmatrix} \frac{a_2^2+a_3^2}{a_1^2+a_2^2+a_3^2} & \frac{-a_1a_2}{a_1^2+a_2^2+a_3^2} & \frac{-a_1a_3}{a_1^2+a_2^2+a_3^2}\\ \frac{-a_1a_2}{a_1^2+a_2^2+a_3^2} & \frac{a_1^2+a_3^2}{a_1^2+a_2^2+a_3^2} & \frac{-a_2a_3}{a_1^2+a_2^2+a_3^2}\\ \frac{-a_1a_3}{a_1^2+a_2^2+a_3^2} & \frac{-a_2a_3}{a_1^2+a_2^2+a_3^2} & \frac{a_1^2+a_2^2}{a_1^2+a_2^2+a_3^2} \end{bmatrix} (\mathbf{g}(t)-\mathbf{b}) = G_2(t), \quad \text{for some function } G_2(t). \end{aligned}$$

□

Therefore, the following corollary is a direct consequence of Proposition 3.1, that is, by taking the partial derivatives of a given parametric surface, we can eliminate the parametrization which is not a surface of revolution.

Corollary 3.1. *Given a parametric surface* $\mathbf{H}(\theta,t) = (H_1,H_2,H_3)$*, if* $|\mathbf{H}_\theta|^2 \neq G_1(t)$ *or* $|\mathbf{H}_t|^2 \neq G_2(t)$ *for some functions* G_1, G_2*, then* $\mathbf{H}(\theta,t)$ *is not a surface of revolution.*

Thus, given a parametric equation of a surface $\mathbf{H}(\theta,t)$, if one of the partial derivative, say $\mathbf{H}_t$, of a parametric surface satisfies Proposition 3.1, then the parametric surface may be a surface of revolution, and the parameter θ may determine angle of the rotation about the axis. To determine whether this surface is indeed a surface of revolution, we need to locate the rotation axis, and determine the directrix curve. Theorem 3.1 below provides a condition for a parametric surface to be a surface of revolution, and formulates the parametric equations for the rotation axis and directrix curve.

Theorem 3.1. *Given a parametric surface* $\mathbf{H}(\theta,t) = (H_1,H_2,H_3)$*, and suppose* $|\mathbf{H}_t|^2 = G_1(t)$ *and* $|\mathbf{H}_\theta|^2 = G_2(t)$ *for some* G_1, G_2*. Suppose there exist* $a_1,a_2,a_3,b_1,b_2,b_3 \in \mathbb{R}$ *satisfying the following two conditions:*

(a) $\sum_{i=1}^{3} a_iH_i(\theta,t) = H(t)$ *for some function* $H(t)$*; and*

(b) *for any* $c \in \mathbb{R}$*, if* t^* *is a solution to* $H(t) = c$*, then*

$$\left|\mathbf{H}(\theta,t^*) - \left(\frac{a_1c - \sum_{i=2}^{3} a_i(a_1b_i - b_1a_i)}{\sum_{i=1}^{3} a_i^2}, \frac{a_2c - \sum_{i=1,i\neq 2}^{3} a_i(a_2b_i - b_2a_i)}{\sum_{i=1}^{3} a_i^2}, \frac{a_3c - \sum_{i=1}^{2} a_i(a_3b_i - b_3a_i)}{\sum_{i=1}^{3} a_i^2}\right)\right|$$

$= \kappa_c$, *a constant depending on* c.

Then $\mathbf{H}(\theta,t)$ *is a surface of revolution generated by rotating the directrix curve* $\mathbf{g}(t)$ *about the axis line* $\mathbf{f}(s) = (a_1 s + b_1, a_2 s + b_2, a_3 s + b_3)$*. Furthermore, the directrix curve is*

$$\mathbf{g}(t) = (\mathbf{R}(\theta))^T (H_1 - b_1, H_2 - b_2, H_3 - b_3)^T + (b_1, b_2, b_3)^T, \quad \text{where } \mathbf{R}(\theta) \text{ is defined in Equation (2.2).}$$

Proof. Suppose there exist $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$ satisfying two conditions in the statement. Let

$$\mathbf{f}(s) = (a_1 s + b_1, a_2 s + b_2, a_3 s + b_3), \quad \text{and} \quad P(x,y,z) = a_1 x + a_2 y + a_3 z = c,$$

where the plane $P(x,y,z) = c$ is perpendicular to the line $\mathbf{f}(s)$. Let $\mathbf{f}(s^*)$ denote the intersection of the line $\mathbf{f}(s)$ and the plane $P(x,y,z) = c$, where s^* is the solution to the equation $P(\mathbf{f}(s)) = c$. Note that

$$P(\mathbf{f}(s)) = a_1(a_1 s + b_1) + a_2(a_2 s + b_2) + a_3(a_3 s + b_3) = \left(\sum_{i=1}^{3} a_i^2\right) s + \sum_{i=1}^{3} a_i b_i = c$$

yields that $s^* = \dfrac{c - \sum_{i=1}^3 a_i b_i}{\sum_{i=1}^3 a_i^2}$, which, in turn, implies that

$$\mathbf{f}(s^*) = \left(\frac{a_1 c - \sum_{i=2}^3 a_i(a_1 b_i - b_1 a_i)}{\sum_{i=1}^3 a_i^2}, \frac{a_2 c - \sum_{i=1, i\neq 2}^3 a_i(a_2 b_i - b_2 a_i)}{\sum_{i=1}^3 a_i^2}, \frac{a_3 c - \sum_{i=1}^2 a_i(a_3 b_i - b_3 a_i)}{\sum_{i=1}^3 a_i^2} \right).$$

As c varies, the point $\mathbf{f}(s^*)$ moves along the line $\mathbf{f}(s)$.

The plane $P(x,y,z) = c$ intersects the parametric surface $\mathbf{H}(\theta,t)$ at a plane curve $\mathcal{C}$. By the first condition $\sum_{i=1}^3 a_i H_i(\theta,t) = H(t)$, the collection of points on the parametric surface $\mathbf{H}(\theta,t)$ under the constraint $P(\mathbf{H}(\theta,t)) = a_1 H_1 + a_2 H_2 + a_3 H_3 = H(t) = c$ corresponds to the plane curve $\mathcal{C}$. That is, the plane curve $\mathcal{C}$ can be parametrically represented as $\mathbf{H}(\theta,t^*)$ where t^* is a solution to $H(t) = c$. Furthermore, the second condition is equivalent to $|\mathbf{H}(\theta,t^*) - \mathbf{f}(s^*)| = \kappa_c$, which implies that the plane curve $\mathcal{C}$ is a circle centered at $\mathbf{f}(s^*)$ with radius κ_c in the plane $P(x,y,z) = c$. Thus, the parametric surface $\mathbf{H}(\theta,t)$ is in fact a collection of the moving plane circles centered on the line $\mathbf{f}(s)$ with radius κ_c in the plane $P(x,y,z) = c$ for all $c \in \mathbb{R}$. Therefore, the surface $\mathbf{H}(\theta,t)$ is a surface of revolution generated by some directrix curve $\mathbf{g}(t)$ about the line of axis $\mathbf{f}(s)$.

Finally, by Theorem 2.2, $\mathbf{H}(\theta,t) = (H_1, H_2, H_3)^T = (b_1, b_2, b_3)^T + \mathbf{R}(\theta)(g_1(t) - b_1, g_2(t) - b_2, g_3(t) - b_3)$. Hence, we finalize the directrix curve $\mathbf{g}(t)$ by solving the above equation for (g_1, g_2, g_3). Thus, the directrix curve

$$\mathbf{g}(t) = (\mathbf{R}(\theta))^T (H_1 - b_1, H_2 - b_2, H_3 - b_3)^T + (b_1, b_2, b_3)^T.$$

□

We will use the next example to illustrate Theorem 3.1.

Example 3.1. Is $\mathbf{H}(\theta,t)$ a surface of revolution, where

$$\mathbf{H}(\theta,t) = \left(\frac{\cos(\theta)\cdot(4t^2+6)}{t^2+1} - 1, \frac{2t}{t^2+1}, \frac{-\sin(\theta)\cdot(4t^2+6)}{t^2+1} \right).$$

First, to eliminate the possibility that the given parametric surface is not a surface of revolution, we take partial derivatives with respect to θ and t, and we verify that

$$\sum_{i=1}^{3}\left(\frac{\partial H_i(\theta,t)}{\partial t}\right)^2 = \frac{[8t(t^2+1) - 2t(4t^2+6)]^2 + (2-2t^2)^2}{(t^2+1)^4} = \frac{4}{(t^2+1)^2}, \quad \sum_{i=1}^{3}\left(\frac{\partial H_i(\theta,t)}{\partial \theta}\right)^2 = \frac{(4t^2+6)^2}{(t^2+1)^2}.$$

Therefore, by Corollary 3.1, it is possible that $\mathbf{H}(\theta,t)$ is a surface of revolution.

To confirm that $\mathbf{H}(\theta,t)$ is a surface of revolution, applying Theorem 3.1, we check that

$$a_1 H_1(\theta,t) + a_2 H_2(\theta,t) + a_3 H_3(\theta,t) = \frac{a_1((4t^2+6)\cos(\theta) - t^2 - 1)}{t^2+1} + \frac{2a_2 t}{t^2+1} + \frac{-a_3(4t^2+6)\sin(\theta)}{t^2+1},$$

and if $a_1 = a_3 = 0$ and $a_2 = 1$, then $a_1 H_1(\theta,t) + a_2 H_2(\theta,t) + a_3 H_3(\theta,t) = \frac{2t}{t^2+1} = H(t)$. Furthermore, to solve the equation $H(t) = \frac{2t}{t^2+1} = c$ for t, we only need to solve $ct^2 - 2t + c = 0$ for t. Thus

$$\begin{cases} t^* = 0, & \text{when } c = 0, \text{ or} \\ t^* = \frac{2\pm\sqrt{4-4c^2}}{2c} = \frac{1\pm\sqrt{1-c^2}}{c}, & \text{when } c \in [-1,0)\cup(0,1]. \end{cases}$$

Let $(a_1, a_2, a_3) = (0, 1, 0)$ and $c = 0$, then $t^* = 0$ and

$$\begin{aligned}
&\left|\mathbf{H}(\theta, t^*) - \left(\frac{a_1 c - \sum_{i=2}^{3} a_i(a_1 b_i - b_1 a_i)}{\sum_{i=1}^{3} a_i^2}, \frac{a_2 c - \sum_{i=1, i\neq 2}^{3} a_i(a_2 b_i - b_2 a_i)}{\sum_{i=1}^{3} a_i^2}, \frac{a_3 c - \sum_{i=1}^{2} a_i(a_3 b_i - b_3 a_i)}{\sum_{i=1}^{3} a_i^2}\right)\right|^2 \\
= \ & |\mathbf{H}(\theta, 0) - (b_1, 0, b_3)|^2 = |(6\cos(\theta) - 1, 0, -6\sin(\theta) - (b_1, 0, b_3)|^2 \\
= \ & (6\cos(\theta) - 1 - b_1)^2 + (-6\sin(\theta) - b_3)^2.
\end{aligned}$$

Now, let $(b_1, b_2, b_3) = (-1, 0, 0)$, then $(6\cos(\theta) - 1 - b_1)^2 + (-6\sin(\theta) - b_3)^2 = 36$ is a constant.
Furthermore, applying $(a_1, a_2, a_3) = (0, 1, 0)$ and $(b_1, b_2, b_3) = (-1, 0, 0)$ to the case $c \in [-1, 0) \cup (0, 1]$,

$$\begin{aligned}
&\left|\mathbf{H}(\theta, t^*) - \left(\frac{a_1 c - \sum_{i=2}^{3} a_i(a_1 b_i - b_1 a_i)}{\sum_{i=1}^{3} a_i^2}, \frac{a_2 c - \sum_{i=1, i\neq 2}^{3} a_i(a_2 b_i - b_2 a_i)}{\sum_{i=1}^{3} a_i^2}, \frac{a_3 c - \sum_{i=1}^{2} a_i(a_3 b_i - b_3 a_i)}{\sum_{i=1}^{3} a_i^2}\right)\right|^2 \\
= \ & |\mathbf{H}(\theta, t^*) - (-1, c, 0)|^2 \\
= \ & \left(\frac{\cos(\theta)\cdot(4(t^*)^2 + 6)}{(t^*)^2 + 1} - 1 + 1\right)^2 + \left(\frac{2t^*}{(t^*)^2 + 1} - c\right)^2 + \left(\frac{-\sin(\theta)\cdot(4(t^*)^2 + 6)}{(t^*)^2 + 1}\right)^2 \\
= \ & \left(\frac{\cos(\theta)\cdot(4(t^*)^2 + 6)}{(t^*)^2 + 1}\right)^2 + \left(\frac{-\sin(\theta)\cdot(4(t^*)^2 + 6)}{(t^*)^2 + 1}\right)^2 \quad \text{the equality holds since } \frac{2t^*}{(t^*)^2 + 1} = c \\
= \ & \left(\frac{4(t^*)^2 + 6}{(t^*)^2 + 1}\right)^2 = \text{constant depending on } c, \quad \text{since } t^* = \frac{1 \pm \sqrt{1 - c^2}}{c} \quad \text{for } c \in [-1, 0) \cup (0, 1].
\end{aligned}$$

Thus, we confirm that $(a_1, a_2, a_3, b_1, b_2, b_3) = (0, 1, 0, -1, 0, 0)$ satisfy the two conditions of Theorem 3.1, hence $\mathbf{H}(\theta, t)$ is a surface of revolution, and the line of axis is $\mathbf{f}(s) = (-1, s, 0)$. Furthermore, since $(a_1, a_2, a_3) = (0, 1, 0)$, by Remark 2.1, $\mathbf{R}(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$. Again, by Theorem 3.1, the directrix curve

$$\mathbf{g}(t) = \mathbf{R}^T(\theta)\begin{bmatrix} H_1 - b_1 \\ H_2 - b_2 \\ H_3 - b_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}\begin{bmatrix} \frac{\cos(\theta)\cdot(4t^2+6)}{t^2+1} \\ \frac{2t}{t^2+1} \\ \frac{-\sin(\theta)\cdot(4t^2+6)}{t^2+1} \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3t^2+5}{t^2+1} \\ \frac{2t}{t^2+1} \\ 0 \end{bmatrix}.$$

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