Results in Nonlinear Analysis **5** (2022) No. 4, 487–498.
https://doi.org/10.53006/rna.1065458
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

Geraghty Proximal Contraction Type Mappings and RJ-Property in $b_v(s)$ -Metric Spaces

Leta Bekere Kumssa[a]

[a] *Department of Mathematics, Madda Walabu University, Bale Robe, Ethiopia.*

Abstract

In this article, we determine the best proximity point results for the Geraghty proximal contraction type mappings in a more general space called the $b_v(s)$-metric space, and prove the existence of the best proximity point for such mappings which satisfy the RJ-property. We also derive some consequences as a justification for the validity of the main result. The results presented here extend, generalize, and integrate many previous results in the literature.

Keywords: best proximity point Geraghty proximal Contraction RJ-property $b_v(s)$-metric space.
2010 MSC: 47H10, 54H25.

1. Introduction

Banach in [7] proposed the important principle which plays a vital role in the advancement of fixed point theory. In this work, he asserted that any contraction self-mapping defined on a complete metric space has only one fixed point. Later, this principle has been generalized and extended in many aspects. Generalization or an extension of the Banach contraction principle is to change the contraction conditions or change the display space. Fifty years later, in 1973, Geraghty [17] became popular by generalizing Banach's result by modifying the contraction constant and replacing it with a function of certain defined properties. Also, Geraghty contraction has been extended and generalized in different aspects in [10, 14, 18, 21]. All of the above assertions are valid only for self-mapping.

In 1969, one of the most important generalization of Banach [7] contraction principle is presented by Fan [15] which is known as best approximation theorem.

Theorem 1.1. *[15] Let A be a nonempty compact convex subset of a normed linear space X and $T : A \to X$ be a continuous function. Then there exists $x \in A$ such that $||x - Tx|| = d(Tx, A) = inf\{||Tx - a|| : a \in A\}$.*

Email address: letabekere@yahoo.com (Leta Bekere Kumssa)

Received :January 30, 2022; Accepted: October 29, 2022; Online: November 10, 2022

In recent studies, this fact has attracted the attention of several authors to deal with non self-mapping. If $\mathcal{N} \neq \emptyset$ and $\mathcal{R} \neq \emptyset$ are subsets of the metric space $(\mathcal{M}, \delta)$ and $S : \mathcal{N} \to \mathcal{R}$ is a mapping, then $\delta(p, Sp) \geq \delta(\mathcal{N}, \mathcal{R})$ for all $p \in \mathcal{N}$. In the map $S : \mathcal{N} \to \mathcal{R}$, if there is no solution for the equation $Sp = p$, then the aim here is to find an element $p \in \mathcal{N}$ which is an approximate solution that minimizes the error $\delta(p, Sp)$, possibly $\delta(p, Sp) = \delta(\mathcal{N}, \mathcal{R})$. In case $\delta(p, Sp) = \delta(\mathcal{N}, \mathcal{R})$, we call p is the best proximity point of S, where $\delta(\mathcal{N}, \mathcal{R}) = \inf\{\delta(p, q) : p \in \mathcal{N}, q \in \mathcal{R}\}$. In recent years, the idea of the best proximity point is the area of attention for many authors in [1, 2, 3, 8, 9, 26]. Geraghty contraction has also been extended to the case of non self-mapping.

In this article, we look at a more general space and prove the best proximity point result for Geraghty proximal contraction type.

2. Preliminaries

In this this study, $\mathbb{R}_+$ and $\mathbb{Z}^+$ represent all sets of non-negative real numbers and all sets of positive integers respectively.

Consider the following notations and definitions. Let $(\mathcal{M}, \delta)$ be the metric space and let $\mathcal{N} \neq \emptyset$ and $\mathcal{R} \neq \emptyset$ are subsets of $\mathcal{M}$.

$$\mathcal{N}_0 := \{p \in \mathcal{N} : \delta(p, q) = \delta(\mathcal{N}, \mathcal{R}) \text{ for some } \ q \in \mathcal{R}\},$$

$$\mathcal{R}_0 := \{q \in \mathcal{R} : \delta(p, q) = \delta(\mathcal{N}, \mathcal{R}) \text{ for some } p \in \mathcal{N}\}.$$

Definition 2.1. *[25]The mapping $S : \mathcal{M} \to \mathcal{M}$ is said to be α-admissible,*

$$\text{if } \alpha(p, q) \geq 1, \text{ then } \alpha(Sp, Sq) \geq 1,$$

provided that $\alpha : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ is a function, where p and q are any points in $\mathcal{M}$.

Definition 2.2. *[22] The mapping $S : \mathcal{M} \to \mathcal{M}$ is called a triangular α-admissible, if for all $p, q, r \in \mathcal{M}$ we have*

(i) S is α-admissible.

(ii) $\alpha(p, q) \geq 1$ and $\alpha(q, r) \geq 1 \implies \alpha(p, r) \geq 1$, provided that $\alpha : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ is a function.

Definition 2.3. *[20] Given that $\mathcal{N} \neq \emptyset$ and $\mathcal{R} \neq \emptyset$ are subsets of the metric space $(\mathcal{M}, \delta)$. Given $\alpha : \mathcal{N} \times \mathcal{N} \to \mathbb{R}_+$ is a function. The mapping $S : \mathcal{N} \to \mathcal{R}$ is said to be $\alpha-$proximal admissible*

$$\text{if } \begin{cases} \alpha(p, q) \geq 1 \\ \delta(u, Sp) = \delta(\mathcal{N}, \mathcal{R}) \\ \delta(v, Sq) = \delta(\mathcal{N}, \mathcal{R}), \end{cases} \text{ then } \alpha(u, v) \geq 1,$$

for all $p, q, u, v \in \mathcal{N}$.

Definition 2.4. *[23] Given that $\mathcal{N} \neq \emptyset$ and $\mathcal{R} \neq \emptyset$ are subsets of the metric space $(\mathcal{M}, \delta)$ and $\alpha : \mathcal{N} \times \mathcal{N} \to \mathbb{R}_+$ be a function. A mapping $S : \mathcal{N} \to \mathcal{R}$ is called a triangular α-proximal admissible if for all $p, q, z, p_1, p_2, u_1, u_2 \in \mathcal{N}$,*

$$(S_1) \begin{cases} \alpha(p_1, p_2) \geq 1 \\ \delta(u_1, Sp_1) = \delta(\mathcal{N}, \mathcal{R}) \\ \delta(u_2, Sp_2) = \delta(\mathcal{N}, \mathcal{R}) \end{cases} \implies \alpha(u_1, u_2) \geq 1,$$

$$(S_2) \begin{cases} \alpha(p, z) \geq 1, \\ \alpha(z, q) \geq 1 \end{cases} \implies \alpha(p, q) \geq 1.$$

In 2016, Hamzehnejadi and Lashkaripour defined RJ-property.

Definition 2.5. *[19] Given that $\mathcal{N} \neq \emptyset$ and $\mathcal{R} \neq \emptyset$ are subsets of the metric space $(\mathcal{M}, \delta)$ and $S : \mathcal{N} \to \mathcal{R}$ be a mapping. For any sequence $\{p_n\} \subseteq \mathcal{N}$, S is said to have the RJ-property,*

$$\text{if } \lim_{n\to\infty} \delta(p_{n+1}, Sp_n) = \delta(\mathcal{N}, \mathcal{R}) \text{ and } \lim_{n\to\infty} p_n = p, \text{ then } p \in \mathcal{N}_0.$$

Definition 2.6. *[11] Let $\emptyset \neq \mathcal{M}$ be a set and $s \geq 1$ is a real number. Suppose that for all $p, q, r \in \mathcal{M}$ the mapping $\delta : \mathcal{M} \times \mathcal{M} \to \mathbb{R}_+$ satisfies the following conditions:*

(δ_1) $\delta(p, q) \geq 0$.

(δ_2) $\delta(p, q) = 0 \iff p = q$.

(δ_3) $\delta(p, q) = \delta(q, p)$.

(δ_4) $\delta(p, r) \leq s[\delta(p, q) + \delta(q, r)]$ *(b-triangular inequality).*

If δ satisfies conditions (δ_1)-(δ_4), then δ is known as b- metric on $\mathcal{M}$. The couple $(\mathcal{M}, \delta)$ is named as b-metric space.

After the introduction of the b-metric spaces, the generalized versions were introduced. These include extended b-metric spaces, rectangular b-metric spaces, $b_v(s)$-metric spaces, and more.

Definition 2.7. *[16] Let $\emptyset \neq \mathcal{M}$ is a set and $s \geq 1$ be a fixed real number. Let $\delta : \mathcal{M} \times \mathcal{M} \to \mathbb{R}_+$ be a mapping such that*

(δ_1) $\delta(p, q) = \delta(q, p) = 0 \iff p = q$,

(δ_2) $\delta(p, q) = \delta(q, p)$,

(δ_3) $\delta(p, q) \leq s[\delta(p, r) + \delta(r, t) + \delta(t, q)]$ *(b-rectangular inequality),*

for every $p, q \in M$ and distinct points $r, t \in \mathcal{M}$, each is different from p and q. In this case, δ is called a b-rectangular metric, and the pair $(\mathcal{M}, \delta)$ is called a b-rectangular metric space.

In this paper, $\mathcal{F}_s$ represents the class of all functions $\{\beta : [0, \infty) \to [0, \frac{1}{s}),\ s \geq 1$ and $\lim_{n\to\infty} \sup \beta(h_n) = \frac{1}{s} \implies \lim_{n\to\infty} h_n = 0\}$.

In the year 2017, Mitrovic and Radenovic [24] introduced a more general version of b-metric space called $b_v(s)$-metric space.

Definition 2.8. *[24] Let $\emptyset \neq \mathcal{M}$ is a set. Let $\delta : \mathcal{M} \times \mathcal{M} \to \mathbb{R}_+$ be a mapping and $v \in \mathbb{Z}^+$, $s \geq 1$ be a constant such that*

(δ_1) $\delta(p, q) = \delta(q, p) = 0$ *if and only if* $p = q$,

(δ_2) $\delta(p, q) = \delta(q, p)$,

(δ_3) $\delta(p, q) \leq s[\delta(p, u_1) + \delta(u_1, u_2) + ... + \delta(u_v, q)]$ *($b_v(s)$-metric inequality),*

for all $p, q \in \mathcal{M}$ and all distinct points $u_1, u_2, ..., u_v \in \mathcal{M}$, each is different from p and q, then we call $(\mathcal{M}, \delta)$ is the $b_v(s)$-metric space.

Example 2.9. *Given that* $\mathcal{M} = \{(0, \frac{1}{n}) : n \in \{2, 3, 4, 5, ...\}\}$,

We define $\delta : \mathcal{M} \times \mathcal{M} \to \mathbb{R}_+$ *by*

$$\delta\big((0, \tfrac{1}{h}), (0, \tfrac{1}{t})\big) = \begin{cases} 0 & \text{if } h = t, \\ 18\lambda & \text{if } h = 2, t = 3, \text{ or if } h = 3, t = 2, \\ \lambda & \text{if } h \in \{2,3,4\}, t \in \{5\} \text{ or} \\ & \text{if } h \in \{5\}, t \in \{2,3,4\}, \\ 3\lambda & \text{if } h \in \{2,3,4,5\}, t \in \{6\} \text{ or} \\ & \text{if } h \in \{6\}, t \in \{2,3,4,5\}, \\ 2\lambda & \text{if } h \in \{2,3,4,5,6\}, t \in \{7\} \text{ or} \\ & \text{if } h \in \{7\}, t \in \{2,3,4,5,6\}, \\ \frac{3}{2}\lambda & \text{if } h \in \{2,3,4,5,6,7\}, t \in \{8\} \text{ or} \\ & \text{if } h \in \{8\}, t \in \{2,3,4,5,6,7\}, \\ 4\lambda & \text{if } h \text{ or } t \notin \{2,3,4,5,6,7,8\}\}, \end{cases}$$

where $\lambda \in (0, \infty)$ *is a constant.*
Now,

$$\begin{aligned} \delta((0, \frac{1}{2}), (0, \frac{1}{3})) = 18\lambda &\le 2[\lambda + 3\lambda + 2\lambda + \frac{3}{2}\lambda + \frac{3}{2}\lambda] \\ &= 2[\delta((0, \frac{1}{2}), (0, \frac{1}{5})) + \delta((0, \frac{1}{5}), (0, \frac{1}{6})) + \delta((0, \frac{1}{6}), (0, \frac{1}{7})) \\ &\quad + \delta((0, \frac{1}{7}), (0, \frac{1}{8})) + \delta((0, \frac{1}{8}), (0, \frac{1}{3}))]. \end{aligned}$$

Therefore, $(\mathcal{M}, \delta)$ *is a* $b_4(2)$*-metric space.*

Definition 2.10. *[24] Let the couple* $(\mathcal{M}, \delta)$ *be the* $b_v(s)$*-metric space,* (p_k) *the sequence of* $\mathcal{M}$*, and* $p \in \mathcal{M}$*. Then*

(i) the sequence (p_k) *converges to* p *in* $(\mathcal{M}, \delta)$ *if for any* $\gamma > 0$ *there is* $N_0 = N_0(\gamma) \in \mathbb{Z}^+$ *such that* $\delta(p_k, p) \le \gamma$ *for all* $k \ge N_0$ *and this fact is expressed as* $\lim_{k\to\infty} p_k = p$,

(ii) the sequence (p_k) *is Cauchy if for any* $\gamma > 0$ *there is* $N_0 = N_0(\gamma) \in \mathbb{Z}^+$ *such that* $\delta(p_k, p_l) \le \gamma$ *for all* $k, l > N_0$,

(iii) $(\mathcal{M}, \delta)$ *is called complete* $b_v(s)$*-metric space if any sequence* $\{p_n\}$ *of* $\mathcal{M}$ *converges to a point* $p \in M$ *as* $n \to \infty$.

Some of the recent fixed point results in $b_v(s)$ can be found on [4, 5, 6, 12, 13] and references cited in these papers.
In line with the definitions of the α-proximal admissible, the triangular α-proximal admissible and the RJ-property in metric spaces, it is possible to define the same in $b_v(s)$-metric spaces.

Definition 2.11. *Given that* $\mathcal{N} \neq \emptyset$ *and* $\mathcal{R} \neq \emptyset$ *are subsets of the* $b_v(s)$ *-metric space* $(\mathcal{M}, \delta)$ *and* $S : \mathcal{N} \to \mathcal{R}$ *be a mapping. For any sequence* $\{p_n\} \subseteq \mathcal{N}$*, we call* S *has the RJ-property*

$$\text{if } \lim_{n\to\infty} \delta(p_{n+1}, Sp_n) = \delta(\mathcal{N}, \mathcal{R}) \text{ and } \lim_{n\to\infty} p_n = p, \text{ then } p \in \mathcal{N}_0.$$

Example 2.12. *Given that* $\mathcal{M} = \{(0, \frac{1}{n}) : n \in \{2, 3, 4, 5, ...\}\} \cup \{(0, 0)\}$,

$$\begin{aligned} &\mathcal{N} = \{(0, \frac{1}{n}) : n \in \{2, 4, 6, 8, ...\}\} \cup \{(0, 0)\} = \mathcal{N}_0 \quad \text{and} \\ &\mathcal{R} = \{(0, \frac{1}{m}) : m \in \{3, 5, 7, ...\}\} = \mathcal{R}_0. \end{aligned}$$

We define $\delta : \mathcal{M} \times \mathcal{M} \to \mathbb{R}_+$ *by*

$$\delta\big((0,h),(0,t)\big) = \begin{cases} 0 & \text{if } h = t, \\ 24 & \text{if } h = \frac{1}{2}, t = \frac{1}{3}, \text{ or if } h = \frac{1}{3}, t = \frac{1}{2}, \\ 4 & \text{if } h \in \{\frac{1}{2},\frac{1}{3},\frac{1}{4}\}, t \in \{\frac{1}{5}\} \text{ or} \\ & \text{if } h \in \{\frac{1}{5}\}, t \in \{\frac{1}{2},\frac{1}{3},\frac{1}{4}\}, \\ 3 & \text{if } h \in \{\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5}\}, t \in \{\frac{1}{6}\} \text{ or} \\ & \text{if } h \in \{\frac{1}{6}\}, t \in \{\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5}\}, \\ \frac{5}{2} & \text{if } h \in \{\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\frac{1}{6}\}, t \in \{\frac{1}{7}\} \text{ or} \\ & \text{if } h \in \{\frac{1}{7}\}, t \in \{\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\frac{1}{6}\}, \\ \frac{3}{2} & \text{if } h \text{ or } t \notin \{\frac{1}{n}\}, \text{where } n = 2,3,4,5,6,7. \end{cases}$$

Now,

$$\begin{aligned} \delta((0,\frac{1}{2}),(0,\frac{1}{3})) = 24 &\le 2[4 + 3 + \frac{5}{2} + \frac{5}{2}] \\ &= 2[\delta((0,\frac{1}{2}),(0,\frac{1}{5})) + \delta((0,\frac{1}{5}),(0,\frac{1}{6})) + \delta((0,\frac{1}{6}),(0,\frac{1}{7})) \\ &\quad + \delta((0,\frac{1}{7}),(0,\frac{1}{3}))]. \end{aligned}$$

Therefore, $(\mathcal{M},\delta)$ *is a* $b_3(2)$*-metric space.*
Define $S : \mathcal{N} \to \mathcal{R}$ *by* $S(0,0) = (0,\frac{1}{3})$ *and* $S(0,\frac{1}{2n}) = (0,\frac{1}{2n+1})$ *for all* $n \ge 1$.
Let $p_n = \{(0,\frac{1}{2n})\} \subseteq N$ *for all* $n \ge 1$.

$$\lim_{n\to\infty} \delta\big((0,p_{n+1}),(0,Sp_n)\big) = \lim_{n\to\infty} \delta\big((0,\frac{1}{2n+2}),(0,\frac{1}{2n+1})\big) = \frac{3}{2} = \delta(\mathcal{N},\mathcal{R}).$$

and $\lim_{n\to\infty}(0,\frac{1}{2n}) = (0,0) \in \mathcal{N}_0$. *Therefore, S has RJ-property.*

In this study, we establish the notion of Geraghty proximal contraction type and prove the existence of best proximity point for mappings which satisfy the RJ-property in $b_v(s)$ -metric spaces.

3. Main Results

Theorem 3.1. *Let* $\mathcal{N} \ne \emptyset$ *and* $\mathcal{R} \ne \emptyset$ *be subsets of a* $b_v(s)$*- metric space*$(\mathcal{M},\delta)$ *with a constant* $s \ge 1$ *and* $\alpha : \mathcal{N} \times \mathcal{N} \to \mathbb{R}_+$ *be a function. Let* $S : \mathcal{N} \to \mathcal{R}$ *is a mapping that has the RJ-property. Suppose we have* $\beta \in \mathcal{F}_s$ *such that the following conditions hold for all* $p,q,u,v \in \mathcal{N}$.

(i)
$$\left.\begin{aligned} \delta(u,Sp) = \delta(\mathcal{N},\mathcal{R}) \\ \delta(v,Sq) = \delta(\mathcal{N},\mathcal{R}) \end{aligned}\right\}$$

$$\Longrightarrow \alpha(p,q)\delta(u,v) \le \beta\big(L(p,q,u,v)\big)L(p,q,u,v), \tag{1}$$

where $L(p,q,u,v) = max\,\{\delta(p,q),\delta(p,u),\delta(q,v)\}$,

(ii) $S(\mathcal{N}_0)$ *is a subset of* $\mathcal{R}_0$ *and S is a triangular* $\alpha-$*proximal admissible,*

(iii) if $\{p_n\}$ *is a sequence of* $\mathcal{N}$ *such that* $\alpha(p_n,p_{n+1}) \ge 1$ *for all n and* $p_n \to p \in \mathcal{N}$ *as* $n \to \infty$*, then for some sub-sequence* $\{p_{n_k}\}$ *of* $\{p_n\}$*, we have* $\alpha(p_{n_k},p) \ge 1$ *for all k,*

(iv) there are points $p_0,p_1 \in \mathcal{N}$ *such that* $\delta(p_1,Sp_0) = \delta(\mathcal{N},\mathcal{R})$ *and* $\alpha(p_0,p_1) \ge 1$.

Then there exists a best proximity point for S.

Proof. According to assumption (iv), there are points $p_0,\ p_1 \in \mathcal{N}$ such that

$$\delta(p_1, Sp_0) = \delta(\mathcal{N}, \mathcal{R}) \text{ and } \alpha(p_0, p_1) \geq 1. \tag{2}$$

Since $Sp_0 \in \mathcal{R}$, by the definition of $\mathcal{N}_0$ and from (2), we have $p_1 \in \mathcal{N}_0$. Since $S(\mathcal{N}_0) \subseteq \mathcal{R}_0$, we have $Sp_1 \in \mathcal{R}_0$. Hence by definition of $\mathcal{R}_0$, there exists an element $p_2 \in \mathcal{N}$ such that

$$\delta(p_2, Sp_1) = \delta(\mathcal{N}, \mathcal{R}). \tag{3}$$

By triangular α-proximal admissibility of S, we obtain $\alpha(p_1, p_2) \geq 1$. Continuing this process, we have that

$$\delta(p_{n+1}, Sp_n) = \delta(\mathcal{N}, \mathcal{R}) \text{ and } \alpha(p_n, p_{n+1}) \geq 1 \tag{4}$$

for all $n \in \mathbb{Z}^+ \cup \{0\}$.
Therefore, for all $n \in \mathbb{Z}^+$, we have

$$\begin{aligned} &\alpha(p_{n-1}, p_n) \geq 1 \\ &\delta(p_n, Sp_{n-1}) = \delta(\mathcal{N}, \mathcal{R}) \\ &\delta(p_{n+1}, Sp_n) = \delta(\mathcal{N}, \mathcal{R}). \end{aligned} \tag{5}$$

According to (1), we have

$$\begin{aligned} \delta(p_n, p_{n+1}) &\leq \alpha(p_{n-1}, p_n)\delta(p_n, p_{n+1}) \\ &\leq \beta\big(L(p_{n-1}, p_n, p_n, p_{n+1})\big)L(p_{n-1}, p_n, p_n, p_{n+1}) \\ &< \frac{1}{s}L(p_{n-1}, p_n, p_n, p_{n+1}), \end{aligned} \tag{6}$$

where

$$\begin{aligned} L(p_{n-1}, p_n, p_n, p_{n+1}) &= \max\{\delta(p_{n-1}, p_n), \delta(p_{n-1}, p_n), \delta(p_n, p_{n+1})\} \\ &= \max\{\delta(p_{n-1}, p_n), \delta(p_n, p_{n+1})\}. \end{aligned} \tag{7}$$

Suppose $p_{n_0-1} = p_{n_0}$ for some $n_0 \in \mathbb{Z}^+$. If possible, assume that $p_{n_0} \neq p_{n_0+1}$.
According to (6) and (7), it follows that

$$\begin{aligned} \delta(p_{n_0}, p_{n_0+1}) &\leq \alpha(p_{n_0-1}, p_{n_0})\delta(p_{n_0}, p_{n_0+1}) \\ &\leq \beta\big(L(p_{n_0-1}, p_{n_0}, p_{n_0}, p_{n_0+1})\big)L(p_{n_0-1}, p_{n_0}, p_{n_0}, p_{n_0+1}), \\ &< \frac{1}{s}L(p_{n_0-1}, p_{n_0}, p_{n_0}, p_{n_0+1}), \end{aligned} \tag{8}$$

where

$$\begin{aligned} L(p_{n_0-1}, p_{n_0}, p_{n_0}, p_{n_0+1}) &= \max\{\delta(p_{n_0-1}, p_{n_0}), \delta(p_{n_0-1}, p_{n_0}), \delta(p_{n_0}, p_{n_0+1})\} \\ &= \delta(p_{n_0}, p_{n_0+1}). \end{aligned} \tag{9}$$

From (8) and (9), we get

$$\delta(p_{n_0}, p_{n_0+1}) < \delta(p_{n_0}, p_{n_0+1}).$$

This is a contradiction. Therefore, $p_{n_0} = p_{n_0+1}$.
Hence $p_{n_0-1} = p_{n_0} = p_{n_0+1}$ and so from (4), it follows that

$$\delta(p_{n_0}, Sp_{n_0}) = \delta(p_{n_0+1}, Sp_{n_0}) = \delta(N, R).$$

That is, p_{n_0} is the best proximity point in the vicinity of S, which is the desired result.

Therefore, assume that $p_{n-1} \neq p_n$ for all $n \in \mathbb{Z}^+$.

If $\max\{\delta(p_{n-1}, p_n), \delta(p_n, p_{n+1})\} = \delta(p_n, p_{n+1})$ in (7), then according to (6) and since $\beta \in \mathcal{F}_s$, we get a contradiction.

Therefore, $\max\{\delta(p_{n-1}, p_n), \delta(p_n, p_{n+1})\} = \delta(p_{n-1}, p_n)$. So, By (6) and (7), we have

$$\delta(p_n, p_{n+1}) < \frac{1}{s}\delta(p_{n-1}, p_n)) \leq \delta(p_{n-1}, p_n).$$

Thus

$$\delta(p_n, p_{n+1}) \leq \delta(p_{n-1}, p_n) \text{ for all } n \in \mathbb{Z}^+.$$

Therefore,we can infer that $\{\delta(p_n, p_{n+1})\}$ is a non-increasing sequence of non-negative real numbers. So, there is $l \geq 0$ such that

$$\lim_{n\to\infty} \delta(p_n, p_{n+1}) = t. \tag{10}$$

If possible, suppose that $t > 0$. Therefor, if the upper limit is taken as $n \to \infty$ in(6), it becomes as follows:

$$t \leq \lim_{n\to\infty} \sup \beta(\delta(p_{n-1}, p_n))t \leq \frac{1}{s}t. \tag{11}$$

Hence it is clear that

$$\frac{1}{s} \leq 1 \leq \lim_{n\to\infty} \sup \beta(\delta(p_{n-1}, p_n)) \leq \frac{1}{s}. \tag{12}$$

Since $\beta \in \mathcal{F}_s$ from (12), we get

$$\lim_{n\to\infty} \sup \beta(\delta(p_{n-1}, p_n)) = \frac{1}{s} \implies 0 = \lim_{n\to\infty} \delta(p_{n-1}, p_n) = t. \tag{13}$$

This is a contradiction. Hence $t = 0$

In the next step, we will show that $\{p_n\}$ is a $b_v(s)$-Cauchy sequence.

Now, assuming the opposite, $\{p_n\}$ is not a $b_v(s)$-Cauchy sequence. That is, $\lim_{n,m\to\infty} \delta(p_n, p_m) \neq 0$. Then there is $\gamma > 0$ and sub sequences $\{m_k\}$ and $\{n_k\}$ of $\{p_n\}$ for which $m_k > n_k + v, \quad n_k > k$,

$$\delta(p_{n_k}, p_{m_k}) \geq \gamma \tag{14}$$

and

$$\delta(p_{n_k+v-1}, p_{m_k-1}) < \gamma. \tag{15}$$

According to (14) and $b_v(s)$ -metric inequality, we obtain

$$\begin{aligned}\gamma \leq \delta(p_{n_k}, p_{m_k}) \leq s[&\delta(p_{n_k}, p_{n_k+1}) + \delta(p_{n_k+1}, p_{n_k+2}) + ... \\ &+ \delta(p_{n_k+v-1}, p_{n_k+v}) + \delta(p_{n_k+v}, p_{m_k})].\end{aligned} \tag{16}$$

Taking the upper limit as $k \to \infty$ in (16), we get

$$\frac{\gamma}{s} \leq \lim_{k\to\infty} \sup \delta(p_{n_k+v}, p_{m_k}). \tag{17}$$

By triangular α-proximal admissibility of S, we show that

$$\alpha(p_{n_k+v-1}, p_{m_k-1}) \geq 1 \text{ for } m_k > n_k + v, n_k \geq k. \tag{18}$$

By triangular α-proximal admissibility of S and if

$$\alpha(p_{n_k}, p_{n_k+1}) \geq 1, \quad \alpha(p_{n_k}, p_{n_k+2}) \geq 1,$$

we have $\alpha(p_{n_k}, p_{n_k+2}) \geq 1$. Hence by extending this process, (18) follows. That is,

$$\begin{aligned} &\alpha(p_{n_k+v-1}, p_{m_k-1}) \geq 1 \\ &\alpha(p_{m_k-2}, p_{m_k-1}) \geq 1 \implies \alpha(p_{n_k+v-1}, p_{m_k-1}) \geq 1. \end{aligned} \tag{19}$$

Now,

$$\begin{aligned} &\alpha(p_{n_k+v-1}, p_{m_k-1}) \geq 1 \\ &\delta(p_{n_k+v}, Sp_{n_k+v-1}) = \delta(\mathcal{N}, \mathcal{R}) \\ &\delta(p_{m_k}, Sp_{m_k-1}) = \delta(\mathcal{N}, \mathcal{R}). \end{aligned} \tag{20}$$

Hence according to (1), we have

$$\begin{aligned} \delta(p_{n_k+v}, p_{m_k}) &\leq \alpha(p_{n_k+v-1}, p_{m_k-1})\delta(p_{n_k+v}, p_{m_k}) \\ &\leq \beta\big(L(p_{n_k+v-1}, p_{m_k-1}, p_{n_k+v}, p_{m_k})\big) \\ &\qquad L(p_{n_k+v-1}, p_{m_k-1}, p_{n_k+v}, p_{m_k}) \\ &< \frac{1}{s} L(p_{n_k+v-1}, p_{m_k-1}, p_{n_k+v}, p_{m_k}), \end{aligned} \tag{21}$$

where

$$\begin{aligned} L(p_{n_k+v-1}, p_{m_k-1}, p_{n_k+v}, p_{m_k}) = \max\{&\delta(p_{n_k+v-1}, p_{m_k-1}), \delta(p_{n_k+v-1}, p_{n_k+v}), \\ &\delta(p_{m_k-1}, p_{m_k})\}. \end{aligned}$$

Thus,

$$\begin{aligned} \lim_{k\to\infty} \sup L(p_{n_k+v-1}, p_{m_k-1}, p_{n_k+v}, p_{m_k}) = \lim_{k\to\infty} \sup \max\{&\delta(p_{n_k+v-1}, p_{m_k-1}), \\ &\delta(p_{n_k+v-1}, p_{n_k+v}), \delta(p_{m_k-1}, p_{m_k})\} \leq \gamma. \end{aligned} \tag{22}$$

Now, taking the upper limit as $k \to \infty$ in (21), and using (17) and (22), we obtain

$$\frac{\gamma}{s} \leq \lim_{k\to\infty} \sup \beta(L(p_{n_k+v-1}, p_{m_k-1}, p_{n_k+v}, p_{m_k}))\gamma \leq \frac{1}{s}\gamma, \tag{23}$$

This implies that

$$\frac{1}{s} \leq \lim_{k\to\infty} \sup \beta(L(p_{n_k+v-1}, p_{m_k-1}, p_{n_k+v}, p_{m_k})) \leq \frac{1}{s}, \tag{24}$$

Since $\beta \in \mathcal{F}_s$ from (24), we get

$$\lim_{k\to\infty} L(p_{n_k+v-1}, p_{m_k-1}, p_{n_k+v}, p_{m_k}) = 0. \tag{25}$$

Therefore, $\{\delta(p_{n_k+v-1}, p_{m_k-1})\}$ converges to 0 as $k \to \infty$.
Now, by using (14) and $b_v(s)$ -metric inequality, we obtain

$$\begin{aligned} \gamma \leq \delta(p_{n_k}, p_{m_k}) \leq s[&\delta(p_{n_k}, p_{n_k+1}) + \delta(p_{n_k+1}, p_{n_k+2}) + ... \\ &+ \delta(p_{n_k+v-1}, p_{m_k-1}) + \delta(p_{m_k-1}, p_{m_k})]. \end{aligned} \tag{26}$$

Taking the limit as $k \to \infty$ in (26), we get

$$\lim_{k\to\infty} \delta(p_{n_k}, p_{m_k}) = 0. \tag{27}$$

This is a contradiction. Hence $\{p_n\}$ is Cauchy.
From completeness of $\mathcal{M}$, there is a point $p* \in \mathcal{M}$ such that $p_n \to p*$ as $n \to \infty$.
Also, since S has the RJ-property, we obtain $p* \in \mathcal{N}_0$. Since $S(p_0) \subseteq \mathcal{R}_0$, we have $S(p*) \in \mathcal{R}_0$. Therefore, there ia a point $q \in \mathcal{N}$ such that
$\delta(q, Sp*) = \delta(\mathcal{N}, \mathcal{R})$. We now prove that $p* = q$. Suppose $p* \neq q$.

$$\delta(p_{n_k+1}, q) \leq s[\delta(p_{n_k+1}, p_{n_k+2}) + ... + \delta(p_{n_k+v-1}, p_{n_k+v}) + \delta(p_{n_k+v}, q)]. \tag{28}$$

Letting $n \to \infty$ in (28), we have

$$\frac{1}{s}\delta(p*, q) \leq \delta(p*, q).$$

According to hypothesis (iii), there is a sub-sequence $\{p_{n_k}\}$ of $\{p_n\}$ such that $\alpha(p_{n_k}, p*) \geq 1$ for all k.
Since

$$\begin{aligned} &\alpha(p_{n_k}, p*) \geq 1 \\ &\delta(p_{n_k+1}, Sp_{n_k}) = \delta(\mathcal{N}, \mathcal{R}) \\ &\delta(q, Sp*) = \delta(\mathcal{N}, \mathcal{R}). \end{aligned} \tag{29}$$

From (1), it follows that

$$\begin{aligned} \delta(p_{n_k+1}, q) &\leq \alpha(p_{n_k}, p*)\delta(p_{n_k+1}, q) \\ &\leq \beta\big(L(p_{n_k}, p*, p_{n_k+1}, q))\big)L(p_{n_k}, p*, p_{n_k+1}, q) \\ &< \frac{1}{s}L(p_{n_k}, p*, p_{n_k+1}, q), \end{aligned} \tag{30}$$

where $L(p_{n_k}, p*, p_{n_k+1}, q) = \max\{\delta(p_{n_k}, p*), \delta(p_{n_k}, p_{n_k+1}), \delta(p*, q)\}$.
Hence

$$\begin{aligned} \lim_{k\to\infty} L(p_{n_k}, p*, p_{n_k+1}, q) &= \lim_{k\to\infty} \{\max\{\delta(p_{n_k}, p*), \delta(p_{n_k}, p_{n_k+1}), \delta(p*, q)\}\} \\ &= \delta(p*, q). \end{aligned}$$

On letting $k \to \infty$ on both sides of (30), we obtain

$$\frac{1}{s} \leq 1 = \frac{\delta(p*, q)}{\delta(p*, q)} \leq \lim_{k\to\infty} \sup \beta(L(p_{n_k}, p*, p_{n_k+1}, q)) \leq \frac{1}{s}.$$

Therefore, $\lim_{k\to\infty} \sup \beta(L(p_{n_k}, p*, p_{n_k+1}, q)) = \frac{1}{s}$. That implies, by property of β,

$$\delta(p*, q) = \lim_{k\to\infty} L(p_{n_k}, p*, p_{n_k+1}, q) = 0.$$

Hence, $\delta(p*, q)) = 0$, that is, $p* = q$, which contradicts the fact that $p* \neq q$. Therefore, $p* = q$. Hence $p*$ is the best proximity point of S. □

Theorem 3.2. *In addition to the hypotheses of Theorem 3.1, let us consider a condition (C): for all $u, w \in P_S(\mathcal{N})$, where $P_S(\mathcal{N})$ indicates the set of all best proximity points of S, $\alpha(u, w) \geq 1$.*
Then there is only one best proximity point for S.

Proof. According to the proof of Theorem 3.1, the best proximity point exists. Here we show that this best proximity point is one and only one. Assume the opposite. That is, consider two different points u and w in such a way

$$\delta(u, Su) = \delta(\mathcal{N}, \mathcal{R})$$
$$\delta(w, Sw) = \delta(\mathcal{N}, \mathcal{R}).$$

Hence, according to condition (C), $\alpha(u, w)\} \geq 1$.
Since

$$\begin{aligned} &\alpha(u, w) \geq 1 \\ &\delta(u, Su) = \delta(\mathcal{N}, \mathcal{R}) \\ &\delta(w, Sw) = \delta(\mathcal{N}, \mathcal{R}). \end{aligned} \tag{31}$$

According to (1), it follows that

$$\begin{aligned} \delta(u, w) &\leq \alpha(u, w)\delta(u, w) \\ &\leq \beta\big(L(u, w, u, w))\big)L(u, w, u, w) < \frac{1}{s}L(u, w, u, w) \\ &= \frac{1}{s}\max\{\delta(u, w), \delta(u, u), \delta(w, w)\} = \frac{1}{s}\delta(u, w). \end{aligned}$$

This is a contradiction. Therefore, $u = w$. □

Corollary 3.3. *Let $\mathcal{N} \neq \emptyset$ and $\mathcal{R} \neq \emptyset$ be subsets of a $b_v(s)$- metric space$(\mathcal{M}, \delta)$ with a constant $s \geq 1$ and $\alpha : \mathcal{N} \times \mathcal{N} \to \mathbb{R}_+$ be a function. Let $S : \mathcal{N} \to \mathcal{R}$ is a mapping that has the RJ-property. Suppose that there exists $\lambda \in (0, \frac{1}{s})$ such that for all $p, q, u, v \in \mathcal{N}$ the following conditions hold:*

(i) $$\left.\begin{aligned} \delta(u, Sp) = \delta(\mathcal{N}, \mathcal{R}) \\ \delta(v, Sq) = \delta(\mathcal{N}, \mathcal{R}) \end{aligned}\right\}$$
$$\implies \alpha(p, q)\delta(u, v) \leq \lambda L(p, q, u, v), \tag{32}$$

where $L(p, q, u, v) = max\ \{\delta(p, q), \delta(p, u), \delta(q, v)\}$;

(ii) $S(\mathcal{N}_0)$ is a subset of $\mathcal{R}_0$ and S is a triangular $\alpha-$proximal admissible;

(iii) if $\{p_n\}$ is a sequence of $\mathcal{N}$ such that $\alpha(p_n, p_{n+1}) \geq 1$ for all n and $p_n \to p \in \mathcal{N}$ as $n \to \infty$, then for some sub-sequence $\{p_{n_k}\}$ of $\{p_n\}$, we have $\alpha(p_{n_k}, p) \geq 1$ for all k;

(iv) there are points p_0 and p_1 in $\mathcal{N}$ such that $\delta(p_1, Sp_0) = \delta(\mathcal{N}, \mathcal{R})$ and $\alpha(p_0, p_1) \geq 1$;

(v) for all $u, w \in P_S(\mathcal{N})$, where $P_S(\mathcal{N})$ indicates the set of all best proximity points of S, $\alpha(u, w) \geq 1$.

Then there is only one best proximity point for S.

Corollary 3.4. *Let $\mathcal{N} \neq \emptyset$ and $\mathcal{R} \neq \emptyset$ be subsets of a $b_v(s)$- metric space$(\mathcal{M}, \delta)$ with a constant $s \geq 1$ and $\alpha : \mathcal{N} \times \mathcal{N} \to \mathbb{R}_+$ be a function. Let $S : \mathcal{N} \to \mathcal{R}$ is a mapping that has the RJ-property. Suppose that there exists $\beta \in \mathcal{F}_s$ such that for all $p, q, u, v \in \mathcal{N}$ the following conditions hold:*

(i) $$\left.\begin{aligned} \delta(u, Sp) = \delta(\mathcal{N}, \mathcal{R}) \\ \delta(v, Sq) = \delta(\mathcal{N}, \mathcal{R}) \end{aligned}\right\}$$
$$\implies \alpha(p, q)\delta(u, v) \leq \beta(\delta(p, q))\delta(p, q) \tag{33}$$

(ii) *$S(\mathcal{N}_0)$ is a subset of $\mathcal{R}_0$ and S is a triangular $\alpha-$proximal admissible;*

(iii) *if $\{p_n\}$ is a sequence of $\mathcal{N}$ such that $\alpha(p_n, p_{n+1}) \geq 1$ for all n and $p_n \to p \in N$ as $n \to \infty$, then for some sub-sequence $\{p_{n_k}\}$ of $\{p_n\}$, we have $\alpha(p_{n_k}, p) \geq 1$ for all k;*

(iv) *there are points p_0 and p_1 in $\mathcal{N}$ such that $\delta(p_1, Sp_0) = \delta(\mathcal{N}, \mathcal{R})$ and $\alpha(p_0, p_1) \geq 1$;*

(v) *for all $u, w \in P_S(\mathcal{N})$, where $P_S(\mathcal{N})$ indicates the set of all best proximity points of S, $\alpha(u, w) \geq 1$.*

Then there is only one best proximity point for S.

If $\mathcal{M} = \mathcal{N} = \mathcal{R}$, we get $\delta(\mathcal{N}, \mathcal{R}) = 0$. That is, $u = Sp$ and $v = Sq$ in Theorem 3.1. Thus we have the following fixed point results.

Corollary 3.5. *Let $\mathcal{M} \neq \emptyset$ be a set and $(\mathcal{M}, \delta)$ be a $b_v(s)$ metric space with a constant $s \geq 1$ and $\alpha : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ be a function. Let $S : \mathcal{M} \times \mathcal{M}$ is a self-mapping. Suppose that there exists $\beta \in \mathcal{F}_s$ such that for all $p, q \in \mathcal{M}$ the following conditions hold:*

(i) $\alpha(p, q)\delta(Sp, Sq) \leq \beta\big(L(p, q)\big)L(p, q)$,

where $L(p, q) = max\ \{\delta(p, q), \delta(p, Sp), \delta(q, Sq)\}$;

(ii) *S is a triangular $\alpha-$ admissible;*

(iii) *if $\{p_n\}$ is a sequence of $\mathcal{M}$ such that $\alpha(p_n, p_{n+1}) \geq 1$ for all n and $p_n \to p \in \mathcal{M}$ as $n \to \infty$, then for some sub-sequence $\{p_{n_k}\}$ of $\{p_n\}$, we have $\alpha(p_{n_k}, p) \geq 1$ for all k;*

(iv) *there is a point $p_0 \in \mathcal{M}$ such that $\alpha(p_0, Sp_0) \geq 1$;*

(v) *for all $u, w \in F_S(\mathcal{M})$, where $F_S(\mathcal{M})$ indicates the set of all fixed points of S, $\alpha(u, w) \geq 1$.*

Then there is only one fixed point for S.

Corollary 3.6. *Let $\mathcal{M} \neq \emptyset$ be a set and $(\mathcal{M}, \delta)$ be a $b_v(s)$ metric space with a constant $s \geq 1$ and $\alpha : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ be a function. Let $S : \mathcal{M} \to \mathcal{M}$ is a self- mapping. Suppose that there exists $\lambda \in (0, \frac{1}{s})$ such that for all $p, q \in \mathcal{M}$ the following conditions hold:*

(i) $\alpha(p, q)\delta(Sp, Sq) \leq \lambda L(p, q)$,

where $L(p, q) = \max\{\delta(p, q), \delta(p, Sp), \delta(q, Sq)\}$;

(ii) *S is a triangular $\alpha-$ admissible;*

(iii) *if $\{p_n\}$ is a sequence for $\mathcal{M}$ such that $\alpha(p_n, p_{n+1}) \geq 1$ for all n and $p_n \to p \in \mathcal{M}$ as $n \to \infty$, then for some sub-sequence $\{p_{n_k}\}$ of $\{p_n\}$, we have $\alpha(p_{n_k}, p) \geq 1$ for all k;*

(iv) *there is a point $p_0 \in \mathcal{M}$ such that $\alpha(p_0, Sp_0) \geq 1$;*

(v) *for all $u, w \in F_S(\mathcal{M})$, where $F_S(\mathcal{M})$ indicates the set of all fixed points of S, $\alpha(u, w) \geq 1$.*

Then there is only one fixed point for S.

Corollary 3.7. *Let $\mathcal{M} \neq \emptyset$ be a set and $(\mathcal{M}, \delta)$ be a $b_v(s)$ metric space with a constant $s \geq 1$ and $\alpha : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ be a function. Let $S : \mathcal{M} \to \mathcal{M}$ is a self-mapping. Suppose that there exists $\beta \in \mathcal{F}_s$ such that for all $p, q \in \mathcal{M}$ the following conditions hold:*

(i) $\alpha(p, q)\delta(Sp, Sq) \leq \beta(\delta(p, q)\delta(p, q)$;

(ii) *S is a triangular $\alpha-$ admissible;*

(iii) *if $\{p_n\}$ is a sequence in $\mathcal{M}$ such that $\alpha(p_n, p_{n+1}) \geq 1$ for all n and $p_n \to p \in \mathcal{M}$ as $n \to \infty$, then for some sub-sequence $\{p_{n_k}\}$ of $\{p_n\}$, we have $\alpha(p_{n_k}, p) \geq 1$ for all k;*

(iv) *there is a point $p_0 \in M$ such that $\alpha(p_0, Sp_0) \geq 1$;*

(v) *for all $u, w \in F_S(\mathcal{M})$, where $F_S(\mathcal{M})$ indicates the set of all fixed points of S, $\alpha(u, w) \geq 1$.*

Then there is only one fixed point for S.

References

[1] A. Abkar, M. Gabeleh, Global optimal solutions of non-cyclic mappings in metric spaces, J. Optim. Theory Appl. 153 (2012) 298–305.

[2] A. Abkar, M. Gabeleh, Best proximity points of non-self mappings. Top 21 (2013) 287–295.

[3] M.A. Al-Thaga, N. Shahzad, Convergence and existence results for best proximity points. Nonlinear Anal. 70(2009) 3665–3671.

[4] S. Aleksic, Z. D. Mitrovic, and S. Radenovic, A fixed point theorem of Jungck in $b_v(s)$-metric spaces. Period. Math. Hung. 77(2) (2018) 224–231.

[5] E. Altiparmak and I. Karahan, Fixed Point Theorems for Geraghty Type Contraction Mappings in Complete Partial $b_v(s)$-Metric Spaces. Sahand Communications in Mathematical Analysis. 18(2) (2021) 45–62.

[6] E. Altiparmak and I. Karahan, Fixed point results in partially ordered partial $b_v(s)$-metric spaces. International Journal of Nonlinear Analysis and Applications. 12(2) (2021) 35–52.

[7] S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations integrales. Fundam. Math. 3(1922) 133–181.

[8] S.S. Basha, Best proximity points optimal solutions. J. Optim. Theory Appl. 151(2011) 210–216.

[9] S. S. Basha, Discrete optimization in partially ordered sets. J. Glob. Optim. 54(2012) 511–517.

[10] P. Charoensawan, Common fixed point theorems for Geraghty's type contraction mapping with two generalized metrics endowed with a directed graph in JS-metric spaces. Carpathian Journal of Mathematics. 34(2018) 305–312.

[11] S. Czerwik, Contraction mappings in b-metric spaces, Acta Mathematica et Informatica Universitatis Ostraviensis. 1(1) (1993) 5-11.

[12] F. Dong, P. Ji, Sheng, and X. Wang, Pata-Type Fixed Point Results in $b_v(s)$-Metric Spaces. International Journal of Analysis and Applications. 17(3) (2019) 342-360.

[13] T. Došenović, Z. Kadelburg, D. Mitrović, and S. Radenović, New fixed point results in $b_v(s)$-metric spaces. Mathematica Slovaca. 70(2) (2020) 441-452.

[14] D. Dukic, Z. Kadelburg, and S. Radenovic, Fixed point of Geraghty-type mappings in various generalized metric spaces. Abstr Appl Anal. in (2011).

[15] K. Fan, Extensions of two fixed point theorems of Browder F.E. Math. Z., 122 (1969), 234–240.

[16] R. George, S. Radenović, K. P. Reshma, and S. Shukla, Rectangular b-metric space and contraction principles. J. Nonlinear Sci. Appl. 8(6) (2015) 1005–1013.

[17] M. Geraghty, On contractive mappings, Proc. Am. Math. Soc. 40(1973) 604–608.

[18] M. Gordji, M. Ramezani, Y. Cho, and S. Pirbavafa, A generalization of Geraghty's theorem in partially ordered metric spaces and applications to ordinary differential equations. Fixed Point Theory and Applications. 74(1) 2012.

[19] J. Hamzehnejadi, and R. Lashkaripour, Best proximity points for generalized $\alpha-\phi-$Geraghty proximal contraction mappings and its application. Fixed point theory and applications. (2016) 2016:72 1-13.

[20] M. Jleli, and B. Samet, Best proximity points for $\alpha-\psi$-proximal contractive type mappings and applications. Bull.Sci.Math. 137(8) (2013) 977-995.

[21] Z. Kadelburg, P. Kumam, S. Radenović, and W. Sintunavarat, Common coupled fixed point theorems for Geraghty-type contraction mappings using monotone property. Fixed Point Theory and Applications. 27(2015).

[22] E. Karapinar, P. Kumam, and P. Salimi, On α-ψ-Meir-Keeler contractive mappings. Fixed Point Theory and Applications. 94(1) (2013) 1–12.

[23] P. Kumam, P. Salimi, and C. Vetro, Best proximity point results for modified α-proximal C-contraction mappings. Fixed Point Theory and Applications. (2014) 2014:99.

[24] Z. D. Mitrović, and S. Radenović, The Banach and Reich contractions in $b_v(s)$-metric spaces. J. Fixed Point Theory Appl. 19(2017) 3087–3095.

[25] B. Samet C. Vetro, and P. Vetro, Fixed point theorems for $\alpha-\psi$-contractive type mappings. Non linear Anal. 75(4) (2012) 2154–2165.

[26] J. Zhang, Y. Su, and Q. Cheng, A note on A best proximity point theorem for Geraghty-contractions. Fixed Point Theory Appl. (2013) 2013:99.