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Bifurcation and Stability of a Discrete-time SIS Epidemic Model with Treatment

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Highlights

- The aim of this work focuses on investigating the spread and control of infection in a SIS model.
- Discrete-time version of the system subject to treatment is proposed to examine the spread of infection.
- Bifurcation theory is applied to achieve the flip bifurcation conditions.
- The Neimark Sacker bifurcation diagram is presented depending on the step size.
- Chaos is controlled via a hybrid controlled method.

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Abstract

The mathematical dynamics are suitable in examining the effect of infective populations. Conditions involving the spread and control of the disease are calculated by analyzing mathematical models so that it is possible to have information about the behavior of the infection. This article includes the dynamics of a discrete SIS endemic model thru treatment. After determining that the fixed point conditions are fulfilled, the stability analysis is completed for those fixed points. The derived endemic fixed point's stability and bifurcation conditions are examined. Depending on the infection coefficient, the flip bifurcation condition is obtained. At the same time, it is determined in which situation Neimark-sacker bifurcation (NSB) may occur depending on the step size, and bifurcation is controlled. Our theoretical findings are supported by a rich dynamical nature.

1. INTRODUCTION

In order to understand the complicated connections within and between species, numerous models based on discrete or continuous time steps have been developed. There is no overlap between successive generations since populations in ecology evolve at different time steps. Therefore, difference models are useful for studying the behavior of populations. Additionally, it is observed that discretization of continuous population models works better when the population is smaller than continuous ones [1-4]. The behavior of discrete-time models has gained the attention to numerous researchers. These studies include the dynamics of models created with discrete-time equations [5-8] and discrete-time systems [9-20] as well as the dynamics of models created forward Euler's scheme [1, 3, 4, 21, 22] non-standard discretization procedure [2] discretization of fractional order systems [23, 24, 25] and discretization with piecewise constant arguments.

Investigating the dynamics of epidemic disease models is essential, which is one of the population interaction dynamics. Epidemic diseases such as influenza, plague, cholera, typhoid, aids, smallpox, malaria, mers, ebola, measles, tuberculosis, and Covid-19 have been problematic for humanity throughout history, affect infected individuals and infected individuals the whole society in many ways. Vaccination and treatment are among the most effective strategies in combating epidemics. The treatment is particularly significant to decrease the blowout of epidemics until the vaccine is developed [26-30]. In recent times, several researchers have investigated the dynamics of endemic models [31-35].

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In this study, our aim is to include a treatment function, which expresses the situation in which the hospital beds and drags are sufficient, into a continuous epidemic disease model; and to examine the dynamics of the discretized model. It is expected that the number of susceptible and infected people will change as a result of treating infected individuals, yielding different findings from the model without treatment.

Li et al. [36], introduced and analyzed the following system taking into account the availability of limited resources

$$\begin{aligned} \frac{dS}{dt} &= \Lambda - \mu S - \beta SI + \gamma I + T(I) \\ \frac{dI}{dt} &= \beta SI - (\mu + \alpha + \gamma) I - T(I) \end{aligned} \tag{1}$$

where S_t and I_t represent the susceptible and infective populations. Furthermore, Λ, μ, γ, α, and β are the recruitment rate, natural death rate, nature recovery rate, disease-related death rate, and coefficient of infection population. All parameters are positive.

When (1) is used with forward Euler's technique, the discrete form of the SIS model is obtained as

$$\begin{aligned} x_{t+1} &= x_t + \delta\left(\Phi - \beta x_t y_t - \varepsilon x_t + \gamma y_t + T(y_t)\right) \\ y_{t+1} &= y_t + \delta\left(\beta x_t y_t - \phi y_t - T(y_t)\right) \end{aligned} \tag{2}$$

with $\delta \frac{dx}{dt} \approx x_{t+1} - x_t$ and $\delta \frac{dy}{dt} \approx y_{t+1} - y_t$ such that $S_t = x_t, I_t = y_t$. Here δ represents the step size. Furthermore, every parameter is a positive constant.

The flip bifurcation conditions are achieved in this study by using the theory of the bifurcation idea. Also, the NSB diagram presented depending on the step size and chaos is controlled via hybrid controlled strategy [37-40]. Assuming that the drugs or beds in the hospital are adequate, the function of treatment is by given

$$T(y_t) = \tau y_t$$

for all y. Then the system (2) becomes

$$\begin{aligned} x_{t+1} &= x_t + \delta\left(\Phi - \beta x_t y_t - \varepsilon x_t + \tau y_t + \gamma y_t\right) \\ y_{t+1} &= y_t + \delta\left(-\tau y_t - \phi y_t + \beta x_t y_t\right). \end{aligned} \tag{3}$$

In case the treatment capacity is exceeded, a fixed treatment is recommended [26, 36]. Studying the discretization of the continuous system with a fixed treatment rate leads to complex operations. In this study, proportional treatment of the discrete system was primarily studied in terms of its dynamical behavior.

The study divided into six sections is made up of like this scheme: The first part is given as an introduction section to have information about the past studies and emphasize our purpose in the study. The second section examined the fixed point's existence requirements and assessed their stability under (3). In the third section, the parametric requirements for flip bifurcation of endemic fixed point are determined. The fourth section examined the existence of NSB and chaos control of endemic fixed points. Finally, brief results are presented in the last section.

Citations must be given in brackets [26]. If there are two citations, use comma to separate [27, 28]. If citations are more than two and in consecutive order, give the starting number and the last number [29-33]. For multiple citations with/without consequence, use the combination of the rules above [4, 10, 11, 21, 23, 34].

2. EXISTENCE OF FIXED POINTS AND STABILITY ANALYSIS OF (3)

The existence of fixed points in the model (3) and an analysis of their local stability are covered in this section. To determine (3)'s fixed points, the discrete model was solved by

$$\Phi-\beta x_t y_t-\varepsilon x_t+\tau y+\gamma y_t=0$$
$$-\phi y_t+\beta x_t y_t-\tau y=0.$$

The solution of the system (3) is obtained by using straight forward calculations.

Lemma 1. The System (3) has disease-free fixed point (DF) $E^1=\left(\frac{\Phi}{\varepsilon},0\right)$ for all positive parameters and a positive endemic fixed point $E^*=\left(\frac{\phi+\tau}{\beta},\frac{(\phi+\tau)\varepsilon}{\beta(\gamma-\phi)}-\frac{\Phi}{(\gamma-\phi)}\right)$ if $\Phi<\frac{(\phi+\tau)\varepsilon}{\beta}$ and $0<\phi<\gamma$ or $\Phi>\frac{(\phi+\tau)\varepsilon}{\beta}$ and $0<\gamma<\phi.$

The Jacobian matrix's characteristic polynomial about $E^1=\left(\frac{\Phi}{\varepsilon},0\right)$ is provided by

$$\mathrm{G}(\mu)=\mu^2+\left(-2+\delta(\varepsilon+\tau+\phi-\frac{\beta\phi}{\varepsilon})\right)\mu+\frac{(-1+\delta\varepsilon)(\varepsilon(-1+\delta(\tau+\phi))-\beta\delta\phi}{\varepsilon}.$$

It is easy to see that the roots are $\mu_1=1-\delta\varepsilon$ ($|\mu_1|<1$ under the assumption $\delta\varepsilon<2$) and $\mu_2=1-\delta(\tau+\phi)+\frac{\beta\delta\phi}{\varepsilon}$. Moreover, we get $|\mu_2|=1-\delta(\tau+\phi)+\frac{\beta\delta\phi}{\varepsilon}$ if and only if $\frac{\beta\Phi}{\varepsilon(\phi+\tau)}<1.$

Remark 1. Suppose that $\delta\phi+\delta\tau+\frac{\beta\Phi}{\varepsilon}<1$ and $\delta\varepsilon<2$. If $\frac{\beta\Phi}{\varepsilon(\phi+\tau)}<1$, then the fixed point (DF) is locally asymptotically stable. The basic reproduction R_0 is referred to as $\frac{\beta\Phi}{\varepsilon(\phi+\tau)}$.

Lemma 2. [20] Let us take $Z(\alpha)=\alpha^2+M\alpha+N$ such that $Z(1)>0$. Also α_1 and α_2 are two roots $F(\lambda)=0$. Then, the subsequent assumptions are true:

i. $Z(-1)>0$ and $N<1$ if and only if $|\alpha_{1,2}|<1$;

ii. $Z(-1)<0$ if and only if $|\alpha_1|<1$ and $|\alpha_2|>1$ (or $|\alpha_1|>1$ and $|\alpha_2|<1$);

iii. $Z(-1) > 0$ and $N > 1$ if and only if $|\alpha_{1,2}| > 1$;

iv. $Z(-1) = 0$ and $M \neq 0, 2$ if and only if $\alpha_1 = -1$ and $|\alpha_2| \neq 1$;

v. $M^2 - 4N < 0$ and $N = 1$ if and only if $\alpha_{1,2}$ complex root and $|\alpha_{1,2}| = 1$.

Let's now examine the stability of the model's fixed points (3). The Jacobian matrix of (3) assessed at points E^1 and E^*.

$$J(x,y) = \begin{bmatrix} 1-\delta(\varepsilon+\beta y) & \delta(\tau-\beta x+\gamma) \\ \delta\beta y & 1-\delta(\phi-\beta x+\tau) \end{bmatrix}. \tag{4}$$

Thus

$$J\left(E^1\right) = \begin{bmatrix} 1-\delta\varepsilon & \delta\left(\gamma+\tau-\dfrac{\beta\Phi}{\varepsilon}\right) \\ 0 & 1+\delta\left(\dfrac{\beta\Phi}{\varepsilon}-\tau-\phi\right) \end{bmatrix}, \tag{5}$$

$$J\left(E^*\right) = \begin{bmatrix} \dfrac{\gamma-\delta\varepsilon(\gamma+\tau)-\phi+\beta\delta\Phi}{\gamma-\phi} & \delta(\gamma-\phi) \\ \dfrac{\delta[\varepsilon(\phi+\tau)-\beta\Phi]}{\gamma-\phi} & 1 \end{bmatrix}. \tag{6}$$

With the help of Lemma 2, the stability analysis of (3) expressed as follows:

Lemma 3. Assume that $\Phi < \dfrac{(\tau+\phi)\varepsilon}{\beta}$. For the exclusion fixed point $E^1 = \left(\dfrac{\Phi}{\varepsilon}, 0\right)$, the following cases hold:

i. E^1 is a sink point if and only if $\delta\varepsilon < 2$ and $\dfrac{\varepsilon(\delta(\tau+\phi)-2)}{\delta\Phi} < \beta$,

ii. E^1 is a saddle point if and only if $\delta\varepsilon > 2$ and $\dfrac{\varepsilon(\delta(\tau+\phi)-2)}{\delta\Phi} < \beta$ or $\delta\varepsilon < 2$ and $\beta < \dfrac{\varepsilon(\delta(\tau+\phi)-2)}{\delta\Phi}$,

iii. E^1 is a source point if and only if $\beta < \dfrac{\varepsilon(\delta(\tau+\phi)-2)}{\delta\Phi}$ and $\delta\varepsilon > 2$,

iv. E^1 is a non-hyperbolic point if and only if $\beta = \dfrac{\varepsilon(\delta(\tau+\phi)-2)}{\delta\Phi}$ or $\delta\varepsilon = 2$,

where $\delta(\tau+\phi) > 2$. Also, if $\gamma < \phi$, E^1 is the unique disease-free fixed point.

Lemma 4. If $\beta > \dfrac{\varepsilon(\tau+\phi)}{\Phi}$ and the endemicpoint $E^* = \left(\dfrac{\tau+\phi}{\beta}, \dfrac{\varepsilon(\tau+\phi)-\beta\Phi}{\beta(\gamma-\phi)}\right)$ such that $\gamma < \phi$, is local asymptotically stable and the following cases hold:

i. If $\beta < \dfrac{2\delta\varepsilon\tau + 4\phi - \delta^2\varepsilon\phi(\tau+\phi) + \gamma(-4+\delta\varepsilon(2+\delta(\tau+\phi)))}{\delta(2+\gamma\delta-\delta\phi)\Phi}$, $0 < \delta \le -\dfrac{1}{\gamma-\phi}$ and $\varepsilon < \dfrac{2}{\delta}$.

ii. If $\beta < \dfrac{\varepsilon(\tau-\delta\phi(\tau+\phi)+\gamma(1+\delta(\tau+\phi)))}{(1+\delta(\gamma-\phi))\Phi}$, $\varepsilon < \dfrac{2}{\delta}$, $\delta = \dfrac{-2}{\gamma-\phi}$.

iii. If $\beta < \dfrac{\varepsilon(\tau-\delta\phi(\tau+\phi)+\gamma(1+\delta(\tau+\phi)))}{(1+\delta(\gamma-\phi))\Phi}$, $\delta > -\dfrac{2}{\gamma-\phi}$, $0 < \varepsilon < \dfrac{2}{\delta}$.

iv. If $\Omega < \beta < \dfrac{\varepsilon(\tau-\delta\phi(\tau+\phi)+\gamma(1+\delta(\tau+\phi)))}{(1+\delta(\gamma-\phi))\Phi}$, $\delta > -\dfrac{2}{\gamma-\phi}$, $\dfrac{2}{\delta} \le \varepsilon$, such that

$$\Omega = \max\left\{\frac{2\delta\varepsilon\tau + 4\phi - \delta^2\varepsilon\phi(\tau+\phi) + \gamma(-4+\delta\varepsilon(2+\delta(\tau+\phi)))}{\delta(2+\gamma\delta-\delta\phi)\Phi}, \frac{\varepsilon(\tau+\phi)}{\Phi}\right\}.$$

Proof. The characteristic polynomial of Jacobian matrix $J\left(E^*\right)$ about $E^* = \left(\dfrac{\tau+\phi}{\beta}, \dfrac{\varepsilon(\tau+\phi)-\beta\Phi}{\beta(\gamma-\phi)}\right)$ is given by

$$Z(\alpha) = \alpha^2 + \frac{\gamma(\delta\varepsilon-2)+\delta\varepsilon\tau+2\phi-\beta\delta\Phi}{\gamma-\phi}\alpha + \frac{\delta^2\phi(\varepsilon(\tau+\phi)-\beta\Phi)-\phi+\delta(\beta\Phi-\varepsilon\tau)-\gamma(\delta\varepsilon-1+\delta^2(\varepsilon(\tau+\phi)-\beta\Phi))}{\gamma-\phi}. \tag{7}$$

When the conditions in Lemma 2 are evaluated, the desired conditions are easily reached.

3. ANALYSIS OF FLIP BIFURCATION

This section uses the bifurcation theory to analyze the flip bifurcation of the system (3) [37-40].

Lemma 5. [6, 41] For system (3), one root of the system is $\alpha = -1$, and the other root lie inside the unit circle if and only if

i. $Z(1) = 1 + M + N > 0$

ii. $Z(-1) = 1 - M + N = 0$

iii. $P_1^+ = 1 + N > 0$

iv. $P_1^- = 1 - N > 0.$

Lemma 6. $\dfrac{-4\phi+\delta^2\phi(\varepsilon(\tau+\phi)-\beta\Phi)+\delta(2\beta\Phi-2\varepsilon\tau)+\gamma(4+\beta\delta\Phi-\delta(\varepsilon(2+\delta(\tau+\phi))))}{\gamma-\phi} = 0$ and $\dfrac{\gamma(\delta\varepsilon-2)+\delta\varepsilon\tau+2\phi-\beta\delta\Phi}{\gamma-\phi} \ne 0, 2$ if and only if $\alpha_1 = -1$ and $|\alpha_2| \ne 1$. (see Lemma 2-(iv)).

Proof. From the condition Lemma 2-(iv), we can easily get that the conditions in Lemma 6.

Theorem 1. Assume that the inequalities are provided;

1. $-\delta^2(\varepsilon(\tau+\phi)-\beta\Phi)>0$,
2. $\dfrac{\delta\varepsilon(\tau-\delta\phi(\tau+\phi)+\gamma(1+\delta(\tau+\phi)))+\beta\delta(\delta(\phi-\gamma)-1)\Phi}{\gamma-\phi}>0$
3. $\dfrac{-2\phi+\delta^2\phi(\varepsilon(\tau+\phi)-\beta\Phi)+\delta(\beta\Phi-\varepsilon\tau)-\gamma(\delta\varepsilon-2+\delta^2(\varepsilon(\tau+\phi)-\beta\Phi))}{\gamma-\phi}>0$ and if $\beta_F=\dfrac{2\delta\varepsilon\tau+4\phi-\delta^2\varepsilon\phi(\tau+\phi)+\gamma(\delta\varepsilon(2+\delta(\tau+\phi))-4)}{\delta(2+\delta(\gamma-\phi))\Phi}$, then one root of the system is $\alpha=-1$

and the other root of the system lie inside the unit circle. The system (3) undertakes a flip bifurcation at the endemic point $E^*=\left(x^*,y^*\right)$ such that

$$x^*=\frac{\delta(\tau+\phi)(2+\delta(\gamma-\phi))\Phi}{2\delta\varepsilon\tau+4\phi-\delta^2\varepsilon\phi(\tau+\phi)+\gamma(\delta\varepsilon(2+\delta(\tau+\phi))-4)}$$
$$y^*=\frac{2(\delta\varepsilon-2)\Phi}{2\delta\varepsilon\tau+4\phi-\delta^2\varepsilon\phi(\tau+\phi)+\gamma(\delta\varepsilon(2+\delta(\tau+\phi))-4)}.$$

Proof. By using the characteristic polynomial, we write that

$$M=\frac{\gamma(\delta\varepsilon-2)+\delta\varepsilon\tau+2\phi-\beta\delta\Phi}{\gamma-\phi},$$
$$N=\frac{\delta^2\phi(\varepsilon(\tau+\phi)-\beta\Phi)-\phi+\delta(\beta\Phi-\varepsilon\tau)-\gamma(\delta\varepsilon-1+\delta^2(\varepsilon(\tau+\phi)-\beta\Phi))}{\gamma-\phi}.$$

By considering Lemma 5 (i) & (ii), we reach the following conditions, respectively

$$\delta^2(\beta\Phi-\varepsilon(\tau+\phi))>0,$$

and

$$\beta=\frac{2\delta\varepsilon\tau+4\phi-\delta^2\varepsilon\phi(\tau+\phi)+\gamma(\delta\varepsilon(2+\delta(\tau+\phi))-4)}{\delta(2+\delta(\gamma-\phi))\Phi}.$$

From Lemma 5 (iii) & (iv), we obtain

$$P_1^+=\frac{\delta^2\phi(\varepsilon(\tau+\phi)-\beta\Phi)-2\phi+\delta(\beta\Phi-\varepsilon\tau)-\gamma(\delta\varepsilon-2+\delta^2(\varepsilon(\tau+\phi)-\beta\Phi))}{\gamma-\phi}>0$$
$$P_1^-=\frac{\delta\varepsilon(\tau-\delta\phi(\tau+\phi)+\gamma(1+\delta(\tau+\phi)))+\beta\delta(\delta(\phi-\gamma)-1)}{\gamma-\phi}>0.$$

Moreover, it is easily seen that the Jacobian matrix Equation (6) has the eigenvalues

$$\alpha_1(\beta_F)=-1$$
$$\alpha_2(\beta_F)=3+\Theta\beta_F \qquad (8)$$

where $\Theta\beta_F = -\delta\varepsilon + \dfrac{2(\delta\varepsilon - 2)}{2 + \delta(\gamma - \phi)}$. Since $|\alpha_2(\beta_F)| \neq 1$, it leads to

$$\Theta\beta_F \neq -2, -4. \tag{9}$$

Using the transformation

$$x = x - \frac{\tau + \phi}{\beta}, \quad y = y - \frac{(\phi + \tau)\varepsilon - \beta\Phi}{(\gamma - \phi)\beta}$$

the endemic point E^* is shifted to zero. By Taylor expansion in around $\left(x^*, y^*\right)$, System (3) has the form:

$$\begin{aligned} x_{t+1} &= \frac{\gamma - \gamma\delta\varepsilon - \delta\varepsilon\tau - \phi + \beta\delta\Phi}{\gamma - \phi} x_t + \delta(\gamma - \phi) y_t + G_1(x, y) \\ y_{t+1} &= \frac{\delta(\varepsilon(\tau + \phi) - \beta\Phi)}{\gamma - \phi} x_t + y_t + G_2(x, y) \end{aligned} \tag{10}$$

or the following map:

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow J(E^*) \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} G_1(x, y) \\ G_2(x, y) \end{bmatrix} \tag{11}$$

where

$$\begin{aligned} G_1(x, y) &= -\beta\delta x y \\ G_2(x, y) &= \beta\delta x y, \end{aligned}$$

such that $\Omega = \left(x, y\right)^T$. The system Equation (11) has been modified as

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} \rightarrow J(E^*) \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \frac{1}{2} \mathrm{T}\left(x_t, y_t\right) + \frac{1}{6} \Upsilon\left(x_t, y_t, u_t\right), x, y, u \in \mathbb{R}^2, \tag{12}$$

here

$$T(x, y) = \begin{bmatrix} T_1(x, y) \\ T_2(x, y) \end{bmatrix},$$

and

$$\Upsilon(x, y, u) = \begin{bmatrix} \Upsilon_1(x, y, u) \\ \Upsilon_2(x, y, u) \end{bmatrix}.$$

These vectors are expressed by

$$T_1(x,y) = \sum_{k,j=1}^{2} \frac{\partial^2 G_1}{\partial \zeta_k \partial \zeta_j}\Big|_{\zeta=0} x_k y_j = -\beta\delta\left(x_1 y_2 + x_2 y_1\right)$$

$$T_2(x,y) = \sum_{k,j=1}^{2} \frac{\partial^2 G_2}{\partial \zeta_k \partial \zeta_j}\Big|_{\zeta=0} x_k y_j = \beta\delta\left(x_1 y_2 + x_2 y_1\right)$$

$$\Upsilon_1(x,y,u) = \sum_{k,j,l=1}^{2} \frac{\partial^3 G_1}{\partial \zeta_k \partial \zeta_j \partial \zeta_l}\Big|_{\zeta=0} x_k y_j u_l = 0$$

$$\Upsilon_2(x,y,u) = \sum_{k,j,l=1}^{2} \frac{\partial^3 G_2}{\partial \zeta_k \partial \zeta_j \partial \zeta_l}\Big|_{\zeta=0} x_k y_j u_l = 0.$$

Suppose that $q, p \in \mathbb{R}^2$ are two eigenvectors of $J\left(E^*_{\beta_G}\right)$ and $J^T\left(E^*_{\beta_G}\right)$, respectively for $\alpha_1(\beta_G) = -1$. Then we obtain, $J\left(E^*_{\beta_G}\right)q = -q$ and $J^T\left(E^*_{\beta_G}\right)p = -p$. By calculation, we get

$$q \approx \left(\frac{2+\delta(\gamma-\phi)}{\delta\varepsilon - 2}, 1\right)^T$$

$$p \approx \left(-\frac{2}{\delta(\gamma-\phi)}, 1\right)^T.$$

With the purpose of normalizing p relating to q , we estimate

$$p = \left(\frac{4-2\delta\varepsilon}{\delta(\delta\varepsilon-4)(\gamma-\phi)-4}, \frac{\delta(\delta\varepsilon-2)(\gamma-\phi)}{\delta(\delta\varepsilon-4)(\gamma-\phi)-4}\right)^T,$$

such that $\langle p,q\rangle = 1$. By using the scalar product in $\mathbb{R}^2 : \langle p,q\rangle = p_1q_1 + p_2q_2$. To determine the direction of the flip bifurcation, we need to get the sign of the coefficient $c\left(\beta_G\right)$ as follows:

$$c\left(\beta_G\right) = \frac{1}{6}\langle p, \Upsilon(q,q,q)\rangle - \frac{1}{2}\left\langle p, T\left(q,(J-I)^{-1}T(q,q)\right)\right\rangle.$$

The following theorem results from the analysis above, section 5.4 in [37], and section 3 in [42, 43].

Theorem 2. If Equation (9) becomes valid, $c\left(\beta_G\right) \neq 0$, and the value of β varies nearby β_G, then system (3) undertakes a flip bifurcation at the endemic point E^*. Also, if $c\left(\beta_G\right) > 0$ (respectively $c\left(\beta_G\right) < 0$), then E^* are stable and bifurcates to the period 2 orbits (unstable).

4. NEIMARK-SACKER BIFURCATIONANALYSIS AND CONTROL OF CHAOS

In this section, the analysis of NSB and how to manage its chaos are covered. Depending on parameter δ, the next lemma is equal to lemma 2. At E^*, a jacobian matrix is

$$J(E^*) = \begin{bmatrix} 1+\delta a_{11} & -\delta a_{12} \\ \delta a_{21} & 1 \end{bmatrix}. \tag{13}$$

Here $a_{11} = \dfrac{\beta\Phi - \varepsilon(\gamma+\tau)}{\gamma-\phi}, a_{12} = \phi - \gamma$ and $a_{21} = \dfrac{\varepsilon(\tau+\phi) - \beta\Phi}{\gamma-\phi}$. The characteristic equation is $Z(\lambda) = \alpha^2 - M\alpha + N$, $M = 2+\delta a_{11}$ and $N = 1+\delta a_{11} + \delta^2 a_{12} a_{21}$. The eigen values are

$$\alpha_{1,2} = 1 + \frac{\delta U}{2} \pm \frac{\delta}{2}\sqrt{U^2 - 4V},$$

while $U = a_{11}$ and $V = a_{12}a_{21}$.

Lemma 7. The unique coexistence endemic fixed point E^* is a

1. sink if
   a) $\mathbb{K}^* < 0$ and $\delta < \delta_3$, or
   b) $\mathbb{K}^* \geq 0$ and $\delta < \delta_2$,
2. source if
   a) $\mathbb{K}^* < 0$ and $\delta > \delta_3$, or
   b) $\mathbb{K}^* \geq 0$ and $\delta > \delta_1$,
3. saddle if $\mathbb{K}^* \geq 0$ and $\delta_2 < \delta < \delta_1$,
4. non-hyperbolic if
   a) $\mathbb{K}^* < 0$ and $\delta = \delta_3$, or
   b) $\mathbb{K}^* > 0$ and $\delta = \delta_1$ or $\delta = \delta_2$.

$\mathbb{K}^* = U^2 - 4V$ and $\delta_1 = \frac{1}{V}\left[\sqrt{U^2-4V} - U\right],\ \delta_2 = -\frac{1}{V}\left[\sqrt{U^2-4V} + U\right],\ \delta_3 = -\frac{U}{V}.$

To analyze the NSB consider the bifurcation parameter as δ. The existence of this bifurcation is confirmed when the roots of the values at an endemic point are complex conjugate with $|\lambda| = 1$ [6]. The quadratic equation obtained from Equation (13) is

$$Z(\alpha) = \alpha^2 - (2+\delta U)\alpha + (1+\delta U + \delta^2 V).$$

From Lemma 7, if $\mathbb{K}^* < 0$ and $\delta = \delta_3$, then the eigen values are

$$\alpha_{1,2} = 1 - \frac{U^2}{2V} \pm i\frac{U}{2V}\sqrt{4V - U^2}.$$

Now we conclude the theorem for the system Equation (3) about the NSB.

Theorem 3. The NSB of (3) ensures when $\mathbb{K}^* < 0$ and $\delta = \delta_3$ and

$$|\alpha_{1,2}| = \left|1 - \frac{U^2}{2V} \pm i\frac{U}{2V}\sqrt{4V - U^2}\right| = 1.$$

Next hybrid controlled method [44, 45] is utilized to control the chaos of the model (3) and is expressed by

$$\begin{aligned} x &= \quad \sigma x + \delta(\Phi + \tau y - \varepsilon x - \beta xy + \gamma y)\sigma + (1-\sigma)x \\ y &= \qquad \sigma y + \delta(-\phi y - \tau y + \beta xy)\sigma + (1-\sigma)y \end{aligned} \tag{14}$$

where $\sigma \in (0,1)$. In (14) the control approach combines parameter perturbation and feedback control, and the right choice of σ leads to the partial or complete removal of NSB. At E^*, Jacobian of (14) is

$$J(E^*) = \begin{bmatrix} 1+\sigma\delta a_{11} & -\sigma\delta a_{12} \\ \sigma\delta a_{21} & 1 \end{bmatrix}. \tag{15}$$

Here a_{11}, a_{12}, a_{21} are the same as given in Equation (13). Asymptotic stability E^* is guaranteed by the roots of Equation (13) being present in the unit disk.

5. NUMERICAL SIMULATIONS

The theoretical investigation is confirmed with appropriate illustrations by taking some distinct cases of (3). Dynamical behavior of (3) around the endemic point under various collections of parameter is exhibited thru MATLAB programming.

Example 1. Consider the parameter values $\Phi = 0.575, \varepsilon = 0.2, \beta = 2.4, \gamma = 0.03, \tau = 0.009, \phi = 0.99$ and $\delta = 0.99$ with the initial conditions $(0.8, 0.2)$. Computation yields $(x^*, y^*) = (0.416, 0.559)$. The Jacobian matrix is $J = \begin{bmatrix} -0.4151 & -0.9504 \\ 1.2128 & 1 \end{bmatrix}$. Here $M = 0.5849$, $N = 0.7375$ and the eigen values are $\alpha_{1,2} = 0.2924 \pm i0.8075$ such that the modulus value is 0.8588, which is less than 1. The conditions for stability are satisfied. Hence, from Figure 1, model (3) is stable. As a result, the phase portrait in Figure 1 depicts a trajectory sinking and spiraling close to the endemic point (x^*, y^*).

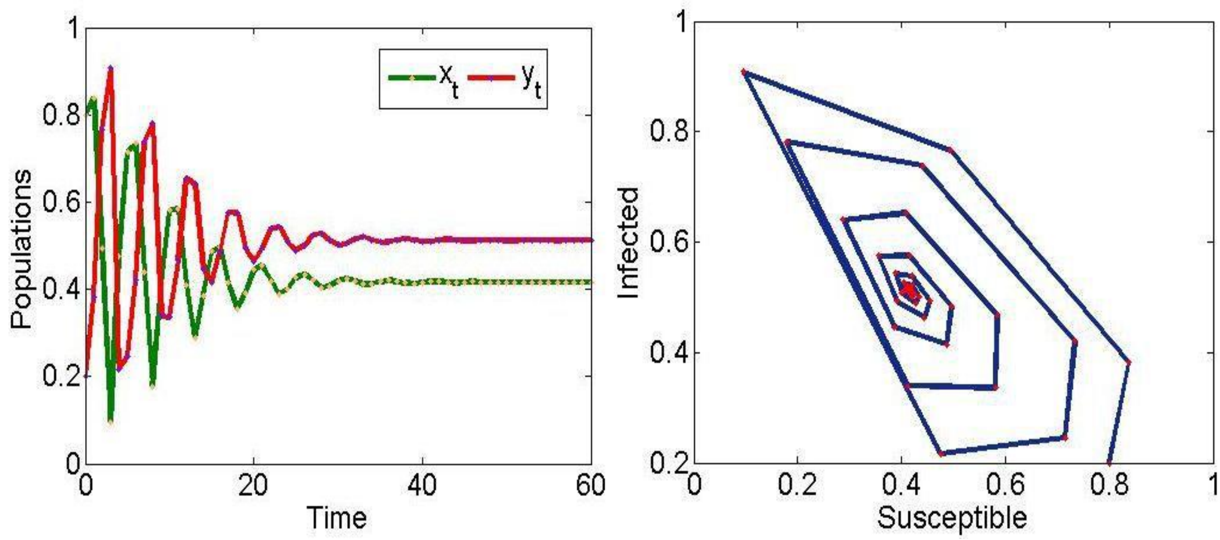


Figure 1. *Stability of the model (3) via timeline and phase plane*

Example 2. Taking the values $\Phi = 0.695, \varepsilon = 0.4, \beta = 1.1, \gamma = 0.02, \tau = 0.3, \phi = 0.9$ and $\delta \in (2.45, 3)$ in the model (3) with the initial value $(0.95, 0.05)$. NSB for the model (3) is discussed. By normal simplification, the endemic value is $(x^* = 1.0909, y^* = 0.2939)$. Also, the conditions are calculated as $U = -0.7233; V = 0.2845; \mathbb{K}^* = -0.6148 < 0$ and $\delta_3 = 2.532$. The Eigenvalues are $\alpha_{1,2} = 0.0806 \pm i0.9967$, and the modulus of the value is one. The requirements for NSB are achieved near the endemic fixed point E^* at bifurcate value δ_3 by using Lemma 7. The endemic point E^* of the model (3) is shown in NSB diagrams in the planes (δ, x) and (δ, y), respectively, in Figures 2(a) and (b). The endemic point of the model (3) is easily seen to be locally asymptotically stable for $\delta < \delta_3 = 2.5423$, when $\delta = \delta_3$, the model becomes unstable and shifts to a stable invariant cycle when $\delta > \delta_3$. Additionally, these orbits take place in the time windows where $\delta > \delta_3$ causes the invariant cycle to shift to a quasi-periodic orbit. Finally, when the bifurcation parameter rises, the orbits tend towards chaos.

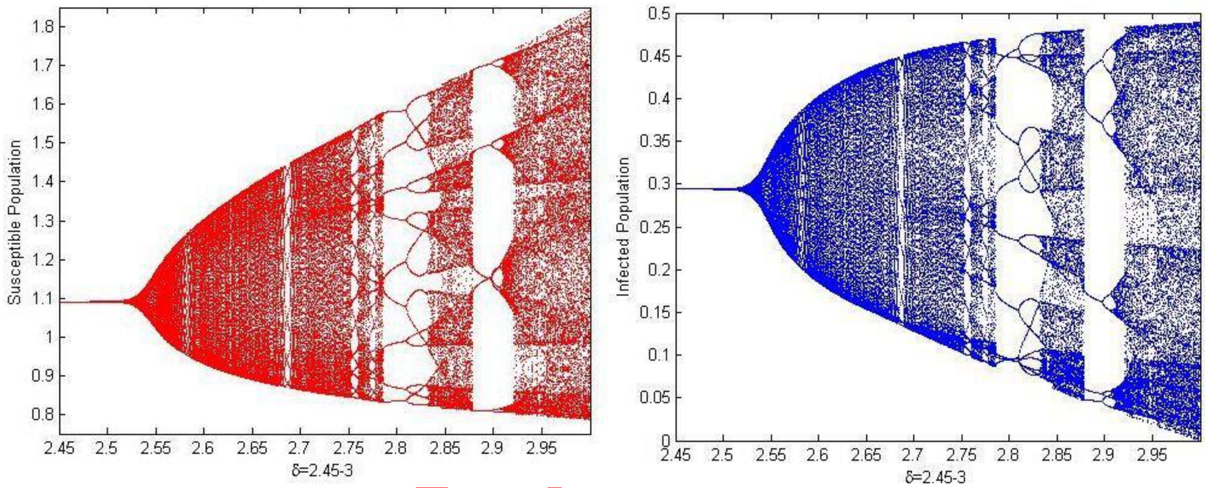
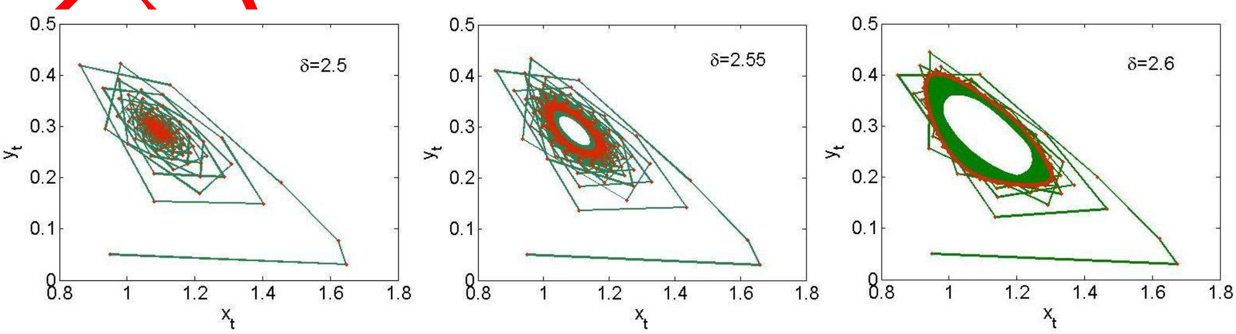


Figure 2. *Neimark-sacker bifurcation of the model (3)*

Figure 3 displays distinct phase planes for a range of values of δ for Model (3). For $\delta = 2.5$, the solution curve spirals inward before settling at stability. The graph shows instability for δ between 2.55 and 2.6 as it settles down as a limit cycle from spirally inwards. The solution curve spirals inward for $\delta = 2.7 - 2.9$ but does not come to a point. Finally, the circle vanishes for $\delta = 2.91 - 2.99$, and chaotic attractors show themselves. We can draw our reasons from the diagrams of the phase and bifurcation planes.



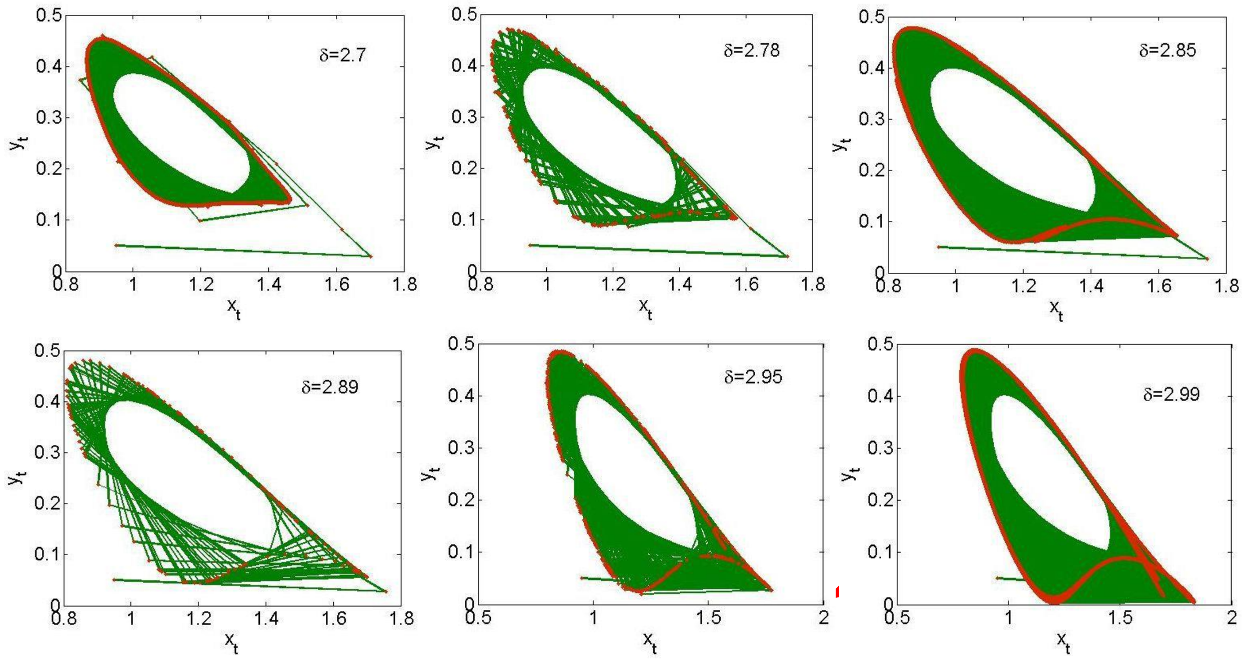


Figure 3. *Various phase plane of the model (3) for different δ values*

Example 3. With the set of parameter values $\Phi = 0.695, \varepsilon = 0.4, \beta = 1.1, \gamma = 0.02, \tau = 0.3, \phi = 0.9$, and takes $\delta = 2.995$ with the initial condition $(x(0), y(0)) = (0.95, 0.05)$, Figure 4 shows the appearance of an unstable fixed point E^* and a closed invariant circle. Thus, the model (14) can be modified as follows:

$$\begin{aligned} x &= \quad x + \delta(\Phi - \beta xy - \varepsilon x + \tau y + \gamma y)\sigma \\ y &= \quad y + \delta(-\tau y - \phi y + \beta xy)\sigma. \end{aligned} \tag{16}$$

where $\Phi = 0.695, \varepsilon = 0.4, \beta = 1.1, \gamma = 0.02, \tau = 0.3, \phi = 0.9$ and $\sigma \in (0,1)$. The Jacobian matrix of the model (Equation 16) is estimated at E^* and is $J(E^*) = \begin{bmatrix} 1-0.7233\sigma & -0.88\sigma \\ 0.4506\sigma & 1 \end{bmatrix}$, and the characteristic equation is $\alpha^2 - (2+0.7233\sigma)\alpha + 0.3965\sigma^2 + 1 - 0.7233\sigma = 0$. Then, the eigen values lie in the unit open disk if and only if $\sigma \in (0, 0.9999)$. Furthermore $\sigma = 0.84$, the plots for x and y of the model (Equation 16) are exhibited in Figure 5. It is clearly observed from Figure 5, the endemic point E^* is stable.

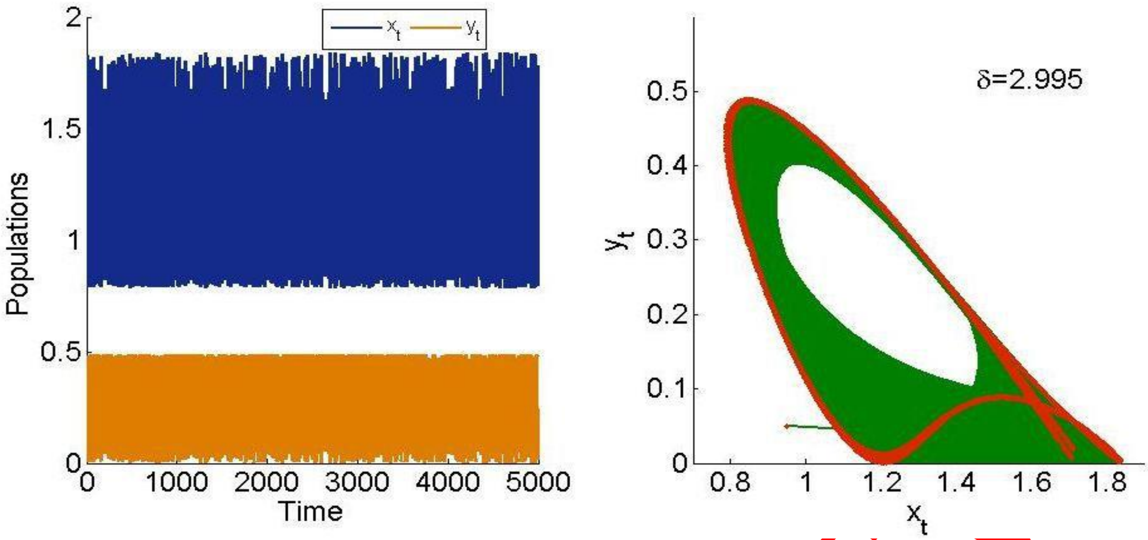


Figure 4. *Unstability of the model (3) via time line and phase plane*

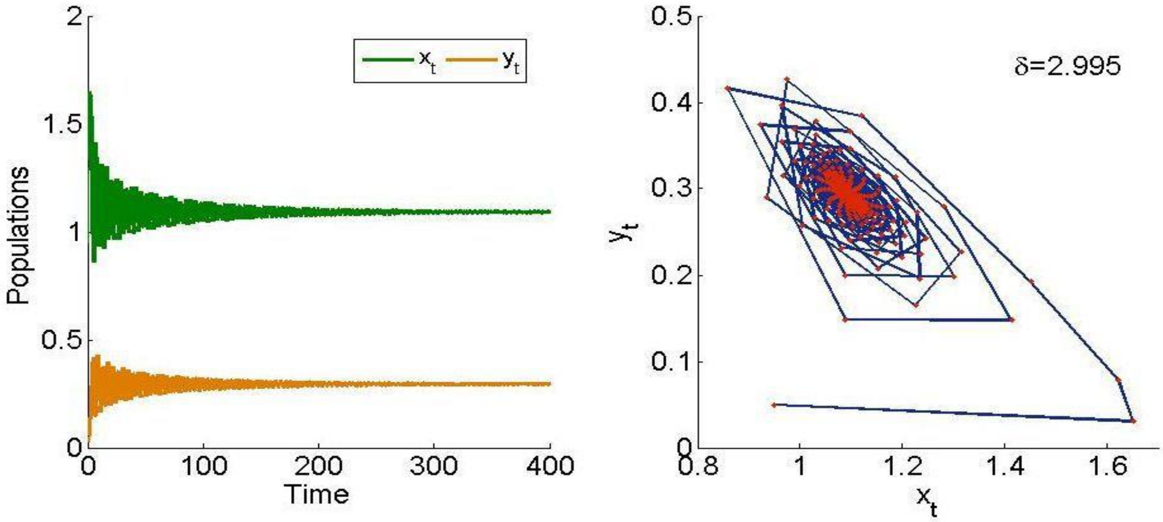


Figure 5. *Stability of the controlled model (16) via time line and phase plane*

Example 4. Consider $\delta = 0.9, \varepsilon = 0.3, \phi = 0.85, \tau = 0.03, \gamma = 0.002$ and $\Phi = 0.45$ with the initial conditions $(0.8, 0.2)$. The bifurcate value is calculated as $\beta_F = 6.44424$.

$$\begin{aligned} x &= x + 0.9(0.45 - 0.3x - 6.44424xy + 0.032y) \\ y &= y + 0.9(6.44424xy - 0.85y - 0.03y). \end{aligned} \tag{17}$$

The endemic fixed point is found as $E^* = (0.13656, 0.48235)$. The Jacobian $J(\beta_F)$ is given by $\begin{bmatrix} -2.06754 & -0.7632 \\ 2.79754 & 1 \end{bmatrix}$. Here, the eigen values are $\alpha_1 = -1$, $\alpha_2 = -0.067542$ such that $|\alpha_2| \neq 1$. The eigenvectors $p, q \in \mathbb{R}^2$ correspond to $\alpha_1(\beta_F) = -1$ and $\alpha_2 = -0.067542$ are $q = (-0.714913, 1)^T$ and $p = (0.62055, 1)^T$ respectively. To normalize p relating to q, we calculate $p = (-3.00018, -1.14487)^T$. The critical coefficient $c(\beta_F) = -1.434625 < 0$. Therefore, the periodic orbits -2 that bifurcate from E^* are unstable see Figure 6.

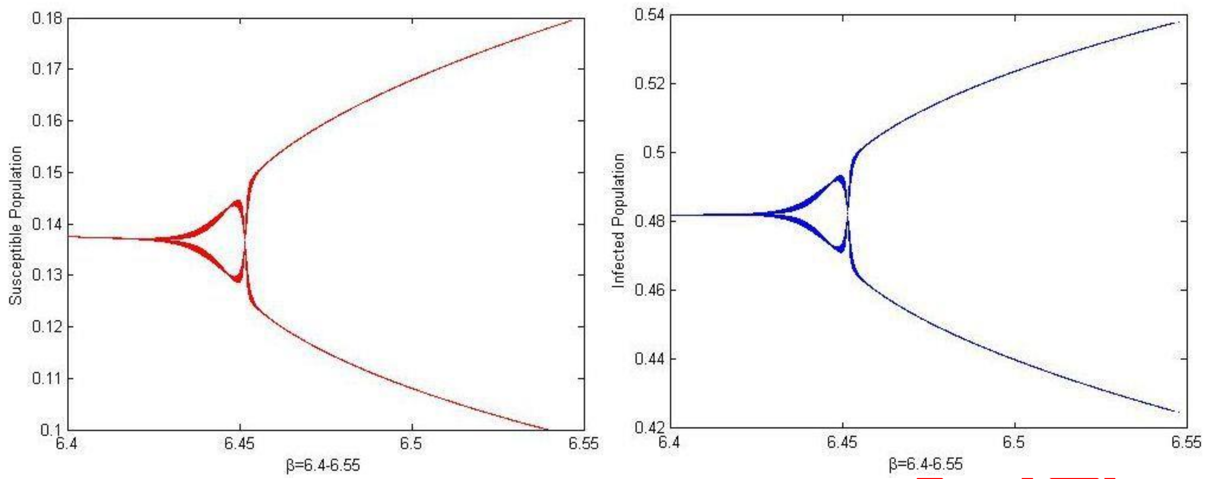


Figure 6. *Flip bifurcation of the model (16)*

Example 5. Consider the SIS model (3) without treatment, then the model (3) modified as

$$\begin{aligned} x_{t+1} &= x_t + \delta\left(\Phi - \varepsilon x_t - \beta x_t y_t + \gamma y_t\right) \\ y_{t+1} &= y_t + \delta\left(-\phi y_t + \beta x_t y_t\right). \end{aligned} \tag{18}$$

here taking the parameter values $\Phi = 0.575, \varepsilon = 0.2, \beta = 2.4, \gamma = 0.03, \phi = 0.99$ and $\delta = 0.99$ with the initial conditions $(0.8, 0.2)$. Computation yields $(x^*, y^*) = (0.4125, 0.513)$. The Jacobian matrix is $J = \begin{bmatrix} -0.4169 & -0.9504 \\ 1.2189 & 1 \end{bmatrix}$. Here $M = 0.5831$, $N = 0.7415$ and the eigen values are $\alpha_{1,2} = 0.2916 \pm i0.8103$ such that the modulus value is 0.8611, which is less than 1. The stability criteria are satisfied. Therefore, based on Figure 7, the model (18) is stable. Also infected people takes time to stable for without treatment compare to the SIS treatment model (3).

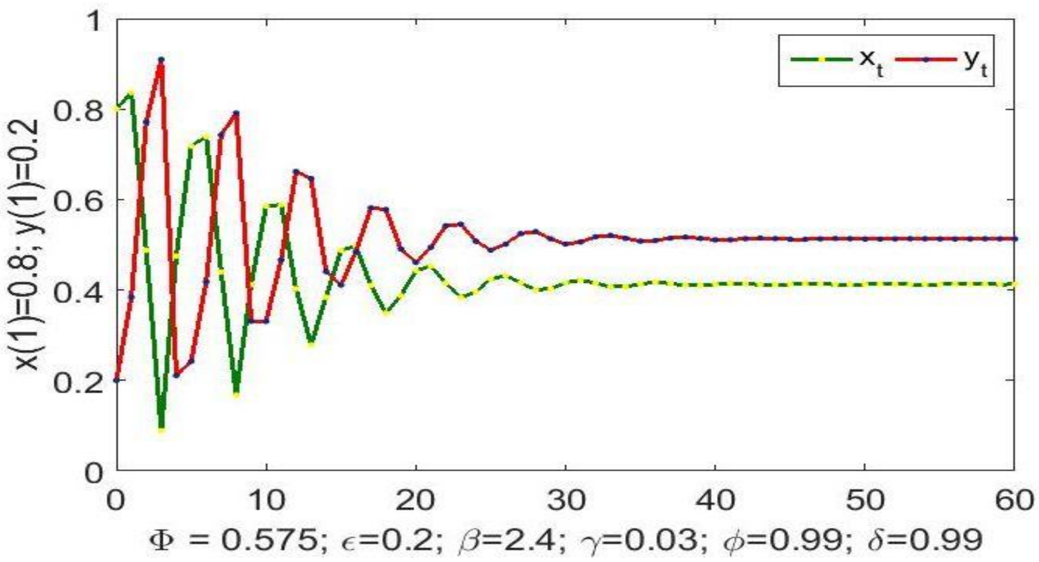


Figure 7. *Stability of the without treatment model (18) via timeline*

6. CONCLUSION

The dynamical behavior of an SIS endemic discrete model is concerned in this work. Assuming that the bifurcation parameter for system (3) is β, the existence conditions of fixed points and the conditions of flip bifurcation are obtained. NSB formation is investigated depending on the step size δ. Chaos is present in the system (3) under NSB, and it is controlled using a hybrid control process.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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