A New Approach on the Strain Energy Function for the Mechanical Behavior of Arteries

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Abstract

In this study, a new constitutive equation that includes the characteristic nonlinear anisotropic response of arteries is proposed. The measurement of the relationship between arterial diameter and arterial pressure is important part of the general problem of blood flow measurements. This relationship was examined in the human thoracic aorta. The clinical data that obtained from literature provide only a pressure-diameter relationship. To determine the parameters of the constitutive formulations, nonlinear regression analysis was used on these data.

Keywords: *human; arteries; constitutive equations.*

Damarların Mekanik Davranışları için Şekil Değiştirme Enerjisi Fonksiyonuna Yeni Bir Yaklaşım

Özet

Bu çalışmada, damarların lineer olmayan, anizotropik davranışını karakterize eden yeni bir bünye denklemi önerilmiştir. Damar çapı ile iç basıncı arasındaki ilişkiyi belirlemek genel kan akışı probleminin önemli bir parçasıdır. Bu ilişki insan torakik aortunda incelenmiştir. Literatürden elde edilen klinik veriler sadece iç basınç-çap ilişkisini sağlamaktadır. Bu veriler kullanılarak lineer olmayan regresyon analizi ile bünye denkleminin parametreleri belirlenmiştir.

Anahtar Kelimeler: *insan; damarlar; bünye denklemleri.*

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1. **Introduction**

In the modern world, human health has become a field where scientists work most intensively. Today, the increase of health threatening factors such as environmental and harmful effects brought about by technology have come with rapid progress in the field of bio-engineering and biomaterials. In this study, a model that is expected to adapt itself to the real artery has been determined by assessing the mechanical properties of artery materials obtained from experimental values.

Many researchers have carried out experiments to determine the mechanical characteristic of human arteries in living tissues. In these studies, they have tried to determine the changing diameter of arteries over time, by means of various visualization methods such as intravenous visualization [1], by the cine-angiography method [2] and measured the diameter of arteries by intravenous artery ultrasonography with the help of a special catheter [3]. It is showed that the artery, independent of internal pressure, is exposed to stable axial strain and force [4-5]. Many researchers described the complex material characteristic of the artery in strain energy function. However, they discussed the mechanical characteristics of the artery physically and ignored the characteristics of the biological tissue [6-11]. As seen directly, the large deformation theory takes us to the nonlinear differential equation in a high order, and the solution of these can only be made in some special cases. A stability issue arises as to whether the equilibrium position of the system, exposed to great deformations and stabilized under external forces, is the only equilibrium position. In recent studies, the artery material is regarded as incompressible, homogeneous, elastic [12], [13] or viscoelastic [14,15,17] and [16], isotropic and in some cases anisotropic [18,9] and [19].

In fact, when morphological structures are taken into account, arteries are observed to have a homogeneous structure and at the same time, their mechanical characteristics are observed to change in both axial and radial directions. Therefore, when the internal structure of the aorta and results of experimental measures are evaluated, strain energy function in this study has been used to characterize the mechanical behavior. The function discussed has been developed as a model in which there are collagen fibers, and as a model that reflects the mechanical characteristics of arteries. A cylindrical unilaminar shell model, in which there are collagen fibers, has been introduced for the artery by using finite elasticity theory. Material parameters that belong to the model that is discussed, as a model of collagen and unilaminar, have been obtained by using the Levenberg–Marquardt algorithm that makes the non-linear parametric functions minimum for the pressure and radius relation given in the experimental study [20]. Finally, stresses made by using radial inflation results and axial extension, under the influence of cylindrical shell physiological forces, belong to this model.

2. Governing equations

Let us assume that the motion and the inverse motion in a three dimensional physical space are described by

$$\mathbf{x}=\mathbf{x}(\mathbf{X},t)\ ,\qquad \mathbf{X}=\mathbf{X}(\mathbf{x},t)\ ,\qquad \det\left(\frac{\partial x_k}{\partial X_K}\right)\neq 0,\ \ k=1,2,3 \tag{1}$$

The velocity and the acceleration vectors are given by

$$v_k=\frac{\partial x_k}{\partial t},\qquad a_k=\frac{\partial v_k}{\partial t}+v_{k,m}v_m \tag{2}$$

Here the comma shows the covariant differentiation with respect to coordinates. Conservation of mass is defined as

$$\frac{\partial \rho}{\partial t}+(\rho v_k)_{,k}=0 \tag{3}$$

Where ρ is the mass density of the body. The balance of linear momentum of the continuous medium is stated as

$$t_{kl,k}+\rho(f_l-a_l)=0,\quad t_{kl}=t_{lk} \tag{4}$$

Where t_{kl} is the symmetric stress tensor, f_l volume force density. The balance of energy may be written in the local form as

$$\rho\dot{\varepsilon}=t_{kl}d_{lk}-q_{k,k}+\rho h \tag{5}$$

Where ε is the internal energy density, q_k is the heat flux vector, h is the volume heat source and d_{lk} is the deformation rate tensor defined by

$$d_{kl}=\frac{1}{2}(v_{k,l}+v_{l,k}) \tag{6}$$

Second law of thermodynamics is written in local form as

$$-\rho(\psi+\eta\dot{\theta})+t_{kl}d_{lk}-\frac{q_k\theta_{,k}}{\theta}\geq 0 \tag{7}$$

Where η is entropy volume density, θ is absolute temperature of the continuous medium and Helmholtz free energy density defined by $\psi=\varepsilon-\theta\eta$.

3. Constitutive equations

Let us assume that the constitutive dependent variables are functions of deformation gradient $F_{kK}\equiv\partial x_k/\partial X_K$, and the temperature.

$$t_{kl}=t_{kl}(F_{kK},\dot{F}_{kK},\theta)\ ;\ q_k=q_k(F_{kK},\dot{F}_{kK},\theta)\ ;\psi=\psi(F_{kK},\dot{F}_{kK},\theta)\ ;\eta=\eta(F_{kK},\dot{F}_{kK},\theta) \tag{8}$$

Introducing (8) into (7) and if nonlinear terms for the variables θ, $\theta_{,k}$ and $\ddot{F}_{kK}$ are neglected, we get

$$-\rho\left(\eta+\frac{\partial\psi}{\partial\theta}\right)\dot{\theta}+\left(t_{kl}-\rho\frac{\partial\psi}{\partial F_{lK}}F_{kK}\right)v_{l,k}-\rho\frac{\partial\psi}{\partial\dot{F}_{kK}}\ddot{F}_{kK}-\frac{q_k\theta_{,k}}{\theta}\geq 0 \tag{9}$$

In order for this inequality to be valid for arbitrary variables, the coefficients

$$\eta=-\frac{\partial\psi}{\partial\theta}\ ,\ q_k=0\ ,\ \frac{\partial\psi}{\partial\dot{F}_{kK}}=0 \tag{10}$$

must vanish. Hence, (9) become

$$\left(t_{kl}-\rho\frac{\partial\psi}{\partial F_{lK}}F_{kK}\right)v_{l,k}\geq 0 \tag{11}$$

Green deformation tensor and Green deformation rate tensor are defined by $C_{KL} = F_{kK}F_{kL}$, $\dot{C}_{KL} = 2d_{kl}F_{kK}F_{kL}$ respectively, introducing into (11) we have

$$t_{kl} = 2\rho \frac{\partial \psi}{\partial C_{KL}} F_{kK}F_{lL} \tag{12}$$

We shall assume that the strain energy function (SEF) is defined by $\Sigma = \Sigma(\mathbf{C}, \theta_0, \mathbf{X})$ and under constant temperature (12) becomes

$$t_{kl} = 2\rho \frac{\partial \Sigma}{\partial C_{KL}} F_{kK}F_{lL} \tag{13}$$

Green deformation tensor C_{KL} is expressed in terms of invariants of itself. Thus, stress tensor takes the following form

$$t_{kl} = 2\rho \sum_{\alpha=1}^{n} \frac{\partial \Sigma}{\partial I_\alpha} \frac{\partial I_\alpha}{\partial C_{KL}} F_{kK}F_{lL} \tag{14}$$

For non-isotropic, hyper-elastic body, different from zero invariants are I_1, I_3, I_4 and I_6. Since the material is assumed to be incompressible, $I_3 = 1$.

We assume that the artery is exposed to a constant axial stretch and an internal pressure. In this condition deformation may be described by

$$r = \left(\frac{R^2}{\lambda} + B \right)^{\frac{1}{2}}, \qquad \theta = \Theta, \qquad z = \lambda Z \tag{16}$$

Where R, Θ, Z and r, θ, z are the cylindrical polar coordinates of a material point before and after final deformation, respectively. B is integral constant, λ is constant axial stretch. Thus green deformation tensor may be given by

$$\mathbf{C} = \begin{bmatrix} {r'}^{2} & 0 & 0 \\ 0 & \dfrac{r^2}{R^2} & 0 \\ 0 & 0 & \lambda^2 \end{bmatrix}, r' \equiv \frac{dr}{dR} \tag{17}$$

4. Constitutive model for the artery

We assume that material of the arteries may be considered as a composite reinforced by two families of (collagen) fibers which are arranged in symmetrical spirals shown in Figure 1.

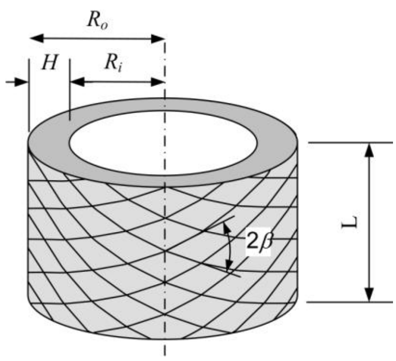

Figure 1. Geometrical properties of arteries.

We suggest two-part SEF that the first part of SEF associated with isotropic deformations and the second part of SEF associated with anisotropic deformations. Hence, SEF is written as

$$\Sigma(\mathbf{C},\mathbf{a}_1,\mathbf{a}_2)=\Sigma_{iso}(\mathbf{C})+\Sigma_{aniso}(\mathbf{C},\mathbf{a}_1,\mathbf{a}_2) \tag{18}$$

Where $\mathbf{a}_1$ and $\mathbf{a}_2$ are introduced to the families of collagenous fibers of direction vectors, and described by

$$\mathbf{a}_1=\begin{bmatrix}0\\ \cos\beta\\ \sin\beta\end{bmatrix},\ \mathbf{a}_2=\begin{bmatrix}0\\ \cos\beta\\ -\sin\beta\end{bmatrix} \tag{19}$$

Hence, invariants of the anisotropic part are written as form

$$I_4=I_6=\mathbf{a_1}\cdot\mathbf{C}\cdot\mathbf{a_1}=\frac{1}{\bar{r}^2}\cos^2\beta+\lambda^2\sin^2\beta=I_{aniso} \tag{20}$$

where $\bar{r}\equiv r/R$ is a parameter without dimensions. Invariant of the isotropic part is written as

$$I_1=\frac{\bar{r}^2}{\lambda^2}+\frac{1}{\bar{r}^2}+\lambda^2 \tag{21}$$

Introducing (20) and (21) into (14) the following result is obtained

$$t_{kl}=P\,\delta_{kl}+2\left[\frac{\partial\Sigma}{\partial I_1}c_{kl}^{-1}+\frac{\partial\Sigma}{\partial I_4}d1_{kl}+\frac{\partial\Sigma}{\partial I_6}d2_{kl}\right] \tag{22}$$

where p is hydrostatic pressure, d_{1kl} and d_{2kl} are defined by

$$d1_{kl}=a_1\otimes a_1\delta_{KL}F_{kK}F_{lL},\qquad d2_{kl}=a_2\otimes a_2\delta_{KL}F_{kK}F_{lL} \tag{23}$$

The general approach is a single strain energy function [21]. However this function is insufficient to describe the material. Another approach is a separation of the strain energy function into isotropic and anisotropic parts [10]. We have proposed a two part strain energy function. One part of the strain energy function is to represent non-linear elastic and isotropic behavior, and the other one to represent non-linear fibrous and anisotropic behavior. The isotropic part and the anisotropic part of (18) are proposed by

$$\Sigma_{iso}=\frac{k_1}{2k_2}\left(\mathrm{e}^{k_2(I_1-3)}-1\right),\ \Sigma_{aniso}=\frac{k_3}{k_4}\left(\mathrm{e}^{k_4(I_{aniso}-1)^2}-1\right) \tag{24}$$

k_1,k_2,k_3 and k_4 are constant material parameters and do not depend on the geometry. Introducing (24) into (22) the physical components of stress tensor we have

$$t_{kl}=P\delta_{kl}+2\left[\frac{k_1}{2}\mathrm{e}^{k_2(I_1-3)}c_{kl}+k_3(I_{aniso}-1)e^{k_4(I_{aniso}-1)^2}(d1_{kl}+d2_{kl})\right] \tag{25}$$

The non-zero physical component in terms of cylindrical coordinates may be given by

$$t_{rr}=P+F(I_1)\frac{\bar{r}^2}{\lambda^2},\ t_{\theta\theta}=P+F(I_1)\frac{1}{\bar{r}^2}+F(I_{aniso})\frac{1}{\bar{r}^2}\cos^2(\beta)$$

$$t_{zz}=P+F(I_1)\lambda^2+F(I_{aniso})\lambda^2\sin^2(\beta) \tag{26}$$

$$F(I_1)=k_1e^{k_2(I_1-3)},\ F(I_{aniso})=4k_3(I_{aniso}-1)e^{k_4(I_{aniso}-1)^2}$$

In the absence of body forces, axial symmetry of geometry the equilibrium equation in cylindrical coordinates is

$$\frac{\partial t_{rr}}{\partial r}+\frac{1}{r}(t_{rr}-t_{\theta\theta})=0 \tag{27}$$

The boundary conditions are

$$t_{rr}\big|_{r=r_i} = -P_i\,,\ t_{rr}\big|_{r=r_o} = 0 \tag{28}$$

Where r_i and r_o are inner and outer radii of the artery respectively. Introducing (26) into (27) under conditions (28) we have,

$$P_i = \int_{r_i}^{r_o} \frac{t_{rr} - t_{\theta\theta}}{r}\,dr \tag{29}$$

The modeled radii were determined by numerically solving (29) for the radius at the corresponding experimental pressure. Pressure-radius relationships were fitted to the experimental data by minimizing the function (least squares)

$$\Omega = \frac{1}{n}\sum_{i=1}^{n} \left(r_i^{\mathrm{mod}} - r_i^{\exp} \right)^2 \tag{30}$$

Where *i* is the data point index and *n* is the total of experimental points measured in the pressure-radius relationship. On the radii *r* the indices mod and exp are used to denote the model and experimental values, respectively. Pressure–diameter (29) relationships were fitted to the experimental data by minimizing the (30). The Levenberg–Marquardt algorithm was used to determine the constitutive parameters as best-fit parameters. Experimental data are taken from a clinical study [3] for the thoracic aorta of a hypertensive subject. In order to investigate pressure- internal radius relations, the material and geometrical data in table 1 are considered. r_i is internal radius before deformation, λ is axial stretch and β is angle between two families of (collagen) fibers for the thoracic aorta are represented in Table 1.

Table 1: Material and geometrical parameters

r_i [mm]	λ [-]	β [o]
9	1.14	29

5. Results

Numerical results were obtained by using the Mathematica 5.0 Wolfram Research, Inc. USA Programme on a Personal Computer. Computed constitutive parameters proposed model k_1, k_2, k_3 and k_4 are stated in Table 2.

Table 2: Computed constitutive parameters in represent of (25)

k_1 [kPa]	k_2 [-]	k_3 [kPa]	k_4 [-]	R^2
1.56	38.45	2.52	21.55	0.92

Figure 2 shows a contour plot of the potential (25) with the material parameters computed and represented in Table 2. SEF is convex for the fitted parameters. Convexity means that the second derivative with respect to **e** is positive definite.

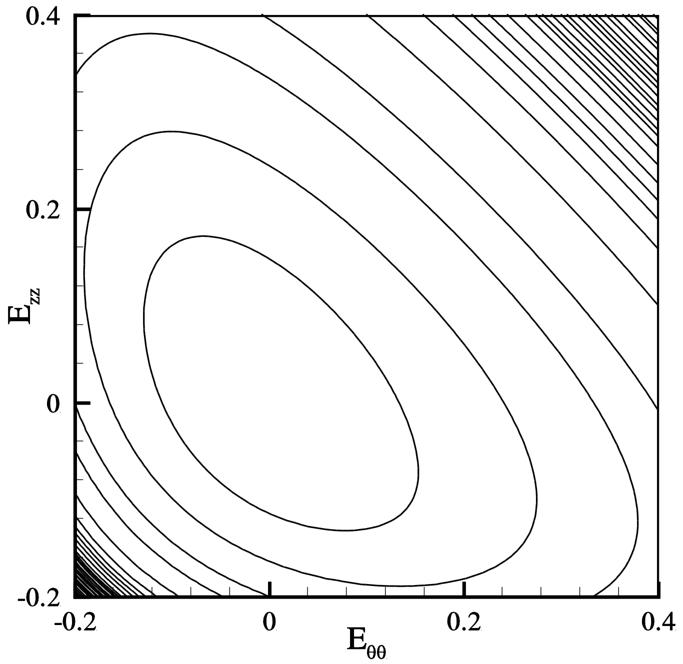

Figure 2. Contour plot of strain energy function.

The recommended approach has been introduced to the results that have been obtained from the clinical experiments carried out on the thoracic aorta, and the pressure-internal diameter relation is stated in Figure 3 [3]. The Figure 3, drawn by using the proposed constitutive equation, is found to be consistent with the data obtained from the clinical studies.

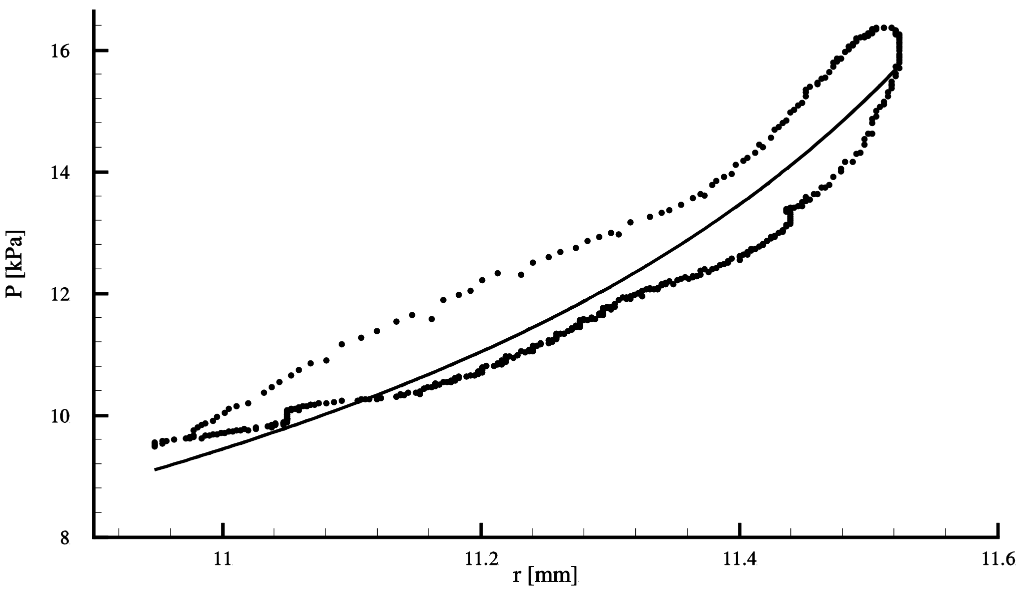

Figure 3. Pressure–internal radius relationship. Circles indicate clinical data; solid line indicates calculated result from constitutive model.

The parameters stated in Table 2 and in (26) Cauchy stress components have been measured according to the deformed artery r-r_i and have been stated in Figure 4. Here r indicates deformed radial coordination and r_i indicates deformed internal radius. Axial stretch has been discussed as λ =1.14. Tangential and axial tension decreases from internal radius to external radius, and radial tension components, as expected, converge to zero.

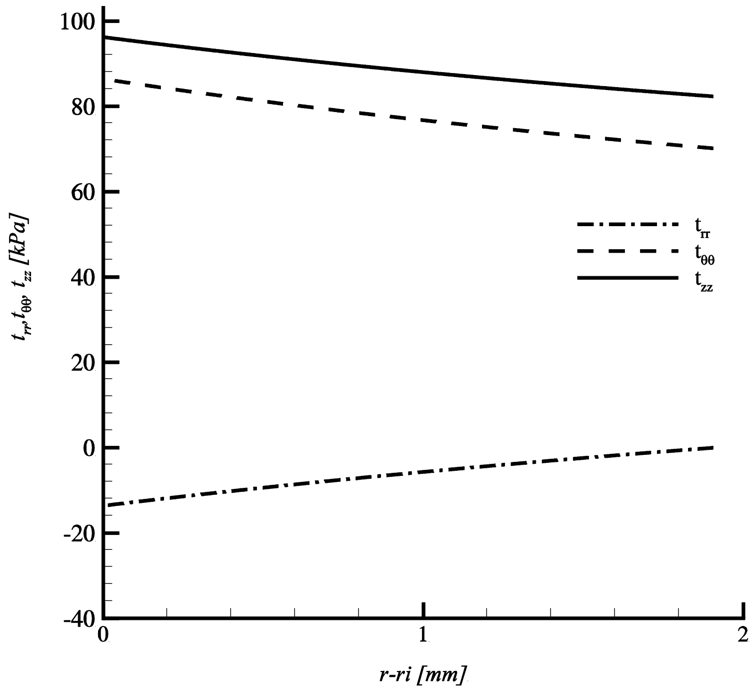

Figure 4. Plots of the principal Cauchy stresses vs. $r - r_i$ at $\lambda = 1.14$.

In this study, the aim is to be able to obtain the mechanical behaviors of arteries in a form that can be used in the field of vascular medical. As a result, by using the results obtained from clinical studies, a new model has been introduced and at the same time rates that have been obtained from experimental studies (axial tension and axial force) have also been used in the studies. These rates have been obtained from the experiments that have been carried out for various purposes and they have been adapted to the human arteries, and determined roughly. If we get these data from actual experiments in the human arteries, we will produce better models.

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