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A characterization of torsion units in integral group of ZS_3

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Abstract

In this study, we give a characterization of all torsion units which are in the unit group of ZS_3 integral group ring of symmetric group S_3, and classify conjugate classes of these units. We used the group of all doubly stochastic matrices in GL(3,Z) in this classification. The investigation of torsion units is not restricted with this study, and the classification of torsion units of bigger ordered groups is open to examine by using the resulted conciliations of this study.

***Key words**: torsion unit, integral group ring, conjugates classes, group representation*

ZS_3 integral grup halkasının sonlu mertebeli elemanlarının bir karakterizasyonu

Özet

Bu çalışmada, S_3 simetrik grubunun ZS_3 integral grup halkasının birimsel grubunda yer alan bütün sonlu mertebeli birimsellerin bir karakterizasyonunu verilmektedir. Ayrıca bu sonlu mertebeli birimsellerin eşlenik sınıflarının bir sınıflandırılması yapılmaktadır. Bu sınıflandırmanın yapılmasında, GL(3,Z) genel lineer grubunda bir alt grup olarak yer alan double stokastik matrisler grubu kullanılmaktadır. Integral grup halkalarının birimsel grubunda yer alan sonlu mertebeli birimsel elemanların sınıflandırılması bu çalışmayla sınırlı değildir. Daha büyük mertebeli grupların integral grup halkalarının sonlu mertebili birimsellerinin sınıflandırılması, bu çalışmadan elde edilen bulguların kullanılmasıyla araştırılmaya açıktır.

***Anahtar kelimeler:** sonlu mertebeli birimsel, integral grup halkası, eşlenik sınıf, grup temsilleri*

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1. Introduction

Hughes and Pearson [2], characterized the group V(ZS_3) of units of augmentation 1 in ZS_3 by showing that V is isomorphic to the subgroup of GL(2, Z). They constructed a 6x6 invertible matrix from a full set of irreducible representations of S_3, and solved a system of six linear congruencies modulo6. Allan-Hobby [1], used a different method to obtain a new description of V(ZS_3) as the group of all doubly stochastic matrices in GL(3, Z). Working in GL(3, Z) instead of GL(2, Z) permits them to exploit the fact that a convex combination of permutation matrices is always doubly stochastic; it is not necessary to invert a 6x6 matrix or to solve a system of linear congruencies.

We represent $S_3 = \langle a,b \mid a^2 = b^3 = I,\ a^{-1}ba = b^2 \rangle$ by

$$\rho\ (a) = A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \rho\ (b) = B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

where $a = (12)$, $b = (132)$ and extend ρ linearly to ZS_3 group ring. Let us write,

$$\alpha = \sum c_{ij} a^i b^j = c_{00} 1 + c_{01} b + c_{02} b^2 + c_{10} a + c_{11} ab + c_{12} ab^2 \in \ ZS_3.$$

It is clear that

$$\rho\ (\alpha) = \begin{bmatrix} c_{00} + c_{12} & c_{01} + c_{10} & c_{02} + c_{11} \\ c_{02} + c_{10} & c_{00} + c_{11} & c_{01} + c_{12} \\ c_{01} + c_{11} & c_{02} + c_{12} & c_{00} + c_{10} \end{bmatrix}.$$

If α is a unit of augmentation 1, than $\rho(\alpha)$ is clearly a doubly stochastic matrix in GL(3, Z). Let us denote the sum of all entries in the location in which there is 1 in $A^i B^j$ matrix of M by $t_{ij}=t_{ij}(M)$ and the number δ_M by

$$\delta_M = \begin{cases} 1\ ; & t_{00}(M) \equiv 1 \ (\mathrm{mod}\, 3)\ ise \\ o\ ; & t_{00}(M) \neq 1 \ (\mathrm{mod}\, 3)\ ise \end{cases}$$

and use the equations

$c_{0j}= (t_{0j}- \delta_M)/3$ and $c_{1j}= (t_{1j}- \delta_M -1)/3$ for j=0,1,2 (1)

to obtain coefficients c_{ij} from M and also $\alpha_M = \sum c_{ij} a^i b^j$ is the corresponding element to M in ZS_3. An arbitrary $M \in GL(3,Z)$ may produce coefficients c_{ij} that are not integers. But if $M \in \rho(V(ZS_3)$, M is a doubly stochastic, than all c_{ij} are integers [1].

2. Torsion elements in ZS_3

If G is a finite group and α an element in ZG, than the order of α, $|\alpha|$, divides the order of G, $|G|$; $|\alpha| \mid |G|$ [3].Thus if α is a torsion unit in ZS_3, the order of α

must be 2,3 and 6. $|\alpha| = 6$ is not possible. If it is so, the minimal polynomial of the matrix of $\rho(\alpha)$ is x^6-1. However the degree of the characteristic polynomial of the matrix is 3. So we need to examine the group ring element the degrees of which are 2 and 3.

2.1. Three-ordered elements

Let α be a three-ordered element in $V(ZS_3)$ and $\rho(\alpha)$=M . Than the order of M must be 3 (ρ is an isomorphism). So we aim to determine the third roots of I in the all double stochastic matrices group. Thus,

$$M^3 = I \Rightarrow |M|^3 = 1 \Rightarrow |M| = 1.$$

The characteristic volumes of M are 1, ω, ω^2 where ω = -1/2+($\sqrt{3/2}$ I ; three characteristic volume of M aren't be 1. If it is so, the characteristic polynomial of M would be $(x-1)^3$ while the minimal polynomial is $(x-1)^3$ or $(x-1)^2$. More over M satisfied the polynomial $x^3-1=0$ and $(x-1)^2 | (x-1)^3$. But these relations are not true. In the same way, three characteristic volume of M don't be ω or ω^2. Hence trM(traceM)=1+ ω+ ω^2 = 0. M is a double stochastic matrix in GL(3, Z), than

$$M = \begin{bmatrix} s & t & 1-s-t \\ u & v & 1-u-v \\ 1-s-u & 1-t-v & -1+s+t+u+v \end{bmatrix},$$

where s,t,u and v are integers. By means of trace and determinant of M we obtain,

$$trM = 2s+2v+t+u-1 = 1+\omega+\omega^2 = 0 \Rightarrow t+u = 1-2s-2v \qquad (2)$$
$$|M| = 3(sv-tu)+(t+u)-(s+v) = 1 \Rightarrow sv-tu = s+v \qquad (3)$$

And since $M^3 = I \Rightarrow M^{-1} = (1/|M|)(adjM) = M^2 \Rightarrow adjM = M^{-1}$.
Implies that

$$M^{-1} = \begin{bmatrix} -s & 1-t & 1+t \\ 1-u & -v & u+v \\ s+u & -v & u+v \end{bmatrix} = M^2$$

M is determined by (2) and (3) in the group $\rho(V(ZS_3))$. Since $t_{00}(M)$ = tr(M) = 0, than the coefficients of $\rho^{-1}(M)$ are

$c_{00} = t_{00}/3 = 0$
$c_{01} = t_{01}/3$ = (t+1-u-v+1-s-u)/3 = (2-3u+(t+u)-s-v)/3 = (2-3u+1-2s-2v-s-v)/3 s-u-v
$c_{02} = t_{02}/3$ = (1-s-t+u+u+1-t-v)/3 = (2-3t+(t+u)-s-v)/3 = (2-3t+1-2s-2v-s-v)/3 = 1-s-t-v
$c_{10} = (t_{t0}-1)/3$ = -s-v
c_{11} = v
c_{12} = s

Thus the three-ordered elements are in the form;

$$\alpha = \alpha_M = \rho^{-1}(M) = (1-s-u-v)b+(1-s-t-v)b^2-(s+v)a+(v)ab+(s)ab^2 \qquad (4)$$

By using (2) and (3) we find,

$u= (-2s^2-ts+s+t-1)/(2t+s-1)$, $v= (t-2ts-t^2+s)/(2t+s-1)$

Now that the form of M is

$$M=(1/2t+s-1)\begin{bmatrix} 2ts+s^2-s & 2t^2+ts-1 & -s^2-2t^2-3ts+2s+st-1 \\ -2s^2-ts+s+t-1 & -t^2-2ts+s+t & 2s^2+t^2+3ts-s \\ s^2-ts+s+t & -t^2+ts+2t-1 & t^2-s^2-t \end{bmatrix} \quad (5)$$

By the necessity of that the entries of M must be integers; $2t+s-1 = \pm 1$.

First, for $2t+s-1 = -1$ than

$$M=\begin{bmatrix} -2t & t & t+1 \\ 6t^2+t+1 & -3t^2+t & -3t^2-2t \\ -6t^2+t & 3t^2-2t+1 & 3t^2+t \end{bmatrix} \quad (6)$$

Form (1) , we get

$$\alpha_M= (-3t^2)b+(3t^2+1)b^2+(3t^2+t)a+(-3t^2+t)ab+(2t)ab^2 \quad (7)$$

Second, for $2t+s-1 = 1$ than

$$M=\begin{bmatrix} -2t+2 & t & t-1 \\ -6t^2+13t-7 & 3t^2-5t+2 & 3t^2-8t+6 \\ 6t^2-11t+6 & -3t^2+4t-1 & -3t^2+7t-4 \end{bmatrix} \quad (8)$$

Using (1) again, we get at the last,

$$\alpha_M= (3t^2-6t+4)b+(-3t^2+6t-3)b^2+(-3t^2+7t-4)a+(3t^{2-}5t+2)ab+(2-2t)ab^2 \quad (9)$$

2.2. Two-ordered elements

Let α be a two-ordered element in $V(ZS_3)$ and $\alpha \neq I$, than $\rho(\alpha) = M$, M^2 and the minimal polynomial of M is (x^2-1). For the characteristic volumes of M, λ_1, λ_2 and λ_3, we find λ_1, $\lambda_2 = \pm 1$ and $|M| = \lambda_1 \lambda_2 \lambda_3$. More over $|M|^2=1 \Rightarrow |M| = \pm 1$. We imply that

$$|M| = -\lambda_3 = 1 \Leftrightarrow trM = \lambda_1+\lambda_2+\lambda_3 = -1 \text{ and } |M| = -\lambda_3 = -1 \Leftrightarrow trM= 1$$

Only one of these two relations is true. First,

$$trM = -1+2s+2v+t+u= -1 \Rightarrow t+u = -2s-2v$$

and we find the relation

$$|M| = 3(sv-tu)+(t+u)-(s+v) = 1 \Leftrightarrow 3(sv-tu)-2s-2v-s-v = 1 \Leftrightarrow 3(sv-tu-s-v) = 1.$$

This has not a solution in Z. So, we conclude that $trM = 1 = 2(s+v)+t+u-1$ and we get,

$$t+u = 2-2s-2v \quad (10)$$

and from $|M| = -1$

$$sv-tu = s+v-1. \quad (11)$$

By using (10) and (11), it is clear that

$$\text{adjM} = \begin{bmatrix} -s & -t & s+t-1 \\ -u & -v & u+v-1 \\ s+u-1 & t+v-1 & s+v-1 \end{bmatrix} = -M$$

and M(-adjM) = I $\Rightarrow$ M^2 = I. Thus, it is determined all square roots of I in $\rho(V(ZS_3)$. From (1), for two-ordered elements in $V(ZS_3)$, We get the form that

$$\alpha = \alpha_M = \rho^{-1}(M) = (1-s-u-v)b+(1-s-t-v)b^2+(1-s-v)a+(v)ab+(s)ab^2 \qquad (12)$$

We can find the general form of the two- ordered elements as unique parameter by using (10) and (11). For example, Selecting s=2, t=3, u= -1 and v= -2, We find that

$$\alpha = \alpha_M = 2b-2b^2+a+2ab+2ab^2 = 2(132)-2(123)+(12)-2(13)+2(23).$$

3. Conjugate classes

All three-ordered torsion units are conjugate to “b”. It can be seen by using the special form of (6) and (9). We will see that if α is an element in $V(ZS_3)$ and α^3 =I, than there is a β element in $V(ZS_3)$ such that $\alpha = \beta^{-1}\alpha\,\beta$. For example, if t= -1 in (6), than

$$M = \begin{bmatrix} 2 & -1 & 0 \\ 6 & -4 & -1 \\ -7 & 6 & 2 \end{bmatrix}$$

and

$$\alpha = \alpha_M = -3b+4b^2+2a-4ab+2ab^2.$$

Since $\rho(b)$ = b, We will find a P matrix that

$$P = \begin{bmatrix} m & n & 1-m-n \\ p & q & 1-p-q \\ 1-m-p & 1-n-q & -1+m+n+p+q \end{bmatrix} \in GL(3,Z),$$

Where $P^{-1}BP$ = M. So, We have the matrix equality that

$$\begin{bmatrix} p & q & 1-p-q \\ 1-m-p & 1-n-q & -1+m+n+p+q \\ m & n & 1-m-n \end{bmatrix} =$$

$$\begin{bmatrix} 9m+13n-7 & -7m-10n+6 & -2m-3n+2 \\ 9p+13q-7 & -7-1q+6 & -2p-3q+2 \\ -9m-13n-9p-13q+15 & 7m+10n+7p+10q-11 & 2m+3n+2p+3q-3 \end{bmatrix}$$

By using the entries of the matrices, we conclude that

1. 6m+10n+q = 6
2. 2m+3n-p-q = 1
3. m+10p+13q = 8

4. n-7p-9q = -5

Since $|P| = \pm 1$, we find that

$$|P| = 3(mq\text{-}np)+n4p\text{-}m\text{-}q = -21p^2-39q^2-57pq+45q-13 = \pm 1 \quad (13)$$

For p=1, q=0 , $|P|$ = -1 in (13) , We find m= -2 , n=2 and

$$P = \begin{bmatrix} -2 & 2 & 1 \\ 1 & 0 & 0 \\ 2 & -1 & 0 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & -1 \\ 1 & -2 & 2 \end{bmatrix}$$

More over, $\beta = \rho^{-1}(P)$ = -1+b+a+ab-ab^2, $\beta^{-1} = \rho^{-1}(P^{-1})$ = 1-b^2+a+ab-ab^2.
By the same way, It can be seen that two-ordered elements in V(ZS_3) has two conjugate classes.
Let G be a group and G(n) = { $g \in G$ | |g| = n }.Let us denote that

$$T^{(n)}(\alpha) = \sum_{g\in G(n)} \alpha(g) \quad \text{ve} \quad \tilde{\alpha}(t) = \sum_{g\sim t} \alpha(g), \quad \text{for} \quad \alpha = \sum_{g\in G} \alpha(g)g \in Z(G)$$

If α is n-ordered unit, than $T^{(n)}(\alpha) = 1$ and $T^{(i)}(\alpha) = 0$ for $i \neq n$ (Bovdi conjecture).
If α is an arbitrary unit, then there is a uniqe $g_0 \in G$ such that $\tilde{\alpha}(g_o) \neq 0$. (Bovdi-Marciniak- Sehgal Conjecture).
It can be seen that These conjecture are true for G = S_3 by using the characterizations in (4), (7), (9) and (12).

4. Results

We have found the characterization of all torsion units in integral group ring ZS_3. We have also seen the structure of conjugate classes the integral group ring of S_3.

a) The three-ordered elements are in the form;

$$\alpha=\alpha_M=\rho^{-1}(M)= (1\text{-}s\text{-}u\text{-}v)b+(1\text{-}s\text{-}t\text{-}v)b^2-(s+v)a+(v)ab+(s)ab^2$$

b) The two-ordered elements are in the torm;

$$\alpha=\alpha_M=\rho^{-1}(M)= (1\text{-}s\text{-}u\text{-}v)b+(1\text{-}s\text{-}t\text{-}v)b^2+(1\text{-}s\text{-}v)a+(v)ab+(s)ab^2$$

c) All three-ordered torsion units are conjugate to “b” and two-ordered elements in V(ZS_3) has two conjugate classes.

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