BAÜ FBE Dergisi
Cilt:10, Sayı:1, 46-51
Temmuz 2008

On the hyperbolic cylindrical Tzitzeica curves in Minkowski 3-space

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Abstract

In this study, we have given necessary and sufficient conditions to be of spacelike, timelike and null curve of the two different hyperbolic cylindrical Tzitzeica curve in Minkowski 3-Space. Here, hyperbolic cylindrical curves satisfying Tzitzeica condition are obtained via the solution of the harmonic equation

Keywords: *Tzitzeica Curve, Hyperbolic Cylindrical Curve, Minkowski 3-Space*

Özet

Bu çalışmada, Minkowski 3-uzayında iki farklı hiperbolik silindirik Tzitzeica eğrisinin, spacelike, timelike ve null olması için gerek ve yeter şartları verdik. Burada, Tzitzeica şartını sağlayan hiperbolik silindirik eğriler, harmonik denklemlerin çözümü yoluyla elde edilir.

Anahtar Kelimeler: *Tzitzeica Eğrisi, Hiperbolik Silindirik Eğri,Minkowski 3-Uzayı.*

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1. Introduction

The Minkowski 3-space E_1^3 is the Euclidean 3-space E^3 provided with the Lorentzian inner product

$$\langle x, y \rangle_L = -x_1 y_1 + x_2 y_2 + x_3 y_3$$

where $x = (x_1, x_2, x_3)$ $y = (y_1, y_2, y_3)$. An arbitrary vector $x = (x_1, x_2, x_3)$ in E_1^3 can have one of three Lorentzian causal characters: it is spacelike if $\langle x, x \rangle_L > 0$ or $x = 0$, timelike if $\langle x, x \rangle_L < 0$ and null (lightlike) if $\langle x, x \rangle_L = 0$ and $x \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in E_1^3 is locally spacelike, timelike or null (lightlike), if all of its velocity vectors (tangents) $\alpha'(s) = T(s)$ are spacelike, timelike or null, respectively, for each $s \in I \subseteq R$. Lorentzian vectoral product of x and y is defined
by

$$x \wedge_L y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_2 y_1 - x_1 y_2).$$

Recall that the pseudo-norm of an arbitrary vector $x \in E_1^3$ is given by $\|x\|_L = \sqrt{|\langle x, x \rangle_L|}$ [2]. If the curve α is non-unit speed, then

$$\kappa(t) = \frac{\|\alpha'(t) \wedge_L \alpha''(t)\|_L}{\|\alpha'(t)\|_L^3}, \quad \tau(t) = \frac{\det(\alpha'(t), \alpha''(t), \alpha'''(t))}{\|\alpha'(t) \wedge_L \alpha''(t)\|_L^2}. \tag{1.1}$$

If the curve α is unit speed, then

$$\kappa(s) = \|\alpha''(s)\|_L, \quad \tau(s) = \|B'(s)\|_L. \tag{1.2}$$

[4].

In this paper, we have interested in hyperbolic cylindrical Tzitzeica curves in Minkowski 3-Space, more precisely we ask in what conditions a cylindrical curve is a Tzitzeica one, namely the function $t \to \dfrac{\tau(t)}{d^2(t)}$ constant, where $d(t)$ is the distance from origin to the osculating plane of curve. The Tzitzeica condition yields a third-order ODE which in our framework admits a direct integration. Therefore the final answer of main problem is given via a second order ODE which in the Hyperbolic case is exactly the equation of a forced harmonic oscillator. In this case, the solution depends of four real constants: one defining the Tzitzeica condition and other three obtained by integration.

2. HYPERBOLIC CYLINDRICAL TZITZEICA CURVES

Proposition 2. 1. Let $\alpha(t)$ be a hyperbolic cylindrical curve in Minkowski 3-space. Then the curve $\alpha(t)$ is Tzitzeica curve if and only if

$$f(t) = f(0)\cosh t + f'(0)\sinh t + \int_0^t G(u)\sinh(u-t)du$$

$$f(t) = f(0)\cosh t + f'(0)\sinh t + \int_0^t \frac{\sinh(u-t)du}{Ku+c}$$

where $f(0), f'(0), K \neq 0$ and c are real constants.

Proof. Let in E_1^3 a curve C given in vectorial form $C : \alpha = \alpha(t)$. This curve is called hyperbolic cylindrical if has the expression $\alpha(t) = (\cosh t, \sinh t, f(t))$ for some $f \in C^\infty$. The torsion function is

$$\tau(t) = \frac{\det(\alpha'(t), \alpha''(t), \alpha'''(t))}{\|\alpha'(t) \wedge_L \alpha''(t)\|_L^2} = \frac{f' - f'''}{f'^2 - f''^2 + 1}.$$

Then the distance from origin to the osculating plane is

$$d(t) = \frac{|f - f''|}{\sqrt{|f'^2 - f''^2 + 1|}}.$$

Let us suppose that the curve is Tzitzeica with the constant $K \neq 0$, because the curve is not contained in a plane

$$K = \frac{\tau(t)}{d^2(t)} = \frac{f' - f'''}{f'^2 - f''^2 + 1} \frac{|f'^2 - f''^2 + 1|^2}{|f'' - f|^2}$$
$$= -\frac{f'''(t) - f'(t)}{(f''(t) - f(t))^2}.$$

Integration gives

$$f''(t) - f(t) = -\frac{1}{Kt + c} \tag{2. 2}$$

where c is a real constant. Then the Laplace transform gives

$$[s^2 Y(s) - sf(0) - f'(0)] + Y(s) = L\{G(t)\} = g(s)$$

where $Y(s)$ and $G(s)$ denote the Laplace transform of $f(t)$ and $g(t)$ respectively, $f(0)$ and $f'(0)$ are arbitrary constants. Hence

$$Y(s) = f(0)\frac{s}{s^2 - 1} + f'(0)\frac{1}{s^2 - 1} + \frac{1}{s^2 - 1} L\{G(t)\}$$
$$= f(0)\frac{s}{s^2 - 1} + f'(0)\frac{1}{s^2 - 1} + \frac{1}{s^2 - 1} g(s)$$

and therefore

$$f(t) = f(0)\cosh t + f'(0)\sinh t + G(t) * \sinh t,$$

where the function denoted by $G(t) * \sinh t$ and defined as

$$G(t) * \sinh t = \int_0^t G(u)\sinh(u-t)du$$

is called the convolution of the functions $\sinh t$ and $G(t)$ or

$$f(t) = f(0)\cosh t + f'(0)\sinh t + \int_0^t G(u)\sinh(u-t)du$$

Theorem 2. 2. If hyperbolic cylindirical Tzitzeica curve is taken as $\alpha(t) = (\cosh t, \sinh t, f(t))$ in Minkowski 3- space, than the curve is spacelike

Proof. The tangent of the curve $\beta(t) = (\sinh t, \cosh t, f(t))$ is $T(t) = (\cosh t, \sinh t, f'(t))$. Since the tangent vector field of the curve $\alpha(t)$ is $\langle T, T \rangle_L = 1 + (f'(t))^2 > 0$. For the all value of $f'(t)$, the curve $\alpha(t)$ is always spacelike curve. Another hyperbolic cylindirical Tzitzeica curve be $\beta(t) = (\sinh t, \cosh t, f(t))$. Straightforward computation gives

$$\tau(t) = \frac{f' - f'''}{f'^2 - f''^2 + 1},$$

$$d(t) = \frac{|f'' - f|}{\sqrt{|f''^2 - f'^2 + 1|}}.$$

For a hyperbolic Tzitzeica curve

$$K = \frac{\tau(t)}{d^2(t)} = \frac{f'''(t) - f'(t)}{(f''(t) - f(t))^2}.$$

And integration gives

$$f''(t) - f(t) = -\frac{1}{Kt + c}.$$

Using the same Formula (2. 3) and the identity

$$f(t) = f(0)\cosh t + f'(0)\sinh t + \int_0^t \frac{\sinh(u-t)du}{Ku + c}$$

it results as required.

Theorem 2. 3. A hyperbolic cylindirical Tzitzeica curve is given as $\beta(t) = (\sinh t, \cosh t, f(t))$ in Minkowski 3-space, than the curve is spacelike, timelike or null curve if and only if $f'^2(0) < 1, f'^2(0) > 1$ and $f'^2(0) = 1$, respectively.

Proof. Since the curve $\beta(t) = (\sinh t, \cosh t, f(t))$, the tangent of the curve is $T(t) = (\cosh t, \sinh t, f'(t))$. The taylor series of the function f in the neighbourhood of zero is

$$f(t) = f(0) + f'(0)t + \frac{f''(0)}{2!}t^2 + \frac{f''(0)}{3!}t^3 + \ldots$$

We take into consideration satisfying

$$f(0) = f'(0) \neq 0, \quad 0 = f''(0) = f'''(0) = \ldots \tag{2. 3}$$

in the neighbourhood of zero. Then we have

$$f(0)\cosh t + f'(0)\sinh t + \int_0^t \frac{\sinh(u-t)du}{Ku+c} = f(0) + f'(0)t.$$

From the last equation, we get

$$\begin{aligned} &f(0)\cosh t = f(0) \\ &f'(0)\sinh t = f'(0)t \\ &\int_0^t \frac{\sinh(u-t)}{Ku+c}du = 0. \end{aligned}$$

Then, for $t \to 0$

$$\begin{aligned} &\cosh t = 1 \\ &\sinh t = t \\ &\lim_{t\to 0} \int_0^t \frac{\sinh(u-t)}{Ku+c}du = 0. \end{aligned}$$

Thus, satisfying the equation (2. 3) as $t \to 0$, $\sinh t = t$ and the function f is written such as $f(t) = f'(0)\sinh t$. If we take the derivative of the last equation for t and square, we have $f'^2(t) = f'^2(0)\cosh^2 t$.

i. The tangent vector field of the spacelike $\beta(t) = (\sinh t, \cosh t, f(t))$ is

$$\langle T(t), T(t)_L \rangle = -1 + f'^2 > 0 .$$

Then, we have

$$\begin{aligned} &f'^2(t) > 1, \\ &f'^2(0)\cosh^2 t > 1. \end{aligned}$$

For $t \to 0$, since $|\cosh t| > 1$, we have $f'^2(0) > 1$.

ii. The curve $\beta(t)$ is timelike curve iff

$$\langle T(t), T(t)_L \rangle = -1 + f'^2 < 0.$$

Then, we have $f'^2(0) > 1$.

iii. The curve $\beta(t)$ is null curve iff

$$\langle T(t), T(t)_L \rangle = -1 + f'^2 = 0.$$

Then, we have $f'^2(0) = 1$.

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