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Matrices with Hyperbolic Number Entries

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Abstract. In this study, firstly, we will present some properties of hyperbolic numbers. Then, we will introduce hyperbolic matrices, which are matrices with hyperbolic number entries. Additionally, we will examine the algebraic properties of these matrices and reveal its difference from other matrix structures such as real, dual, and complex matrices. As a result of comparing the results found in this work with real, dual, and complex matrices, it will be revealed that there are similarities in additive properties and differences in some multiplicative properties. Finally, we will define some special hyperbolic matrices and give their properties and relations with real matrices.

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1. Introduction

Hyperbolic numbers are an extension of real numbers which can be defined as

$$\mathbb{H} = \mathbb{R}[\mathbf{h}] = \{x_0 + x_1\mathbf{h} : x_0, x_1 \in \mathbb{R}, \mathbf{h}^2 = 1, \mathbf{h} \notin \mathbb{R}\}.$$

In the literature, these numbers are also called split-complex numbers, double numbers, or perplex numbers. Hyperbolic numbers are being studied at increasing popularity by many researchers because of their substantial algebraic structures and properties [2, 6, 8]. Beauregard and Suryanarayan examined the Pythagorean triples in terms of hyperbolic numbers [3]. Gutin has studied the matrix decomposition over hyperbolic numbers [7]. Motter and Rosa studied Calculus on hyperbolic numbers [9]. Studies on matrices, which have been examined with different number systems in the literature, can be seen in [1, 4, 5, 10, 12].

For every $h_1 = x_0 + x_1\mathbf{h}$ and $h_2 = y_0 + y_1\mathbf{h}$ hyperbolic numbers, equality, addition and multiplication are defined by

$$\begin{aligned} h_1 = h_2 \quad &\text{iff} \quad x_0 = y_0 \quad \text{and} \quad x_1 = y_1, \\ h_1 + h_2 &= (x_0 + y_0) + (x_1 + y_1)\mathbf{h}, \\ h_1 h_2 &= (x_0y_0 + x_1y_1) + (x_0y_1 + x_1y_0)\mathbf{h}, \end{aligned}$$

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respectively. The hyperbolic conjugation of a hyperbolic number $h = x_0 + x_1\mathbf{h}$ is defined by $\overline{h} = x_0 - x_1\mathbf{h}$. The set of hyperbolic numbers is a commutative, associative ring, and two-dimensional vector space over the real numbers. The set of these numbers also corresponds to $\mathbb{R}^{1,0}$ Clifford algebra [9, 11].

Hyperbolic number can be written as $h = x_0 + x_1\mathbf{h}$ by using the standard base $\{1, \mathbf{h}\}$. Moreover, there is another base $\{\mathbf{h}_+, \mathbf{h}_-\}$ which is called idempotent base regarding with isotropic line which separates the hyperbolic plane. Let $\mathbf{h}_+ = \frac{1}{2}(1 + \mathbf{h})$, $\mathbf{h}_- = \frac{1}{2}(1 - \mathbf{h})$, and $x_+ = x_0 + x_1$, $x_- = x_0 - x_1$, then every hyperbolic number h can be expressed with the help of idempotent base as $h = x_+\mathbf{h}_+ + x_-\mathbf{h}_-$. Moreover, the following statements satisfy;

(1) $\mathbf{h}_+^2 = \mathbf{h}_+$, $\mathbf{h}_-^2 = \mathbf{h}_-$,
(2) $\mathbf{h}_+\mathbf{h}_- = 0$,
(3) $h\mathbf{h}_+ = x_+\mathbf{h}_+$,
(4) $h\mathbf{h}_- = x_+\mathbf{h}_-$, [11].

The powers of hyperbolic numbers can be calculated easily using idempotent bases. Indeed, for every $h \in \mathbb{H}$ and $k \in \mathbb{R}$, we have

$$h^k = (\mathbf{h}_+x_+ + \mathbf{h}_-x_-)^k = (\mathbf{h}_+)^k(x_+)^k + (\mathbf{h}_-)^k(x_-)^k = (\mathbf{h}_+)^k x_+ + (\mathbf{h}_-)^k x_-.$$

Furthermore, it can be defined as a function $f(h) = f(\mathbf{h}_+)x_+ f(\mathbf{h}_-)x_-$, [11].

Example 1.1. Let $f(h) = sin(h)$ and $h = x_0 + x_1\mathbf{h}$, then

$$\begin{aligned} f(h) &= sin(h) = sin(x_0 + x_1\mathbf{h}) \\ &= sin(x_0 + x_1)\frac{1}{2}(1 + \mathbf{h}) + sin(x_0 - x_1)\frac{1}{2}(1 - \mathbf{h}) \\ &= sin(x_0)cos(x_1) + sin(x_1)cos(x_0)\mathbf{h}. \end{aligned}$$

Moreover, hyperbolic numbers provide the following expression

$$\sqrt{x_0 + x_1\mathbf{h}} = \frac{1}{2}[(\sqrt{x_0 + x_1} + \sqrt{x_0 - x_1}) + (\sqrt{x_0 + x_1} - \sqrt{x_0 - x_1})\mathbf{h}]$$

which was given by G. Sobczyk [11]. For further information about hyperbolic numbers and related works, we can refer to reader [11, 13].

The presentation of the rest of this study is planned as follows. In section 2, we will define hyperbolic matrices using the hyperbolic numbers, and their fundamental properties such as matrix operations, the inverse of hyperbolic matrices, and the transpose of these matrices. In section 3, some special hyperbolic matrices will be introduced with their properties. In the final section, some regarding comments are presented.

2. Hyperbolic Matrices

Let $\mathbb{R}_n^m$ be the set of all $m \times n$ real matrices. It is well known that $\mathbb{R}_n^m$ is a vector space with ordinary matrix addition and the matrix multiplication by a scalar over $\mathbb{R}$. Let $\mathbb{H}_n^m$ be the set of $m \times n$ matrices with hyperbolic number entries. A hyperbolic matrix can be shown as follows:

$$\begin{aligned} \hat{A} &= \begin{bmatrix} a_{11} + \mathbf{h}a_{11}^* & a_{12} + \mathbf{h}a_{12}^* & \dots & a_{1n} + \mathbf{h}a_{1n}^* \\ a_{21} + \mathbf{h}a_{21}^* & a_{22} + \mathbf{h}a_{22}^* & \dots & a_{2n} + \mathbf{h}a_{2n}^* \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + \mathbf{h}a_{m1}^* & a_{m2} + \mathbf{h}a_{m2}^* & \dots & a_{mn} + \mathbf{h}a_{mn}^* \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} + \mathbf{h} \begin{bmatrix} a_{11}^* & a_{12}^* & \dots & a_{1n}^* \\ a_{21}^* & a_{22}^* & \dots & a_{2n}^* \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}^* & a_{m2}^* & \dots & a_{mn}^* \end{bmatrix} \\ &= A + \mathbf{h}A^*, \end{aligned}$$

where A and A^* are $m \times n$ real matrices. Note that every hyperbolic matrix can be written as a combination of two real matrices. Thus, the set of hyperbolic matrices can be given

$$\mathbb{H} = \{\hat{A} = A + \mathbf{h}A^* : A, A^* \in \mathbb{R}_n^m, \mathbf{h}^2 = 1, \mathbf{h} \notin \mathbb{R}\}.$$

For instance,

$$\hat{A} = \begin{bmatrix} 1 + \mathbf{h} & 2 - \mathbf{h} & -7\mathbf{h} \\ \mathbf{h} & 5 + 2\mathbf{h} & 4 - \mathbf{h} \end{bmatrix}$$

is a hyperbolic matrix and it can be written as

$$\hat{A} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 5 & 4 \end{bmatrix} + \mathbf{h} \begin{bmatrix} 1 & -1 & -7 \\ 1 & 2 & -1 \end{bmatrix}.$$

Definition 2.1. Some special types of matrices can be given as follows:

(1) Matrices that include just one column(line) are called column(line) hyperbolic matrices.
(2) If all entries of the matrix are 0, then it is called as zero hyperbolic matrix and it is shown as $0_{\mathbb{H}}$.
(3) If the number of line equals the number of columns of a hyperbolic matrix, that is $n \times n$ type, then the matrix is called square hyperbolic matrix.
(4) For square hyperbolic matrices in which all the entries on the diagonal are equals to 1 and others are 0 called unit hyperbolic matrices and denoted by $\mathbf{I}_{\mathbb{H}}$. It can be easily seen the unit hyperbolic matrix and the unit real matrix are equal. Moreover, for all $\hat{A} \in \mathbb{H}_n^n$, $\mathbf{I}_{\mathbb{H}} \cdot \hat{A} = \hat{A} \cdot \mathbf{I}_{\mathbb{H}} = \hat{A}$.

Example 2.2. $\hat{A} = [1 + 3\mathbf{h} \quad 5 - 2\mathbf{h} \quad 7 + \mathbf{h}]$ is a 1×3 line matrix.

Definition 2.3. Equality, addition, and multiplication by a scalar are defined as follows. Let $\hat{A} = [a_{ij} + \mathbf{h}a_{ij}^*]$ and $\hat{B} = [b_{ij} + \mathbf{h}b_{ij}^*]$ be two $m \times n$ hyperbolic matrices and $\lambda \in \mathbb{H}$ be a scalar.

(1) If $a_{ij} + \mathbf{h}a_{ij}^* = b_{ij} + \mathbf{h}b_{ij}^*$ for all i and j, then these hyperbolic matrices are equal.
(2) The sum of $\hat{A}$ and $\hat{B}$ is

$$\hat{A} + \hat{B} = [(a_{ij} + b_{ij}) + \mathbf{h}(a_{ij}^* + b_{ij}^*)] = [a_{ij} + b_{ij}] + \mathbf{h}[a_{ij}^* + b_{ij}^*].$$

(3) The product of a matrix by a scalar is

$$\lambda\hat{A} = [\lambda(a_{ij} + \mathbf{h}a_{ij}^*)] = [\lambda a_{ij} + \mathbf{h}\lambda a_{ij}^*].$$

Example 2.4. For the following matrices $\hat{A}$ and $\hat{B}$,

$$\hat{A} = \begin{bmatrix} 5 - \mathbf{h} & 2 - \mathbf{h} & -7\mathbf{h} \\ \mathbf{h} & 1 - 2\mathbf{h} & 8 - \mathbf{h} \\ 3 - \mathbf{h} & -3\mathbf{h} & 9 + 2\mathbf{h} \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 3 + 4\mathbf{h} & \mathbf{h} & 2 - 3\mathbf{h} \\ 3 & 4 + 3\mathbf{h} & 3 + 2\mathbf{h} \\ 1 + \mathbf{h} & 5 + \mathbf{h} & 4\mathbf{h} \end{bmatrix}$$

the sum of these matrices is

$$\hat{A} + \hat{B} = \begin{bmatrix} 8 + 3\mathbf{h} & 2 & 2 - 10\mathbf{h} \\ 3 + \mathbf{h} & 5 + \mathbf{h} & 11 + \mathbf{h} \\ 4 & 5 - 2\mathbf{h} & 9 + 6\mathbf{h} \end{bmatrix}.$$

Theorem 2.5. *Let $\hat{A}, \hat{B}, \hat{C} \in \mathbb{H}_n^m$ be hyperbolic matrices and $\lambda_1, \lambda_2 \in \mathbb{H}$ be hyperbolic numbers, then the following properties are satisfied:*

(1) $\hat{A} + \hat{B} = \hat{B} + \hat{A}$,
(2) $\hat{A} + (\hat{B} + \hat{C}) = (\hat{A} + \hat{B}) + \hat{C}$,
(3) $\hat{A} + 0_{\mathbb{H}} = \hat{A}$,
(4) $\lambda_1(\hat{A} + \hat{B}) = \lambda_1\hat{A} + \lambda_1\hat{B}$,
(5) $(\lambda_1 + \lambda_2)\hat{A} = \lambda_1\hat{A} + \lambda_2\hat{A}$,
(6) $(\lambda_1.\lambda_2)\hat{A} = \lambda_1(\lambda_2\hat{A})$,
(7) *The additive inverse of* $\hat{A} = A + \mathbf{h}A^*$ *is* $-\hat{A} = -A - \mathbf{h}A^*$.

Proof. Let $\hat{A} = A + \mathbf{h}A^*$, $\hat{B} = B + \mathbf{h}B^*$, and $\hat{C} = C + \mathbf{h}C^*$.

(1) $\hat{A} + \hat{B} = A + \mathbf{h}A^* + B + \mathbf{h}B^* = (A + B) + \mathbf{h}(A^* + B^*) = (B + A) + \mathbf{h}(B^* + A^*) = B + \mathbf{h}B^* + A + \mathbf{h}A^* = \hat{B} + \hat{A}$.
(2) $\hat{A} + (\hat{B} + \hat{C}) = A + \mathbf{h}A^* + (B + \mathbf{h}B^* + C + \mathbf{h}C^*) = (A + \mathbf{h}A^* + B + \mathbf{h}B^*) + C + \mathbf{h}C^* = (\hat{A} + \hat{B}) + \hat{C}$.
(3) $\hat{A} + 0_{\mathbb{H}} = A + \mathbf{h}A^* + 0 + \mathbf{h}0 = (A + 0) + \mathbf{h}(A^* + 0) = A + \mathbf{h}A^* = \hat{A}$.
(4) $\lambda_1(\hat{A} + \hat{B}) = \lambda_1(A + \mathbf{h}A^* + B + \mathbf{h}B^*) = \lambda_1(A + \mathbf{h}A^*) + \lambda_1(B + \mathbf{h}B^*) = \lambda_1\hat{A} + \lambda_1\hat{B}$.
(5) $(\lambda_1 + \lambda_2)\hat{A} = (\lambda_1 + \lambda_2)(A + \mathbf{h}A^*) = \lambda_1(A + \mathbf{h}A^*) + \lambda_2(A + \mathbf{h}A^*) = \lambda_1\hat{A} + \lambda_2\hat{A}$.
(6) $(\lambda_1.\lambda_2)\hat{A} = (\lambda_1.\lambda_2)(A + \mathbf{h}A^*) = \lambda_1.(\lambda_2(A + \mathbf{h}A^*)) = \lambda_1(\lambda_2\hat{A})$.
(7) $(A + \mathbf{h}A^*) + (-A - \mathbf{h}A^*) = (A - A) + \mathbf{h}(A^* - A^*) = 0_{\mathbb{H}}$.

□

Definition 2.6. Let $\hat{A} = [a_{ij} + \mathbf{h}a^*_{ij}] \in \mathbb{H}^m_p$, $\hat{B} = [b_{ij} + \mathbf{h}b^*_{ij}] \in \mathbb{H}^p_n$ be two hyperbolic matrices, then multiplication of these matrices is

$$\hat{A}.\hat{B} = \left[c_{ij} = \sum_{k=1}^{p}(a_{ik} + \mathbf{h}a^*_{ik})(b_{kj} + \mathbf{h}b^*_{kj})\right]_{m\times n} = \left[c_{ij} = \sum_{k=1}^{p}(a_{ik}b_{kj} + a^*_{ik}b^*_{kj} + \mathbf{h}(a_{ik}b^*_{kj} + a^*_{ik}b_{kj})\right]_{m\times n}.$$

If we represent these matrices as a combination of two real matrices $\hat{A} = A + \mathbf{h}A^*$ and $\hat{B} = B + \mathbf{h}B^*$, then we can obtain the multiplication $\hat{A}.\hat{B}$ in the following form

$$\hat{A}.\hat{B} = (A.B + A^*B^*) + \mathbf{h}(A.B^* + A^*B).$$

Example 2.7. For the following matrices $\hat{A}$ and $\hat{B}$,

$$\hat{A} = \begin{bmatrix} 3+\mathbf{h} & 2+5\mathbf{h} \\ 3-\mathbf{h} & 6+\mathbf{h} \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} -1+5\mathbf{h} & 3 \\ 4-2\mathbf{h} & 3-4\mathbf{h} \end{bmatrix}$$

the multiplication of these matrices is

$$\hat{A}\hat{B} = \begin{bmatrix} 30\mathbf{h} & -5+10\mathbf{h} \\ 14+8\mathbf{h} & 23-24\mathbf{h} \end{bmatrix}.$$

Theorem 2.8. *The following properties are provided:*

(1) *If $\hat{A} \in \mathbb{H}^m_p$, $\hat{B} \in \mathbb{H}^p_n$, $\hat{C} \in \mathbb{H}^n_r$, then $\hat{A}(\hat{B}\hat{C}) = (\hat{A}\hat{B})\hat{C}$.*
(2) *If $\hat{A} \in \mathbb{H}^m_p$, $\hat{B}, \hat{C} \in \mathbb{H}^p_n$, then $\hat{A}(\hat{B} + \hat{C}) = \hat{A}\hat{B} + \hat{A}\hat{C}$.*
(3) *If $\hat{A}, \hat{B} \in \mathbb{H}^m_p$, $\hat{C} \in \mathbb{H}^p_n$, then $(\hat{A} + \hat{B})\hat{C} = \hat{A}\hat{C} + \hat{B}\hat{C}$.*
(4) *If $\lambda \in \mathbb{R}$, $\hat{A}, \in \mathbb{H}^m_p$, $\hat{B} \in \mathbb{H}^p_n$, then $\lambda(\hat{A}\hat{B}) = (\lambda\hat{A})\hat{B} = \hat{A}(\lambda\hat{B})$.*

Proof. Straightforward. □

Corollary 2.9. *The set of $\mathbb{H}^n_n$ is a ring with matrix addition and matrix multiplication.*

Definition 2.10. If there is a hyperbolic matrix $\hat{C}$ satisfying the equation $\hat{A}\hat{C} = \hat{C}\hat{A} = \mathbf{I}$, then $\hat{C}$ is called the inverse of $\hat{A}$, and additionally $\hat{A}$ is called an invertible hyperbolic matrix.

Theorem 2.11. *If A and A^* are two real $n \times n$ invertible matrices, then inverse of the matrix $\hat{A} = A + \mathbf{h}A^*$ is*

$$\hat{A}^{-1} = (A - A^*A^{-1}A^*)^{-1} + \mathbf{h}(A^* - A(A^*)^{-1}A)^{-1}.$$

Proof. Since the multiplication of matrices is not commutative, the equation must be provided from both the right and the left sides. Let $C + \mathbf{h}C^*$ be the inverse of the matrix $\hat{A} = A + \mathbf{h}A^*$, then

$$(A + \mathbf{h}A^*)(C + \mathbf{h}C^*) = \mathbf{I},$$
$$(AC + A^*C^*) + \mathbf{h}(AC^* + A^*C) = I + \mathbf{h}0$$

in order to obtain this equality, $AC + A^*C^* = I$ and $AC^* + A^*C = 0$ equations must be provided. Thus, it can be seen that $C = A^{-1} - A^{-1}A^*C^*$ and $C = -(A^*)^{-1}AC^*$. From last two equation, we have

$$\begin{aligned} A^{-1} - A^{-1}A^*C^* &= -(A^*)^{-1}AC^* \\ A^{-1}A^*C^* - (A^*)^{-1}AC^* &= A^{-1} \\ (A^{-1}A^* - (A^*)^{-1}A)C^* &= A^{-1} \\ C^* &= (A^* - A(A^*)^{-1}A)^{-1}. \end{aligned}$$

Similarly, from equations $AC + A^*C^* = I$ and $AC^* + A^*C = 0$, we can obtain $C^* = (A^*)^{-1} - (A^*)^{-1}AC$ and $C^* = -A^{-1}A^*C$. Then, we get

$$\begin{aligned} (A^*)^{-1} - (A^*)^{-1}AC &= -A^{-1}A^*C \\ (A^*)^{-1}AC - A^{-1}A^*C &= (A^*)^{-1} \\ ((A^*)^{-1}A - A^{-1}A^*)C &= (A^*)^{-1} \\ C &= (A - A^*A^{-1}A^*)^{-1}. \end{aligned}$$

When we apply similar steps from the equation $(C + \mathbf{h}C^*)(A + \mathbf{h}A^*) = \mathbf{I}$, we get same C and C^*. □

Definition 2.12. The sum of diagonal elements of a square hyperbolic matrix $\hat{A}$ is called the trace of $\hat{A}$ and denoted by $tr\hat{A}$. Let $\hat{A} = A + \mathbf{h}A^*$, then

$$tr\hat{A} = \sum_{i=1}^{n}(a_{ii} + \mathbf{h}a_{ii}^*) = tr(A) + \mathbf{h}tr(A^*).$$

Theorem 2.13. *The following properties are satisfied;*

(1) *Let $\hat{A}, \hat{B} \in \mathbb{H}_n^n$ be hyperbolic matrices, then $tr(\hat{A} + \hat{B}) = tr(\hat{A}) + tr(\hat{B})$.*
(2) *Let $\hat{A} \in \mathbb{H}_n^n$ be a hyperbolic matrix and $\lambda \in \mathbb{H}$ be a hyperbolic number, then $tr(\lambda\hat{A}) = \lambda tr(\hat{A})$.*
(3) *Let $\hat{A}, \hat{B} \in \mathbb{H}_n^n$, then $tr(\hat{A}\hat{B}) = tr(\hat{B}\hat{A})$.*

Proof. Let $\hat{A} = [a_{ij} + \mathbf{h}a_{ij}^*]$, $\hat{B} = [b_{ij} + \mathbf{h}b_{ij}^*]$ and $\lambda \in \mathbb{H}$.

(1) $tr(\hat{A} + \hat{B}) = \sum_{i=1}^{n}(a_{ii} + \mathbf{h}a_{ii}^* + b_{ii} + \mathbf{h}b_{ii}^*) = \sum_{i=1}^{n}(a_{ii} + \mathbf{h}a_{ii}^*) + \sum_{i=1}^{n}(b_{ii} + \mathbf{h}b_{ii}^*) = tr(\hat{A}) + tr(\hat{B})$.
(2) $tr(\lambda\hat{A}) = \sum_{i=1}^{n}(\lambda(a_{ii} + \mathbf{h}a_{ii}^*)) = \lambda\sum_{i=1}^{n}(a_{ii} + \mathbf{h}a_{ii}^*) = \lambda tr(\hat{A})$.
(3) For $A, C \in \mathbb{R}_n^n$, it is well known that $tr(AC) = tr(CA)$. With this feature, the desired equality can be obtained by using the properties of hyperbolic matrices.

□

Theorem 2.14. *Let $\hat{A} \in \mathbb{H}_p^m$, $\hat{C} \in \mathbb{H}_n^p$, then $\left(\hat{A}.\hat{C}\right)^T = \hat{C}^T.\hat{A}^T$.*

Proof. Let $\hat{A} = A + \mathbf{h}A^*$, $\hat{C} = C + \mathbf{h}C^*$. We know that $(AC)^T = C^T A^T$ for $A, C \in \mathbb{R}_n^n$.

$$\begin{aligned}\left(\hat{A}.\hat{C}\right)^T &= (AC + \mathbf{h}(AC^* + A^*C))^T \\ &= (AC)^T + \mathbf{h}(AC^* + A^*C)^T \\ &= (AC)^T + \mathbf{h}((AC^*)^T + (A^*C)^T) \\ &= C^T A^T + \mathbf{h}((C^*)^T A^T + C^T(A^*)^T) = \hat{C}^T.\hat{A}^T.\end{aligned}$$

□

3. Some Special Hyperbolic Matrices

In this section, we define special types of hyperbolic matrices. They are periodic hyperbolic matrices, symmetric hyperbolic matrices, skew-symmetric hyperbolic matrices, idempotent hyperbolic matrices, Hermitian hyperbolic matrices, skew Hermitian hyperbolic matrices, and unitary hyperbolic matrices. Thus, the difference between hyperbolic matrices and other matrix structures is clearly observed.

Definition 3.1. Let $\hat{A}$ be a square hyperbolic matrix. If $\hat{A}^2 = \hat{A}$, then $\hat{A}$ is called idempotent hyperbolic matrix.

Theorem 3.2. *If $\hat{A} = A + \mathbf{h}A^*$ is an idempotent hyperbolic matrix, then $A + A^*$ is a real idempotent matrix.*

Proof. Let $\hat{A}$ be an idempotent hyperbolic matrix, then

$$(A^2 + (A^*)^2) + \mathbf{h}(AA^* + A^*A) = A + \mathbf{h}A^*$$

From last equation, we have $A^2 + (A^*)^2 = A$ and $AA^* + A^*A = A^*$. If we add these two equations side by side, then

$$\begin{aligned}A^2 + AA^* + (A^*)^2 + A^*A &= A + A^* \\ A(A + A^*) + A^*(A^* + A) &= A + A^* \\ (A + A^*)^2 &= A + A^*.\end{aligned}$$

□

Definition 3.3. A square hyperbolic matrix $\hat{A}$ is called a periodic hyperbolic matrix if $\hat{A}^{p+1} = \hat{A}$. Also, p is called period of the matrix.

Definition 3.4. A square hyperbolic matrix $\hat{A}$ is called a nilpotent hyperbolic matrix if $\hat{A}^q = \mathbf{0}$. Also, q is called index of the matrix.

Definition 3.5. A square hyperbolic matrix $\hat{A}$ is called a symmetric hyperbolic matrix if $\hat{A}^T = \hat{A}$.

Theorem 3.6. *$\hat{A} = A + \mathbf{h}A^*$ is a symmetric hyperbolic matrix iff A and A^* are real symmetric matrices.*

Proof. If $\hat{A}$ is a symmetric matrix, then

$$\begin{aligned} \hat{A}^T = \hat{A} &\iff (A + \mathbf{h}A^*)^T = A + \mathbf{h}A^* \\ &\iff A^T + \mathbf{h}(A^*)^T = A + \mathbf{h}A^* \\ &\iff A^T = A \quad \text{and} \quad (A^*)^T = A^*. \end{aligned}$$

□

Definition 3.7. A square hyperbolic matrix $\hat{A}$ is called a skew symmetric hyperbolic matrix if $\hat{A}^T = -\hat{A}$.

Theorem 3.8. *$\hat{A} = A + \mathbf{h}A^*$ is a skew symmetric hyperbolic matrix iff A and A^* are real skew symmetric matrices.*

Proof. If $\hat{A}$ is a skew symmetric matrix, then

$$\begin{aligned} \hat{A}^T = -\hat{A} &\iff (A + \mathbf{h}A^*)^T = -A - \mathbf{h}A^* \\ &\iff A^T + \mathbf{h}(A^*)^T = -A - \mathbf{h}A^* \\ &\iff A^T = -A \quad \text{and} \quad (A^*)^T = -A^*. \end{aligned}$$

□

Example 3.9. $\hat{A} = \begin{bmatrix} 0 & -2 + \mathbf{h} \\ 2 - \mathbf{h} & 0 \end{bmatrix}$ is a skew symmetric hyperbolic matrix. Indeed, it can be obviously seen that $\hat{A}^T = -\hat{A}$.

Definition 3.10. A square hyperbolic matrix $\hat{A}$ is called a Hermitian hyperbolic matrix if $\overline{(\hat{A}^T)} = \hat{A}$, where $\overline{\hat{A}} = A - \mathbf{h}A^*$ is the conjugate of $\hat{A}$.

Theorem 3.11. *$\hat{A} = A + \mathbf{h}A^*$ is a Hermitian hyperbolic matrix iff A is a real symmetric hyperbolic matrix and A^* is real skew symmetric matrix.*

Proof. If $\hat{A}$ is a Hermitian hyperbolic matrix, then

$$\begin{aligned} \overline{(\hat{A}^T)} = \hat{A} &\iff \overline{(A + \mathbf{h}A^*)^T} = A + \mathbf{h}A^* \\ &\iff \overline{A^T + \mathbf{h}(A^*)^T} = A + \mathbf{h}A^* \\ &\iff A^T - \mathbf{h}(A^*)^T = A + \mathbf{h}A^* \\ &\iff A^T = A \quad \text{and} \quad (A^*)^T = -A^*. \end{aligned}$$

□

Definition 3.12. A square hyperbolic matrix $\hat{A}$ is called a skew Hermitian hyperbolic matrix if $\overline{(\hat{A}^T)} = -\hat{A}$.

Theorem 3.13. *$\hat{A} = A + \mathbf{h}A^*$ is a skew Hermitian hyperbolic matrix iff A is a real skew symmetric hyperbolic matrix and A^* is real symmetric matrix.*

Proof. If $\hat{A}$ is a skew Hermitian hyperbolic matrix, then

$$\begin{aligned} \overline{(\hat{A}^T)} = -\hat{A} &\iff \overline{(A + \mathbf{h}A^*)^T} = -A - \mathbf{h}A^* \\ &\iff \overline{A^T + \mathbf{h}(A^*)^T} = -A - \mathbf{h}A^* \\ &\iff A^T - \mathbf{h}(A^*)^T = -A - \mathbf{h}A^* \\ &\iff A^T = -A \quad \text{and} \quad (A^*)^T = A^*. \end{aligned}$$

□

Definition 3.14. A square hyperbolic matrix $\hat{A}$ is called an involute hyperbolic matrix if $\hat{A}^2 = \mathbf{I}$.

Theorem 3.15. *If $\hat{A} = A + \mathbf{h}A^*$ is an involute hyperbolic matrix, then $A^2 + (A^*)^2 = I$ and $(A^*)^n = (-1)^n A^{-1}(-A^*)^n A$ for $n \in \mathbb{Z}^+$.*

Proof. Let $\hat{A}$ be an involute hyperbolic matrix, then from the equation $(A + \mathbf{h}A^*)^2 = \mathbf{I}$ we get,

$$(A^2 + (A^*)^2) + \mathbf{h}(AA^* + A^*A) = I + \mathbf{h}0$$
$$A^2 + (A^*)^2 = I \quad \text{and} \quad AA^* + A^*A = 0.$$

Since $AA^* + A^*A = 0$, it is clear to see that $AA^* = -A^*A$. Thus, from the last equation desiring result is obtained as follows:

$$\begin{aligned} A^* &= -A^{-1}A^*A \\ (A^*)^2 &= A^{-1}(A^*)^2A \\ (A^*)^3 &= -A^{-1}(A^*)^3A \\ &\vdots \\ (A^*)^n &= (-1)^n A^{-1}(-A^*)^n A. \end{aligned}$$

□

Definition 3.16. Let $\hat{B} = B + \mathbf{h}B^*$ and $\hat{C} = C + \mathbf{h}C^*$ be two $n \times n$ hyperbolic matrices. If $\hat{B}\hat{C} = \hat{C}\hat{B}$, then these matrices are called commutative hyperbolic matrices.

Theorem 3.17. *If $\hat{B}$ and $\hat{C}$ are commutative, then $B + B^*$ and $C + C^*$ matrices are real commutative matrices.*

Proof. Let $\hat{B}\hat{C} = \hat{C}\hat{B}$, then

$$(BC + B^*C^*) + \mathbf{h}(BC^* + B^*C) = (CB + C^*B^*) + \mathbf{h}(CB^* + C^*B).$$

We obtain equalities $BC + B^*C^* = CB + C^*B^*$ and $BC^* + B^*C = CB^* + C^*B$. If we add these two equations side by side, then

$$B(C + C^*) + B^*(C + C^*) = (C + C^*)B + (C + C^*)B^*$$
$$(B + B^*)(C + C^*) = (C + C^*)(B + B^*).$$

□

Definition 3.18. Let $\hat{A} = A + \mathbf{h}A^*$ be a square hyperbolic matrix. If $\overline{\hat{A}^T}\hat{A} = \hat{A}\overline{\hat{A}^T} = \mathbf{I}$, then $\hat{A} = A + \mathbf{h}A^*$ is called unitary hyperbolic matrix.

Theorem 3.19. *Let $\hat{A} = A + \mathbf{h}A^*$ be a unitary hyperbolic matrix. Then, $A + A^*$ is a real unitary matrix.*

Proof. If $\overline{\hat{A}^T}\hat{A} = \mathbf{I}$, then

$$(A^T - \mathbf{h}(A^*)^T)(A + \mathbf{h}A^*) = I + \mathbf{h}0$$
$$(A^TA - (A^*)^TA^*) + \mathbf{h}(A^TA^* - (A^*)^TA) = I + \mathbf{h}0.$$

From the last equation, it can be easily seen that $A^TA - (A^*)^TA^* = I$ and $A^TA^* - (A^*)^TA = 0$. Thus, if we add these equations side by side, then we get

$$A^T(A + A^*) - (A^*)^T(A + A^*) = I$$
$$(A^T - (A^*)^T)(A + A^*) = I.$$

Also, if $\hat{A}\overline{\hat{A}^T} = \mathbf{I}$, then

$$(A + \mathbf{h}A^*)(A^T - \mathbf{h}(A^*)^T) = I + \mathbf{h}0$$
$$(AA^T - A^*(A^*)^T) + h(A^*A^T - A(A^*)^T) = I + \mathbf{h}0.$$

From the last equation, we obtain $AA^T - A^*(A^*)^T = I$ and $A^*A^T - A(A^*)^T = 0$. Therefore, if we sum these equations up side by side, then

$$\begin{aligned}(A + A^*)A^T - (A + A^*)(A^*)^T &= I\\ (A + A^*)(A^T - (A^*)^T) &= I.\end{aligned}$$

□

4. Conclusion

In this study, we examined the properties of hyperbolic matrices. Using these properties, we have defined special hyperbolic matrices and revealed some of the properties they provide. Thus, the difference between these matrices and other matrix structures is regularly considered. Thus, it is aimed to form a basis for further studies on hyperbolic numbers and matrices.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

All authors have contributed sufficiently in the planning, execution, or analysis of this study to be included as authors. All authors have read and agreed to the published version of the manuscript.

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