Differential subordination and harmonic means for the Koebe type $n$-fold symmetric functions

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Abstract
In this research, we develop some differential subordination results involving harmonic means of $f_{b}(z)$, $f_{b}(z) + zf_{b}'(z)$, and $f_{b}(z) + zf_{b}'(z)f_{b}(z)$, where $f_{b}(z) = z(1 - z)^{b}$, $b \geq 0; n \in \mathbb{N} = 1, 2, 3...$ is an $n$-fold symmetric Koebe type functions defined in the unit disk with $f_{b}(0) = 0, f_{b}'(z) \neq 0$. By using the admissibility conditions, we also study several applications in the geometric function theory.

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1. Introduction and definitions
Harmonic mean is a particular case of the power mean and it is used where the average rate is desired. It is Pythagorean mean, along with arithmetic and geometric mean. This is always less than either of them. The harmonic mean denoted by $H$ of positive reals $x_{1}, x_{2}, ..., x_{n}$ is defined as that the reciprocals of the arithmetic average or mean of the reciprocals of $x_{1}, x_{2}, ..., x_{n}$, that is

$$H = \left(\frac{1}{k} \sum_{n=1}^{k} \frac{1}{x_{n}}\right)^{-1}$$

For a special choice, we write $H = \frac{2x_{1}x_{2}}{x_{1}+x_{2}}$. It is related to arithmetic mean $A$ and geometric mean $G$ by $G = \sqrt{AH}$. For more information, we refer [1–7, 9–12]. To obtain our major results, we are introducing some preliminary notations. Let $E$ be the unit open disk in $\mathbb{C}$ defined by $|z| < 1$ and let $\mathcal{H}(E)$ represent the class of functions holomorphic or analytic in $E$. Let $A_{n}$ be the subcollection of $\mathcal{H}(E)$ of functions $f : f(0) = f'(0) - 1 = 0$. A function $f \in A_{n}$ is represented by

$$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + ..., z \in E.$$ (1.1)

A function $f \in A_{n}$ is univalent, if it is one-to-one in $E$. We denote this set by $S$. For $f, g \in \mathcal{H}(E)$, we say that $f \prec g$, if there exists a well-known Schwarz function $w : f(z) = g(w(z))$.

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for \( z \in \mathbb{E} \). If \( g \in \mathcal{S} \), then this \( \prec \) is equivalent to \( f(0) = g(0) \) and \( f(\mathbb{E}) \subset g(\mathbb{E}) \). In the following, we present some definitions. Let \( \mathcal{P} \) be the collection of functions \( p : p \in \mathcal{H}(\mathbb{E}) \), \( p(0) = 1 \), \( \text{Re} \, p(z) > 0 \) and

\[
p(z) = 1 + c_2 z^2 + c_3 z^3 + \ldots, z \in \mathbb{E}.
\]

**Definition 1.1.** For \( \Omega \subseteq \mathbb{C} \), \( n \in \mathbb{N} \) and \( h \in \Omega \), the set of admissible functions \( \Psi_n [\Omega, h] \) consists of \( \Psi : \mathbb{C}^3 \times \mathbb{E} \rightarrow \mathbb{C} \) that meet the admissibility conditions:

\[
\Psi(r, s, t, z) \notin \Omega, \text{ for } r = h(\xi), s = m \xi h'(\xi)
\]

and

\[
\text{Re} \, \frac{t}{s} + 1 \geq m \text{Re} \left( \frac{\xi h''(\xi)}{h'(\xi)} + 1 \right), \quad \xi \in \overline{\mathbb{E}} \setminus \mathbb{E}(h), m \geq n, z \in \mathbb{E},
\]

where the set \( \Omega \) represents analytic or holomorphic and injective functions on \( \partial \mathbb{E} \setminus \mathbb{E}(h) : \mathbb{E}(h) = \{ \xi \in \partial \mathbb{E} : \lim_{z \rightarrow \xi} h(z) = \infty \} \), such that \( h'(\xi) \neq 0 \) for \( \xi \in \partial \mathbb{E} \setminus \mathbb{E}(h) \). If \( h \in \Omega \), then \( h(\mathbb{E}) \) is simply connected.

An \( n \)-fold symmetric Koebe type function \( f_b \) is defined by

\[
f_b(z) = \frac{z}{(1 - z^n)^b}, b \geq 0; n = 1, 2, 3, \ldots.
\]

The function \( f_b \) becomes the Koebe function for \( n = 1 \) and \( b = 2 \). For the above definitions, we refer \([2]\) and \([8]\) respectively.

2. Preliminaries

To discuss our prime theorems, we need the following useful lemma.

**Lemma 2.1 \([8]\).** Let \( h \in \Omega : h(0) = a \), and \( p(z) = a + a_n z^n + \ldots \) with \( p(z) \neq a \) and \( n \geq 1 \). If \( p \not\prec h \), then for \( z_0 = r_0 e^{i\theta} \in \mathbb{E}, \xi_0 \in \partial \mathbb{E} \setminus \mathbb{E}(h) \) and \( m \geq n \geq 1, p(\mathbb{E}_{r_0}) \subset h(\mathbb{E}), p(z_0) = h(\xi_0), h(\xi_0) = m \xi_0 h'(\xi_0) \) and \( \text{Re} \, \frac{z_0 h''(z_0)}{p'(z_0)} + 1 \geq m \text{Re} \left( \frac{\xi_0 h''(\xi_0)}{h'(\xi_0)} + 1 \right) \).

3. Main results

**Theorem 3.1.** Let \( f_b(z) = \frac{z}{(1 - z^n)^b} \) be holomorphic or analytic in \( \mathbb{E} \) with \( f_b(z) \neq 1 \). If

\[
\text{Re} \left\{ \frac{2 f_b(z) [f_b(z) + zf'_b(z)]}{2 f_b(z) + zf'_b(z)} \right\} \in \mathcal{P}, \quad (3.1)
\]

then \( f_b \in \mathcal{P} \).

**Proof.** Let us suppose that

\[
h(z) = \frac{1 - (nb - 1) z}{1 - z} \quad (3.2)
\]

with \( h(\mathbb{E}) = \{ w : \text{Re} \, w > 0 \} \) and \( b > 1 \) such that \( h(0) = 1, h(\mathbb{E}) = \{1\}, h \in \Omega \). Then, from \((3.1)\), we want to prove that \( f_b \not\prec h \). Assume that \( f_b \not\prec h \). The Lemma2.1 shows that there is a point \( z_0 \in \mathbb{E} \) and \( \xi_0 \in \partial \mathbb{E} \setminus \{1\} \) such that \( f_b(z_0) = h(\xi_0) \) and \( \text{Re} \, f_b(z) > 0, z \in \mathbb{E}_{|z|>r} \). Thus \( \text{Re} \, f_b(z_0) = 0 \). Therefore, we choose \( f_b(z_0) = ix \), where \( x \) is real. Because of symmetry, we consider that \( x > 0 \). Take \( h^{-1}(z) = \frac{z - 1}{z - (nb - 1)} \). Then

\[
\xi_0 = h^{-1}(f_b(z_0)) = \frac{1 - f_b(z_0)}{nb - 1 - f_b(z_0)}
\]
or
\[
\xi_0 h'(\xi_0) = \frac{-(1 - f_b(z_0)) (f_b(z_0) - (nb - 1))}{2 - nb} = \frac{-(1 - f_b(z_0)) (f_b(z_0) - (2 \text{Re} f_b(z) - 1))}{2 - 2 \text{Re} f_b(z)}
\]
\[
= \frac{-(f_b(z_0) - 1) (1 - \text{Re} f_b(z))}{2(1 - \text{Re} f_b(z))} = \frac{-(1 - f_b(z_0)) (\text{Re} f_b(z) - 1)}{2(1 - \text{Re} f_b(z))}
\]
\[
\approx \frac{-|f_b(z_0) - 1|^2}{2 (1 - \text{Re} f_b(z))} = \frac{-(1 - f_b(z_0))}{2 \text{Re} (1 - f_b(z))}.
\]
Since for \( \text{Re} f_b(z) \geq \frac{nb}{2}, n \geq 2 \) and \( b > 1 \), if \( z_0 \in \mathbb{E}, \xi_0 \in \partial \mathbb{E} \) and
\[
\text{Re} f_b(z_0) = \min \{\text{Re} f_b(z) : |z| \leq |z_0|\},
\]
then
\[
\frac{z_0 f'_b(z_0)}{f_b(z_0)} < \frac{n}{2} |f_b(z_0) - 1|^2
\]
or we can write
\[
\text{Re} \frac{z_0 f''_b(z_0)}{f_b(z_0)} + 1 \geq 0.
\]
Therefore, we see that
\[
z_0 f'_b(z_0) = m \xi_0 h'(\xi_0) = -\frac{m}{2} |f_b(z_0) - 1|^2.
\]
For \( f_b(z_0) = ix \), the above equation takes the form
\[
z_0 f'_b(z_0) = m \xi_0 h'(\xi_0) = -\frac{m}{2} |ix - 1|^2 = -\frac{m x^2 + 1}{2} = -\frac{m (x^2 + 1)}{2} = y. \tag{3.3}
\]
By using (3.3), we thus prove that
\[
\text{Re} \left\{ \frac{2 f_b(z_0) [f_b(z_0) + z_0 f'_b(z_0)]}{2 f_b(z_0) + z_0 f'_b(z_0)} \right\} = \text{Re} \left[ \frac{2ix (ix + y)}{2ix + y} \right] = \frac{2x^2 y}{4x^2 + y^2} < 0.
\]
This contradicts our hypothesis. Therefore \( f_b \prec h \) and this leads to the desired proof.

From the above result, we have the following corollary as a particular case:

**Corollary 3.2.** If \( \phi(z) = \frac{f_b(z)}{z} \) such that \( \frac{2\phi(z) \phi'(z)}{\phi(z) + z \phi'(z)} \in \mathcal{P} \), then \( \phi \in \mathcal{P} \).

**Theorem 3.3.** Let \( f_b \) be analytic in \( \mathbb{E} \) with \( f_b(z) \neq 1 \). Then \( \frac{2 f_b(z_0) + z_0 f'_b(z_0)}{1 + f_b^2(z_0) + z_0 f_b(z_0) f'_b(z_0)} \in \mathcal{P} \) implies that \( f_b \in \mathcal{P} \).

**Proof.** Following the procedure adopted by Theorem 3.1 and setting \( f_b(z_0) = ix, x > 0 \), \( z_0 f'_b(z_0) = y, y < 0 \), we obtain
\[
\text{Re} \left\{ \frac{2 f_b(z_0) + z_0 f'_b(z_0)}{1 + f_b^2(z_0) + z_0 f_b(z_0) f'_b(z_0)} \right\} = \text{Re} \left[ \frac{2ix + y}{-x^2 + ixy} \right] = \frac{2y}{(1 - x^2)^2 + x^2 y^2} < 0.
\]
This follows the required proof.

**Corollary 3.4.** Let \( \phi(z) = \frac{f_b(z)}{z} \). Then
\[
\frac{2 \phi^{-1}(z) (\phi(z) + z \phi'(z))}{\phi^{-1}(z) + z (\phi(z) + z \phi'(z))} = \frac{2 \frac{z}{f_b(z)} f'_b(z)}{\frac{z}{f_b(z)} + z f'_b(z)} \in \mathcal{P} \text{ implies that } \frac{f_b(z)}{z} \in \mathcal{P}.
\]

**Theorem 3.5.** Let \( f_b \) be analytic in \( \mathbb{E} \) with \( f_b \neq 1 \). Then
\[
\frac{f_b(z) + \frac{z f'_b(z)}{f_b(z)}}{2 + \frac{z f'_b(z)}{f_b(z)}} \in \mathcal{P} \text{ implies that } f_b \in \mathcal{P}.
\]
We just show that there exist following the procedure as in Theorem 3.3, we have a point \( z_0 \in \mathbb{E} : f_b(z_0) = ix, x > 0 \) and \( z_0f_b'(z_0) = y, y < 0 \), so that we can write
\[
\text{Re} \left( \frac{f_b(z_0) + \frac{z_0f_b'(z_0)}{f_b(z_0)}}{2 + \frac{z_0f_b'(z_0)}{f_b(z_0)}} \right) = \text{Re} \left( \frac{ix + \frac{y}{ix}}{2 - \frac{y}{x}} \right) = 0.
\]
Hence the result follows. 

**Corollary 3.6.** Let \( f_b \) be holomorphic or analytic in \( \mathbb{E} \) and \( \frac{zf_b'(z)}{f_b(z)} = \phi(z) \) with \( 1 + \frac{zf_b''(z)}{f_b(z)} = \frac{z\phi'(z)}{\phi(z)} - \phi(z) \). Then
\[
\frac{\phi^2(z) \left( \frac{z\phi'(z)}{\phi(z)} - \phi(z) \right)}{z\phi'(z)} \in \mathbb{P} \text{ implies that } \phi \in \mathbb{P}.
\]

**Theorem 3.7.** Let \( f_b \) be holomorphic or analytic in \( \mathbb{E} \) with \( f_b(z) \neq 1 \). Then
\[
\frac{2f_b(z) + \frac{zf_b'(z)}{f_b(z)}}{1 + f_b^2(z) + zf_b'(z)} \in \mathbb{P} \text{ implies that } \phi \in \mathbb{P}.
\]

**Proof.** Following the procedure as in Theorem 3.1 and setting \( f_b(z_0) = ix, x > 0 \), \( z_0f_b'(z_0) = y, y < 0 \), we obtain
\[
\text{Re} \left( \frac{2f_b(z_0) + \frac{z_0f_b'(z_0)}{f_b(z_0)}}{1 + f_b^2(z_0) + z_0f_b'(z_0)} \right) = \text{Re} \left( \frac{2ix + \frac{y}{ix}}{1 + (ix)^2 + ixy} \right) = 0.
\]
Hence the result follows. 

**Corollary 3.8.** Let \( \phi(z) = \frac{zf_b'(z)}{f_b(z)} \). Then
\[
\frac{2\phi^2(z) \left( \frac{z\phi'(z)}{\phi(z)} - \phi(z) \right)}{z\phi'(z)} \in \mathbb{P} \text{ implies } \phi \in \mathbb{P}.
\]

**Theorem 3.9.** Let \( f_b \) be holomorphic or analytic in \( \mathbb{E} \) with \( f_b(z) \neq 1 \) and \( M \in \mathbb{R}^+ \). Then
\[
\left| \frac{2f_b(z) [f_b(z) + zf_b'(z)]}{2f_b(z) + zf_b'(z)} - 1 \right| < M \text{ implies that } |1 - f_b(z)| < M.
\]

**Proof.** We suppose that
\[
h(z) = 1 + Mz, \quad 0 < M < 1 \tag{3.4}
\]
with \( h(\mathbb{E}) = \{ w : |1 - w| < M \} \), \( h(0) = 1 \) and \( h \in \mathbb{Q} \). Then (3.3) takes the form
\[
\left| \frac{2f_b(z) [f_b(z) + zf_b'(z)]}{2f_b(z) + zf_b'(z)} - 1 \right| < M \text{ which implies that } f_b < h.
\]
Suppose that \( f_b \neq h \). Then Lemma 2.1 shows that there exist \( z_0 \in \mathbb{E}, \xi_0 \in \partial\mathbb{E} \) and \( m \geq 1 \), such that
\[
f_b(z_0) = h(\xi_0), \quad |f_b(z_0) - 1| < M, \quad z \in \mathbb{E}_{|z_0|}
\]
and
\[
|f_b(z_0) - 1| = |h(\xi_0) - 1| = M.
\]
We can select \( f_b(z_0) \) of the form \( f_b(z_0) = 1 + Me^{i\theta} \) where \( \theta \) is real. Using (3.4), we have
\[
\xi_0 = h^{-1}(f_b(z_0)) = \frac{f_b(z_0) - 1}{M} \text{ with } z_0f_b'(z_0) = m\xi_0h'(\xi_0) = mMe^{i\theta}, \quad m \geq 1.
\]
We can also write
\[ \left| \frac{2f_b(z_0) [f_b(z_0) + z_0 f'(z_0)] - 1}{2f_b(z_0) + z_0 f'_b(z_0)} \right| = \left| \frac{2f_b^2(z_0) + 2zf_b(z_0)f'_b(z_0) - 2f_b(z_0) - z_0f'_b(z_0)}{2f_b(z_0) + z_0 f'_b(z_0)} \right| = |M| \left| 1 + \frac{m(Me^{i\theta} + 1)}{2 + (2 + m) Me^{i\theta}} \right|. \]

For contradiction, we show that the last expression is greater than \( M \) which is equivalent to
\[ \left| 1 + \frac{(1 + Me^{i\theta})m}{2 + (2 + m) Me^{i\theta}} \right| \geq 1 \]
or we observe that
\[ 2 + (m + 2) Me^{i\theta} + m \left( Me^{i\theta} + 1 \right) \geq 2 + M (2 + m) e^{i\theta}^2, \]
which further implies that
\[- (3m + 4) \left( \frac{m + 4}{3m + 4} - M \right) (M - 1) \geq 0.\]
For \( M > 0 \) and \( m \geq 1, (3m + 4) \left( \frac{m + 4}{3m + 4} - M \right) (1 - M) > 0 \) which contradicts the assumptions of the theorem. Therefore \( f_b < h \) and thus the proof follows. \( \Box \)

**Corollary 3.10.** Let \( f_b \) be analytic in \( \mathbb{E} \) and \( p(z) = \frac{f_b(z)}{z} \) along with
\[ z \left( p(z) + zp'(z) \right) = \frac{zf'_b(z)}{f_b(z)}. \]
For \( 0 < M < 1 \), the inequality
\[ 2 \left| \frac{zp'(z) + p(z)}{p(z)} + 1 \right| < M \] implies that \( |p(z) - 1| < M. \)

**Corollary 3.11.** Let \( p(z) = f'_b(z) \) and \( M \in (0,1) \). Then
\[ \left| \frac{2f'_b(z) + zf''_b(z)}{2 + zf'_b(z)} - 1 \right| < M \] implies that \( |f'_b(z) - 1| < M. \)

**Corollary 3.12.** Let \( p(z) = \frac{zf'_b(z)}{f_b(z)} \) and \( 0 < M < 1 \). Then
\[ \left| \frac{2zf'_b(z) + zf''_b(z)}{f_b(z)} - \frac{zf'_b(z)}{f_b(z)} - 1 \right| < M \] implies that \( \left| \frac{zf'_b(z)}{f_b(z)} - 1 \right| < M. \)

**Theorem 3.13.** Let \( f_b \) be holomorphic or analytic in \( \mathbb{E} \) with \( f_b(z) \neq 1 \) and \( 0 < \gamma \leq 1 \). Then
\[ \left| \frac{2f_b(z) [f_b(z) + zf'_b(z)]}{2f_b(z) + zf'_b(z)} - 1 \right| < \gamma \frac{\pi}{2} \] implies that \( |f_b(z)| < \gamma \frac{\pi}{2}. \)

**Proof.** Let us suppose that
\[ h(z) = \left[ \frac{(nb - 1) z - 1}{z - 1} \right]^\gamma \] or \( h'(z) = \gamma \left[ \frac{(nb - 1) z - 1}{z - 1} \right]^{\gamma - 1} \frac{2 - nb}{(1 - z)^2}. \)
with \( h(\mathbb{E}) = \{ w : |\arg w| < \frac{\gamma \pi}{2} \} \), \( h(0) = a \), and \( h \in \mathbb{Q} \). Then the condition in (3.5) is rewritten as
\[
\left| \arg \frac{2f_b(z) [f_b(z) + zf'_b(z)]}{2f_b(z) + zf'_b(z)} - 1 \right| < \frac{\gamma \pi}{2} \quad \text{implies that} \quad p < h.
\]

Suppose that \( p \neq h \). Then, from Lemma 2.1, we have \( z_0 \in \mathbb{E}, \xi_0 \in \partial\mathbb{E}\{1\} \) and \( m \geq 1 \), such that
\[
f_b(z_0) = h(\xi_0) \quad \text{or} \quad z_0f'_b(z_0) = m\xi_0h'(\xi_0).
\]
This implies that
\[
f_b(z_0) = h(\xi_0) = (ix)^\gamma = x^\gamma e^{\gamma i\frac{\pi}{2}}, \quad (3.6)
\]
where \( x \in \mathbb{R} \). Because of symmetry, we consider \( x > 0 \). We can write
\[
\xi_0 = h^{-1}(f_b(z_0)) = \frac{[f_b(z_0)]^{\frac{1}{\gamma}} - 1}{[f_b(z_0)]^{\frac{1}{\gamma}} - (nb - 1)},
\]
and
\[
h'(z) = \frac{(1 - (nb - 1) z)^{\gamma - 1}}{(1 - z)^{\gamma + 1}} [2 - nb]
\]
By using (3.6), we see that
\[
\xi_0h'(\xi_0) = \frac{\gamma \left( [f_b(z_0)]^{\frac{1}{\gamma}} - 1 \right) \left( [f_b(z_0)]^{\frac{1}{\gamma}} - (nb - 1) \right)}{(2 - nb)}
\]
or
\[
z_0f'_b(z_0) = m\xi_0h'(\xi_0)
\]
We also see that
\[
\frac{z_0f'_b(z_0)}{f_b(z_0)} = \frac{m\gamma}{2} \left( \frac{(nb - 1) - x^2 - nbxi}{x} \right) e^{-i\frac{\pi}{2}} = \frac{im\gamma}{2} \left( \frac{x^2 - (nb - 1) + nbxi}{x} \right) = iu.
\]
Thus we can write
\[
\left| \arg \frac{2f_b(z_0)}{f_b(z_0) + z_0f_b'(z_0)} \right| = \left| \arg \frac{2f_b(z_0)\left[1 + \frac{z_0f_b'(z_0)}{f_b(z_0)}\right]}{2f_b(z_0) + z_0f_b'(z_0)} - 1 \right| = \left| \arg \frac{2f_b(z_0)\left[1 + \frac{z_0f_b'(z_0)}{f_b(z_0)}\right]}{2 + z_0f_b'(z_0)} \right|
\]
\[
= \arg f_b(z_0) + \arg \frac{1 + \frac{z_0f_b'(z_0)}{f_b(z_0)}}{2 + \frac{z_0f_b'(z_0)}{f_b(z_0)}} = \gamma \frac{\pi}{2} + \arg \frac{1 + iu}{2 + iu} \geq \gamma \frac{\pi}{2}.
\]
This is a contradiction to our hypothesis. Therefore \( p < h \) and we have the desired proof. \( \square \)

**Corollary 3.14.** Let \( f_b \) be analytic in \( \mathbb{E} \) and \( p(z) = \frac{f_b(z)}{z} \). Then
\[
\left| \arg \frac{2f_b(z)f_b'(z)}{f_b(z) + zf_b'(z)} \right| < \gamma \frac{\pi}{2}
\]
implies that \( \left| \arg \frac{f_b(z)}{z} \right| < \gamma \frac{\pi}{2} \).

4. Concluding remarks and observations

In this research, we developed differential subordination associated with the harmonic means of \( f_b(z), f_b(z) + zf_b'(z) \) along with \( f_b(z) + \frac{zf_b'(z)}{f_b(z)} \), where \( f_b(z) = \frac{z}{1 - z^b}, \ b \geq 0; \ n \in \mathbb{N} \) is an \( n \)-fold symmetric Koebe type function defined in \( \mathbb{E} : f_b(0) = 0, f_b'(z) \neq 0 \). By using the admissibility conditions, we also studied some applications in geometric function theory.

**References**


