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On the a New Family of *k*-Fibonacci Numbers

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ABSTRACT

In this paper, it was showed that Fibonacci sequences $\left\{F_n^{(k,p)}\right\}$ of a new family defined in work of (Mikkawy and Sogabe, 2010) are simply periodic sequences according to modulo p. We gave some relationship between the new family and ordinary Fibonacci numbers. Also, we proved some theorems concerning the new family and Lucas numbers.

Mathematics Subject Classification: 11B39, 11B50

Keywords: Fibonacci numbers, Lucas numbers, Period

k-Fibonacci Sayılarının Yeni Bir Ailesi Üzerine

ÖZET

Bu çalışmada, (Mikkawy ve Sogabe, 2010) çalışmasında tanımlanan yeni bir ailenin, $\left\{F_n^{(k,p)}\right\}$ Fibonacci dizilerinin p moduna göre basit periyodik diziler olduğu gösterildi. Yeni aile ve bilinen Fibonacci sayıları arasındaki bazı ilişkileri verdik. Ayrıca, yeni aile ve Lucas sayıları ile ilgili bazı teoremleri ispatladık.

Anahtar Kelimeler: Fibonacci sayıları, Lucas sayıları, Periyot

1. Introduction

Many of the obtained numbers by using homogeneous lineer recurrence relations and their the miscellaneous properties have been stuied; see, for example, (Lee, 2000), (Taher and Rachidi, 2003). Fibonacci numbers are one of the most well-known numbers, and it has many important applications to diverse fields such as mathematics, computer science, physics, biology and statistics. We can see applications of Fibonacci sequences in group theory in (C. Campbell and P. Campbell, 2005), (Deveci, 2011, 2015) and also see some generalized Fibonacci and Lucas sequences in (Arı 2015), (Grabowski and Wojtecki, 2004), (İpek and Arı, 2015), (Kılıç and Taşçı, 2006), (Lee, 2000), (P. Stanimirovic, Nikolov and I. Stanimirovic 2008), (Taher and Rachidi, 2003), (Taşçı and Kılıç, 2004). Y. Taşyurdu and İ. Gültekin obtain the period of generalized Fibonacci sequence in finite rings with identity of order

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p^2 (Taşyurdu and Gültekin 2013, 2016). The Fibonacci numbers F_n are the terms of the sequence $0,1,1,2,3,5,8,13,\ldots$ where $F_n = F_{n-1} + F_{n-2}$ with the initial values $F_0 = 0$ and $F_1 = 1$. The Lucas numbers L_n are the term of the sequence $2,1,3,4,7,11,18,29,47,\ldots$ where $L_n = L_{n-1} + L_{n-1}$ with initial conditions $L_0 = 2$ and $L_1 = 1$. A. İpek and K. Arı obtain several new connections between the generalizations of Fibonacci and Lucas sequences (İpek and Arı 2014). Generalized Fibonacci sequence have been intensively studied for many years and have become into an interesting topic in Applied Mathematics. Fibonacci sequences and their related higher-order (tribonacci, k-nacci) sequences are generally viewed as sequences of integers. The notion of Wall number was first proposed by D. D. Wall (Wall 1960) in 1960 and gave some theorems and properties concerning Wall number of the Fibonacci sequence. K. Lü and J. Wang (Lü and Wang 2007) contributed to the study of the Wall number for the k-step Fibonacci sequence.

A sequence is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is called the period of the sequence. For example, the sequence $a,b,c,d,b,c,d,b,c,d,\ldots$ is periodic after the initial element and has period 3. A sequence of group elements is simply periodic with period k if the first k elements in the sequence form a repeating subsequence. For example, the sequence $a,b,c,d,e,a,b,c,d,e,a,b,c,d,e,\ldots$ is simply periodic with period 5 (Knox, 1992).

The minumum period lenght of $(F_i mod n)_{i=-\infty}^{\infty}$ sequence is staded by $k(n)$ and is named Wall number of n (Wall 1960).

An k-step Fibonacci sequence $\left\{F_n^{(k)}\right\}_{n=1}^{\infty}$ is defined by letting $F_n^{(k)} = 0$ for $n \le 0$, $F_1^{(k)} = 1$, $F_2^{(k)} = 1$, and other terms according to the linear recurrence equation $F_n^{(k)} = \sum_{i=1}^{k} F_{n-i}^{(k)}$ for $k \ge 2$.

It is well known that the Fibonacci numbers F_n for $n = 0,1,\ldots$ are defined by the Binet's formula as follows:

$$F_n = \frac{1}{\sqrt{5}}(\alpha^{n+1} - \beta^{n+1}), \quad n = 0,1,\ldots \qquad (1)$$

where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. The first few Fibonacci numbers are $1,1,2,3,5,8,13,21,34,55,89,\ldots$ The numbers F_n satisfy the second order linear recurrence relation

$$F_n = F_{n-1} + F_{n-2}, \quad n = 1,2,\ldots$$

with the initial conditional $F_{-1} = 0$, $F_0 = 1$. It is also widely know that the F_n is related by the determinant of the special tridiagonal matrix of the form

$$T_n = \begin{pmatrix} 1 & 1 & & & \\ -1 & 1 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 1 & 1 \\ & & & -1 & 1 \end{pmatrix} \in R^{nxn}$$

The Lucas numbers L_n are closely related to the Fibonacci numbers F_n. The Lucas numbers are defined by

$$L_n = L_{n\text{-}1} + L_{n\text{-}1} \quad n = 2,3,\dots$$

with initial conditions $L_0 = 2$ and $L_1 = 1$. The firs few Lucas numbers are $2, 1, 3, 4, 7, 11, 18, 29, 47, 76, \dots$ The Binet's formula for the Lucas numbers L_n is $L_n = \alpha^n + \beta^n$, $n = 0, 1, \dots$ Than, we see that the Lucas and Fibonacci numbers are related by

$$L_n = F_n + F_{n\text{-}2} = \frac{F_{2n\text{-}1}}{F_{n\text{-}1}}.$$

Now, we give definition of a new generalized k-Fibonacci number in (Mikkawy and Sogabe 2010) as follows.

Definition 1.1. Let n and$k$ $(\neq 0)$ be natural numbers, then exist unique numbers m and n such that $n = mk + r$ $(0 \leq r \leq k)$. Using these parameters, we define generalized k-Fibonacci numbers $F_n^{(k)}$ by

$$F_n^{(k)} = \frac{1}{(\sqrt{5})^k}(\alpha^{m+2}\text{-}\beta^{m+2})^r(\alpha^{m+1}\text{-}\beta^{m+1})^{k\text{-}r},$$
$$n = mk + r. \qquad (2)$$

The first few numbers of the new family for $k = 2, 3$ are as follows:

$$\left\{F_n^{(2)}\right\}_{n=0}^{10} = \{1, 1,1,2,4,6,9,15,25,40,64\},$$

$$\left\{F_n^{(3)}\right\}_{n=0}^{10} = \{1,1,2,4,8,12,18,27,45\}.$$

From (1) and Definition1.1, the generalized k-Fibonacci and Fibonacci numbers are related by

$$F_n^{(k)} = (F_m)^{k\text{-}r}(F_{m+1})^r, \quad n = mk + r.$$

Considering the case $k = 1$ in (2), we see that $m = n$ and $r = 0$. Therefore, $F_n^{(1)}$ is the ordinary Fibonacci numbers F_n (Mikkawy and Sogabe 2010).

In the present paper, we shall focus on a new family of these numbers defined in (Mikkawy and Sogabe 2010). We will prove some theorems concerning the family. We will give some relationship between the family and ordinary Fibonacci number. Also, we prove some theorems concerning a new family and Lucas numbers. We will also prove that Fibonaci sequences $\left\{F_n^{(k,p)}\right\}$ of the family are simply periodic sequences.

2. Main Results

Reducing the a new family of k-Fibonacci numbers by modulo p, we can get a repeating. Sequences denoted by

$$\left\{F_n^{(k,p)}\right\} = \left\{F_0^{(k,p)}, F_1^{(k,p)}, \dots, F_n^{(k,p)}, \dots\right\}$$

where $F_n^{(k,p)} = F_n^{(k)}(mod p)$. Let $h_{(k,p)}$ denote the smallest period of $\left\{F_n^{(k,p)}\right\}$, called the period of a new family of k-Fibonacci numbers by modulo p.

Theorem 2.1. $\left\{F_n^{(k,p)}\right\}$ is a simply periodic sequences.

Proof. From Definition 1.1, we have

$$F_n^{(k,p)} = \left(F_m^p\right)^{k\text{-}r}\left(F_{m+1}^p\right)^r, \quad n = mk + r$$

where $\left(F_m^p\right)^{k\text{-}r}$ and $\left(F_{m+1}^p\right)^r$ are ordinary Fibonacci numbers. So, we can write

$$F_t^{(k,P)} = \left(F_s^p\right)^{k\text{-}r}\left(F_{s+1}^p\right)^r, \quad t = sk + r$$

Because the sequence $\left(F_m^p\right)^{k\text{-}r}$ is a simply periodic, we have

$$\left(F_{m-1}^{p}\right)^{k\text{-}r} \equiv \left(F_{s-1}^{p}\right)^{k\text{-}r}$$

where $\left(F_{m+1}^{p}\right)^{k\text{-}r} \equiv \left(F_{s+1}^{p}\right)^{k\text{-}r}$ and $\left(F_{m}^{p}\right)^{k\text{-}r} \equiv \left(F_{s}^{p}\right)^{k\text{-}r}$. Since also the sequence $\left(F_{m+1}^{p}\right)^{r}$ is a simply periodic, we obtain

$$\left(F_{m}^{p}\right)^{r} \equiv \left(F_{s}^{p}\right)^{r}$$

where $\left(F_{m+2}^{p}\right)^{r} \equiv \left(F_{s+2}^{p}\right)^{r}$ and $\left(F_{m+1}^{p}\right)^{r} \equiv \left(F_{s+1}^{p}\right)^{r}$.
So, we can write

$$(F_{m-1})^{k\text{-}r}(F_m)^r \equiv (F_{s-1})^{k\text{-}r}(F_s)^r \Rightarrow F_{n-1}^{(k,p)} \equiv F_{t-1}^{(k,p)}.$$

We can write $F_{n+1}^{(k,p)} \equiv F_{t+1}^{(k,p)}$ because we have $\left(F_{m+1}^{p}\right)^{k\text{-}r}\left(F_{m+2}^{p}\right)^{r} \equiv \left(F_{s+1}^{p}\right)^{k\text{-}r}\left(F_{s+2}^{p}\right)^{r}$.
Similarly, we can write $F_{n}^{(k,p)} \equiv F_{t}^{(k,p)}$ because we have $\left(F_{m}^{p}\right)^{k\text{-}r}\left(F_{m+1}^{p}\right)^{r} \equiv \left(F_{s}^{p}\right)^{k\text{-}r}\left(F_{s+1}^{p}\right)^{r}$.

That is, we have

$$F_{n-1}^{(k,p)} \equiv F_{t-1}^{(k,p)}$$

where $F_{n+1}^{(k,p)} \equiv F_{t+1}^{(k,p)}$ and $F_{n}^{(k,p)} \equiv F_{t}^{(k,p)}$.

So $\left\{F_n^{(k,p)}\right\}$ is a simply periodic sequences.

Theorem 2.2. Let $n \in \{1,2,\ldots\}$. For fixed n, the generalized 3-Fibonacci numbers satisfy

$$F_{3n-1}^{(3)} + F_{3n}^{(3)} = F_{3n+1}^{(3)}.$$

Proof. We have $F_m^{(k)} = F_{nk+r}^{(k)} = F_n^{(k-r)}F_{n+1}^{(r)}$ for $n = mk + r$ $(0 \leq r < k)$. For $k = 3$ and $m = n - 1$, we obtain

$$F_{3n-1}^{(3)} = F_{3(n-1)+2}^{(3)} = (F_{n-1})^1(F_n)^2$$

and

$$F_{3n}^{(3)} = (F_n)^3(F_{n+1})^0 = (F_n)^3.$$

By using the last two equalities, the proof is completed as fallows,

$$F_{3n-1}^{(3)} + F_{3n}^{(3)} = (F_{n-1})^1(F_n)^2 + (F_n)^3$$

$$= (F_n)^2\{F_{n-1} + F_n\}$$

$$= F_{n+1}(F_n)^2$$

$$= F_{3n+1}^{(3)}$$

Theorem 2.3. Let $n \in \{2,3\ldots\}$, $n = 2k+1$. For fixed n, the generalized 2-Fibonacci numbers satisfy

$$F_{2n+1}^{(2)} - F_{2n+2}^{(2)} + F_{2n}^{(2)} = (-1)^n$$

Proof. We have the following equalities

$$F_{2n}^{(2)} = (F_n)^2$$

$$F_{2n+2}^{(2)} = (F_{n+1})^2$$

$$F_{2n+1}^{(2)} = (F_n)(F_{n+1}).$$

From the last three equalities, we can write

$$F_{2n+1}^{(2)} - F_{2n+2}^{(2)} + F_{2n}^{(2)} = (F_n)^2 - (F_{n+1})^2 + (F_n)(F_{n+1})$$

$$= (F_n)\{F_n + F_{n+1}\} - (F_{n+1})^2$$

$$= (F_n)(F_{n+1}) - (F_{n+1})^2$$

$$= -[(F_{n+1})^2 - (F_n)(F_{n+1})]$$

$$= -(-1)^{n+1}$$

$$= (-1)^n$$

Theorem 2.4. Let $k, n \in \{1,2,\ldots\}$. For fixed n, the generalized k-Fibonacci numbers satisfy

$$F_{nk+k-1}^{(k)} + F_{nk+k}^{(k)} = F_{nk+k+1}^{(k)}$$

Proof.

$$F_{nk+k-1}^{(k)} + F_{nk+k}^{(k)} = [(F_n)^1(F_{n+1})^{k-1}] + [\ (F_{n+1})^k(F_{n+2})^0]$$

$= [(F_n)^1 (F_{n+1})^{k-1}] + (F_{n+1})^k$

$= (F_{n+1})^{k-1}[(F_n) + (F_{n+1})]$

$= (F_{n+1})^{k-1} F_{n+2}$

$= F_{nk+k+1}^{(k)}$

Theorem 2.5.

$$F_n . L_n = F_{2n-1}^{(2)} + F_{2n+1}^{(2)}$$

Proof. We have $L_n = F_{n-1} + F_{n+1}$ between Fibonacci numbers and Lucas numbers. Using by that equalities, we obtain,

$$F_n L_n = F_n (F_{n-1} + F_{n+1})$$

$$= F_n F_{n-1} + F_n F_{n+1}$$

$$= F_{2n-1}^{(2)} + F_{2n+1}^{(2)}$$

Theorem 2.6.

$$L_n L_{n+1} - F_{n-2} F_{n+1} = F_{2n+1}^{(2)} + F_{2n-1}^{(2)} + F_{2n-3}^{(2)}$$

Proof. We have $L_n = F_n + F_{n-2}$, between Fibonacci numbers and Lucas numbers. Using by that equalities, we can write,

$L_n L_{n+1} - F_{n-2} F_{n+1}$

$$= (F_n + F_{n-2})(F_{n+1} + F_{n-1}) - F_{n-2} F_{n+1}$$

$$= F_n F_{n+1} + F_n F_{n-1} + F_{n-2} F_{n+1} F_{n-2} F_{n+1}$$

$$= F_n F_{n+1} + F_n F_{n-1} + F_{n-2} F_{n-1}$$

$$= F_{2n+1}^{(2)} + F_{2n-1}^{(2)} + F_{2n-3}^{(2)}$$

Theorem 2.7. $L_n F_{n+1} = F_{2n-1}^{(2)} + F_{2n-3}^{(2)}$

Proof. We have $L_n = F_n + F_{n-2}$, between Fibonacci numbers and Lucas numbers. Using by that equalities, we can write,

$$L_n F_{n-1} = (F_n + F_{n-2}) F_{n-1}$$

$$= F_n F_{n-1} + F_{n-2} F_{n-1}$$

$$= F_{2n-1}^{(2)} + F_{2n-3}^{(2)}$$

Table 1 The period and$k$-Fibonacci sequences $\left\{F_n^{(k,p)}\right\}$ for $k = 2,3,4$ and $p = 2,3,4,5$.

$\left\{F_n^{(k,p)}\right\}$	$\left\{F_0^{(k,p)}, F_1^{(k,p)}, \ldots, F_n^{(k,p)}, \ldots\right\}$	Period
$\left\{F_n^{(2,2)}\right\}_{n=0}^{8}$	$\{1,1,1,0,0,0,1,1,1, \ldots\}$	$h_{(2,2)} = 6$
$\left\{F_n^{(2,3)}\right\}_{n=0}^{10}$	$\{1,1,1,2, 1, 0, 0, 0, 1, 1, 1, \ldots\}$	$h_{(2,3)} = 8$
$\left\{F_n^{(2,4)}\right\}_{n=0}^{14}$	$\{1,1, 1, 2, 0, 2, 1, 3, 1, 0, 0, 0,1, 1,1, \ldots\}$	$h_{(2,4)} = 12$
$\left\{F_n^{(2,5)}\right\}_{n=0}^{22}$	$\{1,1, 1, 2, 4,1, 4, 0, 0, 0, 4, 4, 4, 3, 1, 4, 1, 0, 0, 0, 1,1, 1, \ldots\}$	$h_{(2,5)} = 20$
$\left\{F_n^{(3,2)}\right\}_{n=0}^{12}$	$\{1,1,1,1,0,0,0,0,0,1,1,1,1, \ldots\}$	$h_{(3,2)} = 9$

$\left\{F_n^{(3,3)}\right\}_{n=0}^{27}$	{1,1,1,1,2,1,2,0,0,0,0,0,2,2,2,2,1,2,1,0,0,0,0,0,1,1,1,1, …}	$h_{(3,3)} = 24$
$\left\{F_n^{(3,4)}\right\}_{n=0}^{21}$	{1,1,1,1,2,0,0,0,2,3,1,3,1,0,0,0,0,0,1,1,1,1, …}	$h_{(3,4)} = 18$
$\left\{F_n^{(3,5)}\right\}_{n=0}^{64}$	{1,1,1,1,2,4,3,2,3,2,0,0,0,0,0,2,2,2,2,4,3,1,4,1,4,0,0,0,0,0,4,4,4, 4, 3,1,2,3,2,3,0,0,0,0,0,3,3,3,3,1,2,4,1,4,1,0,0,0,0,0,1,1,1,1,1 …}	$h_{(3,5)} = 60$
$\left\{F_n^{(4,2)}\right\}_{n=0}^{16}$	{1,1,1,1,1,0,0,0,0,0,0,0,1,1,1,1,1, …}	$h_{(4,2)} = 12$
$\left\{F_n^{(4,3)}\right\}_{n=0}^{20}$	{1,1,1,1,1,2,1,2,1,0,0,0,0,0,0,0,1,1,1,1,1, …}	$h_{(4,3)} = 16$
$\left\{F_n^{(4,4)}\right\}_{n=0}^{28}$	{1,1,1,1,1,2,0,0,0,0,0,2,1,3,1,3,1,0,0,0,0,0,0,0,1,1,1,1,1, …}	$h_{(4,4)} = 24$
$\left\{F_n^{(4,5)}\right\}_{n=0}^{24}$	{1,1,1,1,1,2,4,3,1,4,1,4,1,0,0,0,0,0,0,0,1,1,1,1,1, …}	$h_{(4,5)} = 20$

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