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Investigation of a non-linear Cramér-Lundberg risk model

Zulfiye Hanalioglu[1], Yusup Allyyev [2], Tahir Khaniyev [2, 3, *]

[1]Department of Actuarial Sciences, Karabuk University, Karabuk, Turkey
e-mail: zulfiyyamammadova@karabuk.edu.tr, ORCID No: http://orcid.org/0000-0003-1197-9421
[2] Department of Industrial Engineering, TOBB University of Economics and Technology, Ankara, Turkey
e-mails: yusupallyyev@etu.edu.tr; tahirkhaniyev@etu.edu.tr, ORCID No: http://orcid.org/0000-0001-5410-2705
[3] Department of Digital Technology and Applied Informatics, Azerbaijan State University of Economics, Baku, Azerbaijan
e-mail: tahirkhaniyev@etu.edu.tr, ORCID No: http://orcid.org/0000-0003-1974-0140

[*] Corresponding author

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Abstract

In this study, a non-linear version of a Cramér-Lundberg risk model is examined. The objective of this work is to evaluate the ruin probability of the non-linear risk model. The classical linear Cramér-Lundberg model has been widely studied in the literature. However, the linear model is not always realistic. Because an insurance company's premium income cannot always increase linearly. Therefore, it is recommended to adapt premium income as a function which increases monotonically and yet its rate of growth decreases over time. Thus, to account for this, a more realistic non-linear mathematical model has been constructed and investigated, when the premium income function is $p(t) = c\sqrt{t}$. Then Lundberg type upper bound was calculated for the ruin probability for the model under investigation.

1. Introduction

Insurance is a concept that concerns and affects almost everyone in our daily lives. Therefore, insurance is an inevitable part of the developed economies. Contemporary economies and modern states would hardly operate without insurance companies. Because these institutions guarantee compensation to almost any actors of the society at the individualistic, company, or the organizational level at an unfortunate time when catastrophes such as fires, floods, earthquakes, accidents and riots befalls onto them. Therefore, the examination of risk and ruin problems of insurance company has a vital role in actuarial science. Swedish mathematicians and the pioneers in this area, Lundberg (1903) and Cramér (1930) laid foundations of modern risk theory based on the probability theory, statistics, and stochastic processes. They found out that insurance business can be aptly modelled via stochastic processes framework. Cramér-Lundberg model is one of the pillars of non-life insurance mathematics (Mikosch, 2004). This model has been extended and adapted to various domains of applied probability: financial mathematics, renewal theory, queuing theory, branching processes, reliability, and extreme value theory are just some of them. Many valuable studies have been done in the literature on this subject (see, Cramér, 1930; Lundberg, 1903; Malinovski, 2014; Mikosch, 2004; Mishura, 2014; Yang, 1998). The question of how much of initial capital is required to keep the probability of ruin above some predefined threshold value is answered in the paper by Malinovski (2014). Evaluation of ruin probabilities largely depends on the distribution of demand quantities, and given two or more demand distributions, it is important which one gives greater ruin probabilities in finite or infinite time horizon for a given initial capital value of u. This issue is addressed in (Asmussen and Rolski, 1994; Chadjiconstantinidis and Politis, 2007; Cohen and Young, 2020; Gaier et al., 2003 ; Gauchonab et al., 2020; Gerber, 1998; Kaas et al., 2001; Rolski, 1999; Straub, 1988; and etc.). Most of the research is done on variations of premium income function $p(t)$ and its relation to the total claim process $S(t)$. In the paper by Boikov (2002), the premium income process is studied by assuming that premium income is stochastic and also independent of the risk process. Constantinescu et al. (2018), investigated ruin probabilities in classical risk model with gamma claims.

A scenario when the total claim amount process is the same as in the classical model while the premium income – unlike the classical case – is a stochastic process, called as random premiums risk process, is investigated by Temnov (2004) and ruin probabilities are estimated numerically. Similarly, Zang and Yang (2009) considered risk model with stochastic premium income. In their work, some specific dependence structure among the claim sizes, inter-claim times, and premium sizes is assumed.

The studies mentioned above are all valuable, each focusing on specific sets of conditions. In the literature, the premium income function $p(t)$ is generally modeled as a linear function. However, the linear model is not always realistic because an insurance company's premium collection cannot always increment in a linear fashion. This is particularly the case for companies (markets) saturated with insurance policyholders. Accordinggly, a reasonable approach would be to formulate the premium income as a monotonically increasing function whose growth-rate slows with time. To this end, a more realistic mathematical non-linear Cramér-Lundberg risk model is formulated and studied, which is defined as $V(t) = u + c\sum_{i=1}^{\nu(t)} g(W_i) - S(t)$. Here, $g(t)$ is a non-linear function that is monotonically increasing whose growth-rate is decreasing with time; $W_i,\ i = 1, 2, 3\dots$ are positive-valued independent and identically distributed random variables describing inter-arrival times of claims; u – the initial capital of the company; c – the premium rate; $S(t) = \sum_{i=1}^{\nu(t} X_i$ is the renewal-reward process that expresses the insurer's capital outflow; $\nu(t)$ – the process counting the total number of claims and $X_i,\ i = 1, 2, 3\dots$ are independent and identically distributed random variables representing the amount of payment for the i^{th} claim, for $i = 1, 2, 3, \dots$. In other words, $V(t)$ expresses insurance company's capital balance at any time t. The main goal of this research is to investigate the ruin probability of the non-linear risk model $V(t)$. For this aim, Lundberg type upper bound is obtained for the ruin probability of this non-linear risk model, when the income function is defined as $p(t) = c\sqrt{t}$.

The primary contribution of this study is the mathematical construction of the Cramér-Lundberg model under the assumption that the premium income function is a non-linear function. Finding a Lundberg type upper bound for the ruin probability of this non-linear risk process is a secondary contribution.

The remaining part of the article is organized as follows. In Section 2, construction of the model is given. In Section 3, relevant definitions from the risk theory are provided. In Section 4, Lundberg-type upper bound is obtained for the ruin probability. In Section 5, the generalized Lundberg adjustment coefficient r is analyzed in detail. In Section 6, conclusion is provided.

Now, we proceed to the mathematical construction of the process $V(t)$.

2. Mathematical Construction of the Non-Linear Cramér-Lundberg Risk Model

Let sequences of random variables $\{W_i\}$ and $\{X_i\}, i = 1, 2, 3, \dots$ be defined on the same probability space $(\Omega, \mathcal{F}, P)$ and let variables in each sequence be independent and identically distributed. Also suppose that $W_i's$ and $X_i's$ take only positive values.

$W_i,\ i = 1, 2, \dots$ are independent and identically distributed random variables describing inter-arrival times of claims, similarly, $X_i,\ i = 1, 2, \dots$ are independent and identically distributed random variables denoting the amount of the payment for the i^{th} claim, for $i = 1, 2, 3, \dots$.

Define the renewal sequences $\{T_n\}$ and $\{S_n\}$ as follows:

$T_n = \sum_{i=1}^{n} W_i;\ S_n = \sum_{i=1}^{n} X_i,\ \ i = 1, 2, 3, \dots;\ T_0 = S_0 = 0$

Here, T_n is the n^{th} claim time; S_n represents the total payment amount in the first n claims.

Moreover, to count the total number of claims occurred in $[0, t]$, define a renewal process as follows: $\nu(t) \equiv max\{n \geq 0 : T_n \leq t\}$, $t \geq 0$. Now we can define the stochastic process $V(t)$, which represents the insurance company's capital at time t, as follows:

$$V(t) = u + c\sum_{i=1}^{V(t)} g(W_i) - \sum_{i=1}^{V(t)} X_i \ , \tag{1}$$

Here, u – company's initial capital; c – premium rate per unit time; $g(t)$ is a monotonically increasing non-linear function whose growth-rate declines with time and $g(0) = 0; g(1) = 1$.

The stochastic process $V(t)$ is called a Non-Linear Risk process. Moreover, the model expressed by means of this process is called the Non-Linear Cramér-Lundberg Risk Model.

Additionally, let's define the following boundary functionals:

$$N_1 \equiv N_1(u) = min\{n \geq 1: u + \sum_{i=1}^{n}[cg(W_i) - X_i] < 0\}; \quad \tau_1 = \sum_{i=1}^{N_1} W_i. \tag{2}$$

Here, N_1, represents the number of claims until ruin, and τ_1 representes the time of ruin.

A sample graph of the process $V(t)$ is shown in Figure 1.

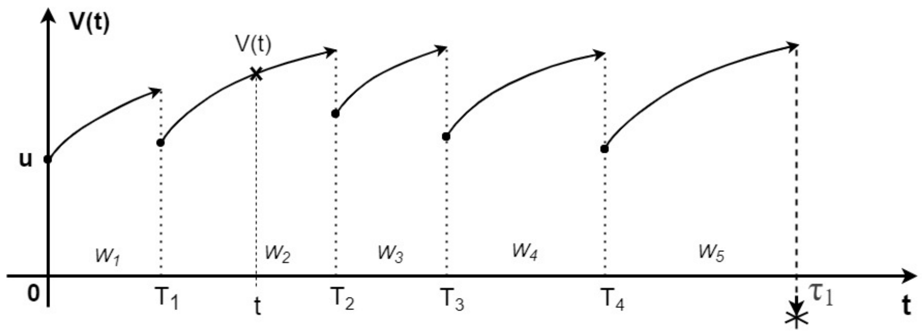

Figure 1. A sample path of non–linear risk process $V(t)$

The key objective of this study is to determine the ruin probability of the non-linear risk model in Eq.(1). For this purpose, it is necessary to give the following preliminary definitions.

3. Ruin Probability of Non-Linear Lundberg Risk Model

Let us provide the following definitions in a similar manner as in Mikosch (2004).

Definition 3.1: i. The event that $V(t)$ ever falls below zero is called *ruin*, i.e.,

$$\boldsymbol{Ruin} = \{V(t) < 0 \ \ for\ some\ t > 0\}$$

ii. The first time when the non-linear process falls below zero is called ***ruin time,*** $\boldsymbol{\tau_1}$:

$$\boldsymbol{\tau_1} = \inf\{t > 0: V(t) < 0\}$$

Note: The τ_1 defined here is the same as the random variable τ_1 defined in the Eq.(2), with probability 1.

iii. Then $\boldsymbol{\psi(u)} - the\ probability\ of\ ruin$, is then given by:

$$\boldsymbol{\psi(u)} \equiv P\{(Ruin \mid V(0) = u\} = P_u\{\tau_1 < \infty\}, \ u > 0.$$

Observe that the occurrence of ruin is possible only at discrete times $t = T_n$ for some $n = 1,2, \ldots$. Therefore, we can rewrite

$$Ruin = \left\{\inf_{t>0} V(t) < 0\right\} = \left\{\inf_{n\geq1} V(T_n) < 0\right\} = \left\{\inf_{n\geq1} [u + c\sum_{i=1}^{n} g(W_i) - \sum_{i=1}^{n} X_i] < 0\right\}.$$

For shortness of notation, let's define the following random variables:
$Z_n = X_n - cg(W_n), \quad D_n = \sum_{i=1}^{n} Z_i \ , n = 1,2,3, \dots, \quad D_0 = 0$.
Then we have the following alternative expression for the ruin probability $\psi(u)$ with the initial capital u:

$$\psi(u) = P\left\{\inf_{n\geq 1} (u - D_n) < 0\right\} = P\left\{\sup_{n\geq 1} D_n > u\right\} \tag{3}$$

Definition 3.2: The non-linear Cramér-Lundberg model satisfies the net profit condition if

$$E(Z_1) = E(X_1) - cE(g(W_1)) < 0 \tag{4}$$

The net profit condition can be contemplated as follows:

The expected amount of a claim $E(X_1)$ should always be less than the expected premium income $cE(g(W_1))$ in one period. In other words, more average premium income should be earned than the average loss the company paid in each period. However, this does not imply that the ruin of the company is completely averted. Because the Net Profit Condition does not consider the fluctuating behavior of the stochastic process.

In this model, a *small* claim condition is assumed, meaning that, there exists a moment generating function of the claim size distribution in a neighborhood of the origin, i.e., for each $h \in (-h_0, h_0)$, $M_Z(h) \equiv E(e^{hZ_1})$ exists, $for\ some\ h_0 > 0$.

Definition 3.3: Assume that there exists a moment generating function of Z_1 in some neighborhood of $(-h_0, h_0)$, for $h_0 > 0$, of the origin. If the equation below,

$$E\left(e^{h(X_1 - cg(W_1))}\right) = 1 \tag{5}$$

has a positive solution $h = r$, then this solution r is called the generalized Lundberg coefficient or adjustment coefficient and can be represented as follows:

$$r \equiv inf\left\{h > 0: M_Z(h) = E(e^{hZ_1}) = E\left(e^{h(X_1 - cg(W_1))}\right) = 1\right\}$$

4. Lundberg-Type Upper Bound for Ruin Probability

In this part of the study, a Lundberg type upper bound is found for the non-linear risk model constructed in Section 2 (see, Eq.(1)). The following theorem establishes this upper bound.

Theorem 4.1: Assume the non-linear risk model given in Eq.(1) with Net Profit Condition satisfied. Also suppose that the generalized adjustment coefficient (r) exists for this model. Then, for all $u > 0$, the following inequality holds:

$$\psi(u) \leq e^{-ru} \tag{6}$$

Proof: Using the method in Mikosch (2004), let us show that for each $u > 0$, the inequality (6) is satisfied for the non-linear risk model.

Let's denote the probability of ruin in the n^{th} claim by $\psi_n(u)$. By definition, $\psi_n(u)$ can be written as:

$$\psi_n(u) = P\left\{\sup_{1\leq k\leq n} (D_k) > u\right\}.$$

Here, $D_n = \sum_{i=1}^{n} Z_i\,;\ Z_n = X_n - cg(W_n),\ n = 1,2,3, \dots$.

Now, for every $n = 1,2,3, \dots$ let us prove that

$$\psi_n(u) \leq e^{-ru} \tag{7}$$

by the method of induction.

Using the Markov Inequality for $n = 1$, the following inequality can be written for the ruin probability:

$$\psi_1(u) = P\{D_1 > u\} = P\{rD_1 > ru\} = P\{e^{rD_1} > e^{ru}\} = P\{e^{rZ_1} > e^{ru}\} \leq \frac{E(e^{rZ_1})}{e^{ru}}$$

$$= e^{-ru}E(e^{rZ_1}) = e^{-ru}M_Z(r).$$

Here, $M_Z(r) \equiv E(e^{rZ_1}), r > 0$.

r exists when net profit condition is satisfied and this coefficient is defined as:

$$r \equiv inf\{k > 0: M_Z(k) = 1\} \quad (8)$$

In other words, the coefficient r is the first positive solution of the equation $M_Z(k) = 1$.

Thus, we have shown that for $n = 1, \psi_1(u) \leq e^{-ru}$. Now let's prove the proposition (7) for $n = 2$. In this case, the ruin probability can be calculated as follows:

$$\psi_2(u) = P\left\{\sup_{1 \leq k \leq 2} D_k > u\right\} = P\{D_1 > u \; or \; D_2 > u\} = P\{D_1 > u\} + P\{D_1 \leq u; D_2 > u\}$$

$$= \int_u^\infty P\{Z_1 \in dx\} + \int_{-\infty}^{u} P\{D_1 > u - x\}P\{Z_1 \in dx\} = \int_u^\infty dF_Z(x) + \int_{-\infty}^{u} \psi_1(u - x)dF_Z(x)$$

$$\leq \int_u^\infty e^{-r(u-x)}dF_Z(x) + \int_{-\infty}^{u} e^{-r(u-x)}dF_Z(x) = \int_{-\infty}^{\infty} e^{-r(u-x)}dF_Z(x) = e^{-ru}\int_{-\infty}^{\infty} e^{rx}dF_Z(x)$$

$$= e^{-ru}E(e^{rZ_1}) = e^{-ru}M_Z(r) \, . \quad (9)$$

Here, $F_Z(x) \equiv P\{Z_1 \leq x\}$. Note that, in Eq.(9), it is considered that $e^{-r(u-x)} \geq 1$, for $x > u$.

By the definition of r, the equality $M_Z(r) = 1$ must hold. Therefore, from Eq.(9) we get:

$\psi_2(u) \leq e^{-ru}$.

Similarly, the proposition in (7) can be proved for $n = 3, 4, \ldots$ by induction.

In other words, it is shown that for every $n = 1, 2, 3, 4, \ldots$ we have:

$\psi_n(u) \leq e^{-ru}$

The following inequality can be written, using the property of the supremum:

$\psi_1(u) \leq \psi_2(u) \leq \cdots \leq \psi_n(u) \leq \cdots \leq e^{-ru}$

In other words, $\psi_n(.)$ is a positive valued, monotonically non-decreasing, bounded above sequence. In this case, $\lim_{n\to\infty} \psi_n(u)$ exists and $\lim_{n\to\infty} \psi_n(u) \equiv \psi(u) \leq e^{-ru}$.

Thus, for every $u > 0$: $\psi(u) \leq e^{-ru}$.

Therefore, Theorem 4.1 is proved for every $u > 0$. ■

Remark: According to Eq.(6), when the company's initial capital u is large, the ruin probability will be decrease. Similarly, the larger the adjustment coefficient r, the smaller the probability of ruin.

Since the initial capital u is known, the generalized Lundberg coefficient r needs to be examined to find a Lundberg-type upper bound. This issue is addressed in detail in the following section.

5. Analysis of the Generalized Lundberg Coefficient r

It follows from the previous section that the investigation of the probability of ruin is equivalent to examining the adjustment coefficient r.

Therefore, let's reconsider Eq.(5) to find the adjustment coefficient r.

$$M_Z(r) = E(e^{rZ_1}) = E\left(e^{r(X_1 - cg(W_1))}\right) = E(e^{rX_1})E\left(e^{-crg(W_1}\right) = 1 \quad (10)$$

Here, X_i, $i = 1, 2, \ldots$ are independent and identically distributed random variables denoting the amount of payment for i^{th} claim, for $i = 1,2,3, \ldots$, and $cg(t)$ represents the income function. After introducing a notation $Y_i \equiv g(W_i)$, Eq.(10) can be rewritten as follows:

$$M_Z(r) = E(e^{rX_1})E\left(e^{-crg(W_1}\right) = M_X(r)M_Y(-cr) = 1 \quad (11)$$

Here,

$$M_X(r) = E(e^{rX_1}) = \int_0^\infty e^{rx} f_X(x)dx; \; M_Y(-cr) = E\left(e^{-crg(W_1)}\right) = \int_0^\infty e^{-cry} f_Y(y)dy \, .$$

In summary, the adjustment coefficient r can be found from the integral in Eq.(11). However, it is very difficult to find the coefficient r from the integral equation (11) in general. Therefore, a special case is discussed and studied below.

A Special Case: In this case, the income function is taken as $c\sqrt{t}$ and the corresponding non-linear risk model is examined. In addition, in this special case, the random variables $X_i, i = 1,2,3, \ldots$ representing the payment amount in the i^{th} claim have Exponential distribution with $\mu > 0$ parameter; $W_i, i = 1,2,3, ..$ random variables expressing the time between claims are assumed to have Exponential distribution with $\lambda > 0$ parameter. The probability density functions of these random variables are as follows, respectively:

$$f_X(x) = \mu e^{-\mu x}, x \geq 0; \; f_W(t) = \lambda e^{-\lambda t}, t \geq 0 \quad (12)$$

Before solving Eq.(11), let's examine the distribution and Moment Generating Function of random variables $Y_n = \sqrt{W}_n$.

Since the random variable W_n has the Exponential distribution with parameter λ, then distribution of the random variable Y_n can be written as follows:

$$F_Y(t) \equiv P\{Y_n \leq t\} = P\{\sqrt{W}_n \leq t\} = P\{W_n \leq t^2\} = 1 - \exp(-\lambda t^2)\, , t \geq 0 \quad (13)$$

As can be seen from Eq.(13), Y_n has Weibull distribution with parameter set $(\alpha = 2; \lambda)$ whose probability density function is as follows:

$$f_Y(t) = 2\lambda t \exp(-\lambda t^2)\, , t \geq 0. \quad (14)$$

Now let's write the Moment Generating Function of the random variable Y_n.

$$M_Y(k) = E(e^{kY_1}) = \int_0^\infty e^{kt} f_Y(t)dt = \int_0^\infty e^{kt} 2\lambda t e^{-\lambda t^2} dt = 2\lambda \int_0^\infty texp\left\{-\lambda\left[t^2 - \frac{k}{\lambda}t\right]\right\}dt$$

$$= 2\lambda \exp\left(\frac{k^2}{4\lambda}\right)\int_0^\infty t \exp\left\{-\lambda[t - \frac{k}{2\lambda}]^2\right\}dt \quad (15)$$

By changing the variable $v = t - \frac{k}{2\lambda}$ in the integral in Eq.(15), the Moment Generating Function of the random variable Y_n, can be written as follows:

$$M_Y(k) = 2\lambda \exp\left(\frac{k^2}{4\lambda}\right) \int_{-\frac{k}{2\lambda}}^\infty \left(v + \frac{k}{2\lambda}\right) \exp(-\lambda v^2)\, dv$$

$$= 2\lambda \exp\left(\frac{k^2}{4\lambda}\right)\left\{\int_{-\frac{k}{2\lambda}}^{\infty} v\exp\left(-\lambda v^2\right)dv + \frac{k}{2\lambda}\int_{-\frac{k}{2\lambda}}^{\infty} \exp\left(-\lambda v^2\right)dv\right\} \tag{16}$$

Let's rewrite Eq.(16) by change of the variable $x = \sqrt{2\lambda}v$:

$$M_Y(k) = 2\lambda \exp\left(\frac{k^2}{4\lambda}\right)\left\{\int_{-\frac{k}{\sqrt{2\lambda}}}^{\infty} \frac{x}{\sqrt{2\lambda}} exp\left(-\frac{x^2}{2}\right)\frac{dx}{\sqrt{2\lambda}} + \frac{k}{2\lambda}\int_{-\frac{k}{\sqrt{2\lambda}}}^{\infty} exp\left(-\frac{x^2}{2}\right)\frac{dx}{\sqrt{2\lambda}}\right\} \tag{17}$$

For brevity, let's introduce the following notations: $T \equiv -\frac{k}{\sqrt{2\lambda}}$; $\varphi(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right), x \in R$.

Here φ(x) is the probability density function of the standard normal distribution.

Considering the above accepted notations in Eq.(17), the Moment Generating Function $M_Y(k)$ of the random variable Y_n can be written as follows:

$$M_Y(k) = 2\lambda \exp\left(\frac{T^2}{2}\right)\left\{\frac{1}{2\lambda}\int_T^{\infty} x \exp\left(-\frac{x^2}{2}\right)dx + \frac{k}{2\lambda\sqrt{2\lambda}}\int_T^{\infty} \exp\left(-\frac{x^2}{2}\right)dx\right\}$$

$$= 2\lambda \exp\left(\frac{T^2}{2}\right)\left\{\frac{1}{2\lambda}\int_T^{\infty} x\sqrt{2\pi}\,\varphi(x)dx + \frac{k}{2\lambda\sqrt{2\lambda}}\int_T^{\infty} \sqrt{2\pi}\,\varphi(x)dx\right\}$$

$$= \sqrt{2\pi} exp\left(\frac{T^2}{2}\right)\left\{\int_T^{\infty} x\,\varphi(x)dx + \frac{k}{\sqrt{2\lambda}}\int_T^{\infty} \varphi(x)dx\right\} = \frac{1}{\varphi(T)}\left\{\int_T^{\infty} x\,\varphi(x)dx - T\int_T^{\infty} \varphi(x)dx\right\}$$

$$= \frac{1}{\varphi(T)}\{\varphi(T) - T\bar{\Phi}(T)\} = 1 - \frac{T}{\varphi(T)}\bar{\Phi}(T)$$

Here $\bar{\Phi}(T) \equiv \int_T^{\infty} \varphi(x)dx$.

In summary, we have:

$$M_Y(k) \equiv 1 - \frac{T}{\varphi(T)}\bar{\Phi}(T), \qquad T \equiv -\frac{k}{\sqrt{2\lambda}}. \tag{18}$$

Considering that $k = -cr$ in Eq.(18), it becomes $T \equiv \frac{cr}{\sqrt{2\lambda}}$. In this case, the following equation is obtained: $M_Y(-cr) \equiv 1 - \frac{T}{\varphi(T)}\bar{\Phi}(T)$

On the other hand, since the random variables $X_i, i = 1,2,3,...$, which represent the payment amount, have Exponential distribution with parameter $\mu > 0$, the Moment Generating Function can be presented as follows:

$$M_X(r) \equiv \frac{\mu}{\mu - r} \tag{19}$$

Inserting Eq.(18) and Eq.(19) into Eq.(11), the following equation is obtained:

$$\frac{\mu}{\mu - r}\left[1 - \frac{T}{\varphi(T)}\bar{\Phi}(T)\right] = 1 \Leftrightarrow 1 - \frac{T}{\varphi(T)}\bar{\Phi}(T) = \frac{\mu - r}{\mu} \Leftrightarrow 1 - \frac{T}{\varphi(T)}\bar{\Phi}(T) = 1 - \frac{r}{\mu}$$

In other words,

$$\frac{T}{\varphi(T)}\bar{\Phi}(T) = \frac{r}{\mu} \tag{20}$$

It can be rewritten as $r = \frac{\sqrt{2\lambda}}{c}T$, from the definition of $T \equiv \frac{cr}{\sqrt{2\lambda}}$.

Inserting $r = \frac{\sqrt{2\lambda}}{c}T$ in Eq.(20), we obtain the following equation:

$$\frac{T}{\varphi(T)}\bar{\Phi}(T) = \frac{\sqrt{2\lambda}}{c\mu}T \Leftrightarrow \frac{\bar{\Phi}(T)}{\varphi(T)} = \frac{\sqrt{2\lambda}}{c\mu} \Leftrightarrow \varphi(T) = \frac{c\mu}{\sqrt{2\lambda}}\bar{\Phi}(T) \tag{21}$$

For brevity we put $K \equiv \frac{c\mu}{\sqrt{2\lambda}}$. In this case, Eq.(21) becomes:

$$\varphi(T) = K\bar{\Phi}(T) \tag{22}$$

Here, $\varphi(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right), x \in R; \bar{\Phi}(T) \equiv \int_T^\infty \varphi(x)dx; K \equiv \frac{c\mu}{\sqrt{2\lambda}}; T \equiv \frac{cr}{\sqrt{2\lambda}}$.

Note that, when net profit condition is satisfied, an interval for variation of the constant K can be found. We know that the expected value of the claim amount is $E(X_1) = \frac{1}{\mu}$. Also, since the random variable Y_1 representing time dependent income has a Weibull distribution with $(\alpha = 2; \lambda)$ parameter, its expected value is given as follows:

$$E(Y_1) = \frac{1}{\sqrt{\lambda}}\Gamma\left(1 + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2\sqrt{\lambda}} \tag{23}$$

Net profit condition demands that $E(X_1) < cE(Y_1)$. In this case, the following inequality

$$\frac{1}{\mu} < c\frac{\sqrt{\pi}}{2\sqrt{\lambda}} \tag{24}$$

holds. Thus, from the inequality (24), the following interval of variation is found for the coefficient K:

$$\sqrt{\frac{2}{\pi}} < K < \infty \tag{25}$$

When $K > \sqrt{\frac{2}{\pi}}$, the graph of the functions $y = \varphi(T)$ and $y = K\bar{\Phi}(T)$ can be drawn on the same coordinate system and the intersection point of these functions can be found (see, Figure 2). As can be seen in the graph below, the graphs of these functions intersect at a single positive abscissa point. Therefore, the first positive solution of Eq.(22) can be found by numerical methods.

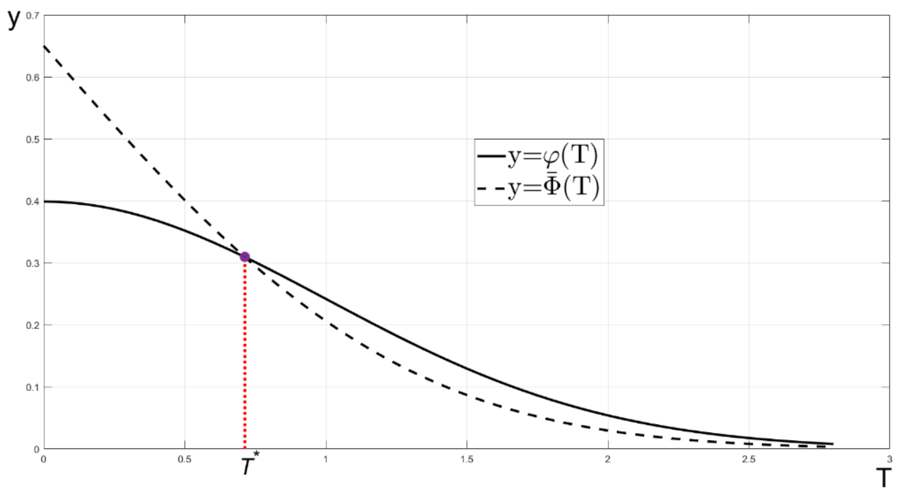

Figure 2. Intersection of $y = \varphi(T)$ and $y = \bar{\Phi}(T)$ at T^*

The T^* parameter, which is the abscissa of the intersection point, is the first positive value that satisfies the Eq.(22). To find the upper bound for the probability of ruin, the value of r corresponding to the T^* value can be found from the following formula:

$$r = \frac{\sqrt{2\lambda}}{c}T^* \tag{26}$$

Let us demonstrate the calculation of r using the above algorithm with the following example.

Example 1: Let the random variable X_1 express the claim amount which has exponential distribution with the $\mu = \frac{1}{2}$ parameter. Let the random variable W_1 represent the time between claims which has Exponential distribution with $\lambda = \frac{1}{2}$ parameter. Assume that the premium rate is $c = 2$ per unit time. Then what is the upper bound for the ruin probability?

Solution: For $\mu = \frac{1}{2}; \lambda = \frac{1}{2}; c = 2$ values, $K \equiv \frac{c\mu}{\sqrt{2\lambda}} = 1$.

Using the above algorithm for $K \equiv 1$, we get $\varphi(T) \equiv \overline{\Phi}(T)$.

Solving this equation by numerical method, the value $T^* = 0.22885$ is obtained. A corresponding adjustment coefficient r to this value of T^* is found as follows:

$$r = \frac{\sqrt{2\lambda}}{c}T^* = \frac{0.22885}{2} = 0.114425$$

Here, $T^* = inf\{T > 0: \varphi(T) \equiv \overline{\Phi}(T)\}$.

Thus, the following inequality can be written for the ruin probability – for the non-linear model being considered – of the insurance company with the initial capital $u > 0$ as follows:

$$\Psi(u) = P_u\{\tau_1 < +\infty\} \leq e^{-ru} = e^{-(0.114425)u}.$$

Question: What should be the initial capital of the insurance company considered in Example 1 so that the probability of ruin is less than 5%.

Answer: When the adjustment coefficient is $r = 0.114425$, the upper bound of the probability of ruin of the considered insurance company is $e^{-(0.114425)u}$**.** If the upper bound of the probability of ruin is 5%, the following equation can be written:

$$e^{-(0.114425)u} = 0.05$$

From above equation, we get:

$$u = \frac{\ln(0.05)}{-0.114425} = \frac{-2.995732}{-0.114425} = 26.18075 \approx 26\, units \approx 13E(X_1)$$

is found. In summary, for the company to have a probability of ruin less than %5, the initial capital of the company should be at least approximately 13 times the payment in a claim. Note that in this example 1 $unit = \frac{1}{2}E(X_1)$.

6. Conclusion

In this paper, a special non-linear risk model is constructed and analyzed. In this model, the premium income function of an insurance company is expressed as $p(t) = cg(t) = c\sqrt{t}$, a function which increases slower than the linear function as in the classical model, which yields the Non-Linear Lundberg Risk Model as a result. Furthermore, a Lundberg type upper bound was found for this non-linear model. To calculate an upper bound, the adjustment coefficient r was estimated with numerical methods by solving integral equations.

Applying the mathematical techniques presented in this study, similar problems can be investigated by expressing the premium income function different than $p(t) = c\sqrt{t}$, which are monotonically increasing with the decreasing rate of growth. For instance, allowing $p(t) = cln(1 + t)$, an analogous work could be conducted. Because, the rate of growth of the logarithmic function is slower than square-root function, logarithmic modeling can be preferred for some specific cases.

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Conflicts of Interest

The authors declared that there is no conflict of interest.

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