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Construction of Pure Metrics on Almost Complex Metallic Manifolds

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ABSTRACT: This paper aims to present a way of pure metrics construction on a pseudo-Riemannian manifold of neutral signature with respect to almost metallic structures and give some examples about these pure metrics.

Keywords: Almost Complex Structure, Metallic Structure, Pure Metric

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INTRODUCTION

Let M_{2n} be a Riemannian manifolds with a neutral metric, i.e., with a pseudo-Riemannian metric g of signature (n,n). The set of all tensor fields of type (p,q) on M_{2n} is denoted by $\Im_q^p(M_{2n})$. Tensor fields and manifolds are always assumed to be differentiable and of class C^∞.

In this paper, since we will emphasize on pure metrics and almost metallic structures, we give some explanations about them.

Let M_{2n} be a differentiable manifold. J is called an almost complex structure if $J^2 = -I$ for the an affinor field $J \in \Im_1^1(M_{2n})$, where I is a field of identity endomorphisms.

A pseudo-Riemannian metric g is called a Norden metric on M_{2n} if

$$g(JX,JY) = -g(X,Y) \tag{1}$$

for any $X,Y \in \Im_0^1(M_{2n})$. (M_{2n},J,g) is said to be an almost Norden manifold if (M_{2n},J) is an almost complex manifold with Norden metric g. If J is integrable, (M_{2n},J,g) is called a Norden manifold. It is known that the condition of integrability of an almost complex structure is $N_J = 0$, where N is Nijenhuis tensor field on M_{2n}. Also, this condition is equivalant to $\nabla J = 0$, where ∇ is a torsion free linear connection. The Norden metrics are also studied under the names pure, anti-Hermitian and B-metrics in (Tachibana, 1960; Kruchkovich, 1972; Salimov, 1983; Ganchev and Borisov, 1986; Etayo and Santamaria, 2000; Vishnevskii, 2002; Salimov and Iscan, 2010).

The golden mean is represented by the positive root of the equation $x^2 - x - 1 = 0$ and has the most important two generalizations. One of these generalizations is a positive root of the equation $x^{r+1} - x^r - 1 = 0$, $(r = 0,1,2,\cdots)$ which is named the golden r −proportions (Stakhov, 1977). The other generalization is named metallic proportions or metallic means family (de Spinadel, 1999; de Spinadel, 1999; de Spinadel, 2000; de Spinadel,2002). The positive solution of the equation $x^2 - rx - s = 0$ for the positive integers r and s, is called members of the metallic means family. This family's all members are $\sigma_{r,s} = \frac{r+\sqrt{r^2+4s}}{2}$ which are positive quadratic irrational numbers. These numbers $\sigma_{r,s}$ are also called (r,s) −metallic numbers.

In (Turanli et all., 2021) equation $x^2 - rx + \frac{3}{2}s = 0$ is considered, in which equation $s \geq 0$, $-\sqrt{6s} < r < \sqrt{6s}$ and both of them are real numbers. Then, $\sigma_{r,s}^c = \frac{r\pm\sqrt{r^2-6s}}{2}$ are complex roots of the equation. They call the complex numbers $\sigma_{r,s}^c = \frac{r+\sqrt{r^2-6s}}{2}$ as complex metallic means family. Based on the complex metallic means family, an almost complex metallic structure is established on a Riemannin manifold. The almost complex metallic structure denoted by J_M is a $(1,1)$ −tensor field and holds ${J_M}^2 - rJ_M + \frac{3}{2}sI = 0$, where I is the identity oparator. Therefore, the pair (M_{2n},J_M) is called an almost complex manifold if the manifold M_{2n} has the almost complex metallic structure J_M . Moreover, in this study relationships between the J_M and the J on M_{2n} is examined and the following equation

$$J_M = \frac{r}{2}I \pm \left(\frac{2\sigma_{r,s}^c - r}{2}\right)J \tag{2}$$

is obtained, where J is an almost complex structure.

MATERIALS AND METHODS

Let (M_{2n},g) be a pseudo-Riemannian manifold with the neutral metric g, J and J' are the almost complex structures, in terms of an orthonormal frame $\{e_1,e_2,e_3,e_4\}$ of vectors and its dual frame $\{e^1,e^2,e^3,e^4\}$ of 1 −forms, almost complex structures J and $J'$are given as follows:

$$J=\begin{pmatrix}0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\end{pmatrix} \tag{3}$$

$$J'=\begin{pmatrix}0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{pmatrix} \tag{4}$$

where J and J' also commute with each other as $J^2={J'}^2=-I$, $JJ'=J'J$ (Bonome et al., 2005). Substituting (3) in (2), we obtain the following almost metallic structures

$$J_{M_1}=\begin{pmatrix} r/2 & -\left(\sqrt{r^2-6s}\right)/2 & 0 & 0\\ \left(\sqrt{r^2-6s}\right)/2 & r/2 & 0 & 0\\ 0 & 0 & r/2 & -\left(\sqrt{r^2-6s}\right)/2\\ 0 & 0 & \left(\sqrt{r^2-6s}\right)/2 & r/2\end{pmatrix} \tag{5}$$

and

$$J_{M_2}=\begin{pmatrix} r/2 & \left(\sqrt{r^2-6s}\right)/2 & 0 & 0\\ -\left(\sqrt{r^2-6s}\right)/2 & r/2 & 0 & 0\\ 0 & 0 & r/2 & \left(\sqrt{r^2-6s}\right)/2\\ 0 & 0 & -\left(\sqrt{r^2-6s}\right)/2 & r/2\end{pmatrix} \tag{6}$$

Similarly, substituting (4) in (2), we obtain the following almost metallic structures

$$J'_{M_1}=\begin{pmatrix} r/2 & -\left(\sqrt{r^2-6s}\right)/2 & 0 & 0\\ \left(\sqrt{r^2-6s}\right)/2 & r/2 & 0 & 0\\ 0 & 0 & r/2 & \left(\sqrt{r^2-6s}\right)/2\\ 0 & 0 & -\left(\sqrt{r^2-6s}\right)/2 & r/2\end{pmatrix} \tag{7}$$

and

$$J'_{M_2}=\begin{pmatrix} r/2 & \left(\sqrt{r^2-6s}\right)/2 & 0 & 0\\ -\left(\sqrt{r^2-6s}\right)/2 & r/2 & 0 & 0\\ 0 & 0 & r/2 & -\left(\sqrt{r^2-6s}\right)/2\\ 0 & 0 & \left(\sqrt{r^2-6s}\right)/2 & r/2\end{pmatrix} \tag{8}$$

RESULTS AND DISCUSSION

In this section, we shall focus on to find a new pure metric g with respect to almost complex metallic structures J_{M_1}, J_{M_2}, J'_{M_1} and J'_{M_2}. Let (M_{2n}, g) be a pseudo-Riemannian manifold with the neutral metric g. It is possible to write the pure metric g in equation (1), as the matrix equation:

$$J^T g = gJ \tag{9}$$

where J^T is transpose matrix of the matrix J (Savas et al., 2016). We put

$$g = (g_{ij}) = \begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix} \tag{10}$$

and determine the components $g_{ij} = (e_i, e_j)$ with respect to the orthonormal frame $\{e_1, e_2, e_3, e_4\}$. With the substitution of (5) and (10) in (9), for the metric g in (10), we obtain a new pure metric g^1 as follows:

$$g^1 = (g^1_{ij}) = \begin{pmatrix} \gamma & \alpha & \mu & \beta \\ \alpha & -\gamma & \beta & -\mu \\ \mu & \beta & \vartheta & \delta \\ \beta & -\mu & \delta & -\vartheta \end{pmatrix}$$

where $\det g^1 \neq 0$ and $\gamma, \alpha, \mu, \beta, \vartheta, \delta$ are functions on M_{2n}. Moreover, with the substitution of (6) and (10) in (9), for the metric g in (10), we obtain also the pure metric g^1. Thus, we have the theorem below.

Theorem 1: A neutral metric on M_{2n} is pure if and only if it has the form

$$g^1 = (g^1_{ij}) = \begin{pmatrix} \gamma & \alpha & \mu & \beta \\ \alpha & -\gamma & \beta & -\mu \\ \mu & \beta & \vartheta & \delta \\ \beta & -\mu & \delta & -\vartheta \end{pmatrix}, \quad \det g^1 \neq 0 \tag{11}$$

with respect to the orthonormal frame $\{e_1, e_2, e_3, e_4\}$.

Similarly, with the substitution of (7) and (10) in (9), for the metric g in (10), we obtain a new pure metric g^2 as follows:

$$g^2 = (g^2_{ij}) = \begin{pmatrix} \gamma & \alpha & \mu & \beta \\ \alpha & -\gamma & -\beta & \mu \\ \mu & -\beta & \vartheta & \delta \\ \beta & \mu & \delta & -\vartheta \end{pmatrix}$$

where $\det g^2 \neq 0$ and $\gamma, \alpha, \mu, \beta, \vartheta, \delta$ are functions on M_{2n}. Moreover, with the substitution of (8) and (10) in (9), for the metric g in (10), we obtain also the pure metric g^2. Thus, we have the theorem below.

Theorem 2: A neutral metric on M_{2n} is pure if and only if it has the form

$$g^2 = (g^2_{ij}) = \begin{pmatrix} \gamma & \alpha & \mu & \beta \\ \alpha & -\gamma & -\beta & \mu \\ \mu & -\beta & \vartheta & \delta \\ \beta & \mu & \delta & -\vartheta \end{pmatrix}, \quad \det g^2 \neq 0 \tag{12}$$

with respect to the orthonormal frame $\{e_1, e_2, e_3, e_4\}$.

Some Examles of Pure Metrics

From (11), two examles of pure metrics can be written as follows:

$$g^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{13}$$

or

$$g^1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{14}$$

Similarly, from (12), two examles of pure metrics can be written as follows:

$$g^2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{15}$$

or

$$g^2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{16}$$

The construction of pure (Norden) metrics is also studied by many authors (Bonome et al., 2005; Salimov and Iscan, 2010; Salimov et al., 2011; Savas et al., 2016). In the paper (Bonome et al., 2005), a method of construction of Norden metrics for almost complex structures on a neutral 4-manifold is investigated and the method is applied to construction of Norden metrics on a Walker 4-manifold. In addition, some examples similar to the ones we found in our study are given about constructed Norden metrics. In the paper (Salimov and Iscan, 2010), for a proper almost complex structure J on Walker 4-manifold, an almost Norden structure (g^{N+}, J) is constructed, where g^{N+} is a Norden metric with respect to the J and called an almost Norden Walker metric. In the paper (Salimov et al., 2011) an other Norden Walker metric is constructed for a proper almost complex structure φ and denoted by G^{N+}. In the paper (Savas et al., 2016), a way of Norden metrics construction on a semi-Riemannian 4-manifold of neutral signature with respect to Golden structures is given and some examples about constructed Norden metrics are presented.

CONCLUSION

In this paper, new pure metrics are constructed with respect to almost complex metallic structures on M_{2n} and some typical examples of these metrics are given.

Conflict of Interest

The article authors declare that there is no conflict of interest between them.

Author's Contributions

The author declares that se has contributed to the article alone.

REFERENCES

Bonome A, Castro R, Hervella LM, MatsushitaY, 2005. Construction of Norden structures on neutral 4-manifolds. JP J. Geom. Topol.5 (2):121-140.

De Spinadel VW, 1999. The Metallic Means Family and Multifractal Spectra. Nonlinear Anal. Ser. B: Real World Appl. 36 (6): 721-745.

De Spinadel VW, 1999. The Family of Metallic Means. Vis. Math. 1, 3. http//members.tripod.com/vismath1/spinadel/.

De Spinadel VW, 2000. The Metallic Means Family and Renormalization Group Techniques. Proc. Steklov Inst. Math. Control in Dynamic Systems, suppl. 1, 194-209.

De Spinadel VW, 2002. The Metallic Means Family and Forbidden Symmetries. Int. Math. J. 2 (3): 279-288.

Etayo F, Santamaria R, 2000. ($J^2 = \pm 1$)-Metric Manifolds. Publ. Math. Debrecen 57 (3-4): 435-444.

Ganchev GT, Borisov AV, 1986. Note on the Almost Complex Manifolds with a Norden Metric. C. R. Acad. Bulgarie Sci. 39 (5): 31-34.

Kruchkovich GI, 1972. Hypercomplex Structure on a Manifold. I, Tr. Sem. Vect. Tens. Anal. Moscow Univ. 16: 174-201.

Salimov AA, 1983. Almost Analyticity of a Riemannian Metric and Integrability of a Structure. Trudy Geom. Sem. Kazan. Univ. 15:72-78.

Salimov AA,Iscan M, 2010. Some Properties of Norden-Walker Metrics. Kodai Math. J. 33(2):283-293.

Salimov AA, Iscan M, Turanli S, 2011. Differential Geometry of Walker Manifolds. Uchenye Zapiski Kazanskogo Universiteta. Seriya Fiziko-Matematicheskie Nauki. 153(3): 264–271

Savas M, Ozkan M, Iscan M, 2016. On 4-Dimensional Golden-Walker Structures. Journal of Science and Arts, 16(2), 89.

Stakhov AP, 1977. Introduction into Algorithmic Measurement Theory. Moscow in Russian.

Tachibana S, 1960. Analytic Tensor and İts Generalization. Tohoku Math. J. 12 (2):208-221.

Turanli S, Gezer A, Cakicioglu H, 2021. Metallic Kahler and Nearly Metallic Kahler Manifolds. Int. J. Geom. Methods Mod. Phys. 18 (9): 2150146.

Vishnevskii VV, 2002. Integrable Affinor Structures and Their Plural İnterpretations. J. of Math. Sciences, 108 (2): 151-187.