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# On the Solution of Mathematical Model Including Space-Time Fractional Diffusion Equation in Conformable Derivative, Via Weighted Inner Product

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## Abstract

This research aims to accomplish an analytic solution to mathematical models involving space-time fractional differential equations in the conformable sense in series form through the weighted inner product and separation of variables method. The main advantage of this method is that various linear problems of any kind of differential equations can be solved by using this method. First, the corresponding eigenfunctions are established by solving the Sturm-Liouville eigenvalue problem. Secondly, the coefficients of the eigenfunctions are determined by employing weighted inner product and initial condition. Thirdly, the analytic solution to the problem is constructed in the series form. Finally, an illustrative example is presented to show how this method is implemented for fractional problems and exhibit its effectiveness and accuracy.

## 1. Introduction

Since the role of fractional partial differential equations has come into prominence, the focus of numerous scientists in various fields is directed to this subject. As a result, fractional differential equations are employed in the modeling of processes in diverse research areas such as applied mathematics, physics chemistry, power systems, control theory, system theory, optimization, signal processing, epidemic model of childhood disease, epidemic system of HIV/AIDS transmission etc., [1-13]. The main reason for this increase in interest is that the fractional derivative is a non-local operator, which allows us to analyze the behavior of the complex non-linear processes much better than by using the alternatives.

Dealing with fractional derivatives is more complicated than ordinary derivatives as several difficulties are encountered when solving fractional differential equations. Essential properties of ordinary derivatives such as the product rule and the chain rule are not held by the majority of the fractional derivatives. However, conformable fractional derivative holds almost all properties of ordinary derivative, which allows us to handle and accomplish the solution of mathematical models, including fractional differential equations in the conformable sense without any difficulty [14-16]. Therefore, in this research, we look for the solution to models given in the following form :

$$\frac{\partial^{\alpha} u(x,t;\alpha,\beta)}{\partial t^{\alpha}} = A\frac{\partial^{2\beta} u(x,t;\alpha,\beta)}{\partial x^{2\beta}} + B\frac{\partial^{\beta} u(x,t;\alpha,\beta)}{\partial x^{\beta}} + C \qquad (1)$$

$$u(0,t) = u(L,t) = 0, \qquad (2)$$

$$u(x,0) = f(x,\beta), \qquad (3)$$

where $0 \le x \le L,\ 0 \le t \le T,\ 0 < \alpha \le 1,\ 1 < 2\beta \le 2,$ $A, B, C \in R, A \ne 0.$

The primary motivation for this study is the fact that diverse scientific processes are modeled by fractional diffusion equations. Subsequently, solving this kind of fractional diffusion problem has drawn the attention of

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many scientists in various branches.

## 2. Preliminary Results

The definition and fundamental properties of the conformable fractional derivative are presented in this section.

Definition: The conformable fractional derivative of function $f(x)$ for $0 < \alpha \leq 1$ is introduced as follows [11]: $\frac{d^\alpha f(x)}{dx^\alpha} = \frac{f(x+\epsilon x^{1-\alpha})-f(x)}{\epsilon}$.

We list fundamental properties of the conformable fractional derivatives of certain functions as:

1. If $f$ is differentiable then $\frac{d^\alpha}{dx^\alpha} f(x) = x^{1-\alpha} \frac{d}{dx} f(x)$.
2. $\frac{d^\alpha}{dx^\alpha}(af(x)+bg(x)) = a\frac{d^\alpha f(x)}{dx^\alpha} + b\frac{d^\alpha g(x)}{dx^\alpha}$.
3. $\frac{d^\alpha}{dx^\alpha}(x^p) = px^{p-\alpha}$ for all $p \in R$.
4. $\frac{d^\alpha}{dx^\alpha}\big(f(x)g(x)\big) = f(x)\frac{d^\alpha g(x)}{dx^\alpha} + g(x)\frac{d^\alpha f(x)}{dx^\alpha}$.
5. $\frac{d^\alpha}{dx^\alpha}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d^\alpha f(x)}{dx^\alpha} - f(x)\frac{d^\alpha g(x)}{dx^\alpha}}{g^2(x)}$.
6. $\frac{d^\alpha}{dx^\alpha}(\lambda) = 0$, for all constant functions $f(x) = \lambda$.

Additionally, the conformable fractional derivative of basic functions are computed in the following form:

$$\frac{d^\alpha}{dx^\alpha} x^p = px^{p-\alpha},$$
$$\frac{d^\alpha}{dx^\alpha} sin\left(\frac{1}{\alpha}x^\alpha\right) = cos\left(\frac{1}{\alpha}x^\alpha\right),$$
$$\frac{d^\alpha}{dx^\alpha} cos\left(\frac{1}{\alpha}x^\alpha\right) = -sin\left(\frac{1}{\alpha}x^\alpha\right),$$
$$\frac{d^\alpha}{dx^\alpha} exp\left(\frac{1}{\alpha}x^\alpha\right) = exp\left(\frac{1}{\alpha}x^\alpha\right).$$

Notice that the corresponding ordinary derivatives of the functions given above are obtained for $\alpha = 1$.

Even though some functions such as $f(x) = 2\sqrt{x}$ are not differentiable, they could be $\alpha-$differentiable in the conformable sense, for example, $\frac{d^{\frac{1}{2}} f(x)}{dx^{\frac{1}{2}}} = 1$ and $\frac{d^{\frac{1}{2}} f(0)}{dx^{\frac{1}{2}}} = 1$. This is another distinct property of conformable fractional derivative.

## 3. Main Results

By means of the separation of variables method, the generalized solution to problem (1)-(3) is constructed in the following form:

$$u(x,t;\alpha,\beta) = X(x;\beta)\, T(t;\alpha,\beta) \tag{4}$$

where $0 \leq x \leq L$, $0 \leq t \leq T$.

As indicated above, the function $X$ depends on the spatial fractional order $\beta$, while the function $T$ depends on both time and spatial fractional orders $\alpha$ and $\beta$, respectively.

Substituting (4) into (1) and rearranging yields

$$\frac{d^\alpha T(t;\alpha,\beta)}{dt^\alpha}\frac{1}{T(t;\alpha,\beta)} = A\frac{d^{2\beta}X(x;\beta)}{dx^{2\beta}}\frac{1}{X(x;\beta)} + B\frac{d^\beta X(x;\beta)}{dx^\beta}\frac{1}{X(x;\beta)} + C = -\lambda(\beta) \tag{5}$$

Notice that $\lambda$ is a function of $\beta$. Taking equation (5) into account, two fractional differential equations of unknown functions $X$ and $T$ are obtained separately. The fractional differential equation of X subject to the boundary conditions is given as follows:

$$A\frac{d^{2\beta}X(x;\beta)}{dx^{2\beta}}\frac{1}{X(x;\beta)} + B\frac{d^\beta X(x;\beta)}{dx^\beta}\frac{1}{X(x;\beta)} + \lambda(\beta)X(x;\beta) = 0, \tag{6}$$

$$X(0;\beta) = X(L;\beta) = 0. \tag{7}$$

This problem is called the fractional Sturm-Liouville problem, and its solutions are obtained by applying the exponential function in the following form:

$$X(x;\beta) = exp\left(r\frac{x^\beta}{\beta}\right). \tag{8}$$

This yields the following characteristic equation:

$$Ar^2 + Br + \lambda(\beta) = 0. \tag{9}$$

In order to solve this equation, the following 3 cases are taken into account:

Case 1, $B^2 - 4A\lambda(\beta) > 0$:

There are two real and distinct solutions $r_1, r_2$ providing the solution to problem (6)-(7) in the following form:

$$X(x;\beta) = c_1\, exp\left(r_1\frac{x^\beta}{\beta}\right) + c_2 exp\left(r_2\frac{x^\beta}{\beta}\right).$$

First boundary condition yields

$$X(0;\beta) = c_1 + c_2 = 0 \Longrightarrow c_2 = -c_1. \tag{10}$$

Hence, the solution becomes

$$X(x;\beta) = c_1\left(exp\left(r_1\frac{x^\beta}{\beta}\right) - exp\left(r_2\frac{x^\beta}{\beta}\right)\right). \tag{11}$$

Similarly, last boundary condition yields

$$X(L;\beta) = c_1\left(exp\left(r_1\frac{L^\beta}{\beta}\right) - exp\left(r_2\frac{L^\beta}{\beta}\right)\right) = 0, \qquad (12)$$

which indicates that

$$exp\left(r_1\frac{L^\beta}{\beta}\right) \neq exp\left(r_2\frac{L^\beta}{\beta}\right). \qquad (13)$$

As a result, we have

$$c_1 = 0 \qquad (14)$$

which implies that there is no solution for Case 1.

Case 2, $B^2 - 4A\lambda(\beta) = 0$:

There is a corresponding coincident as the solutions are equal $r_1 = r_2$, which provides the solution to problem (6)-(7) in the following form:

$$X(x;\beta) = c_1\, exp\left(r_1\frac{x^\beta}{\beta}\right) + c_2\frac{x^\beta}{\beta} exp\left(r_1\frac{x^\beta}{\beta}\right). \qquad (15)$$

First boundary condition yields

$$X(0) = c_1 = 0. \qquad (16)$$

Hence the solution becomes

$$X(x;\beta) = c_2\frac{x^\beta}{\beta} exp\left(r_1\frac{x^\beta}{\beta}\right). \qquad (17)$$

Similarly last boundary condition yields

$$X(l) = c_2\frac{L^\beta}{\beta} exp\left(r_1\frac{L^\beta}{\beta}\right) \Longrightarrow c_2 = 0, \qquad (18)$$

which implies that there is no solution for Case 2.

Case 3, $B^2 - 4A\lambda(\beta) < 0$:

There are two complex roots $-\frac{B}{2A} \pm i\frac{\sqrt{4A\lambda(\beta)-B^2}}{2A}$ which provides the solution to problem (6)-(7) in the following form:

$$X(x;\beta) = exp\left(-\frac{B}{2A}\frac{x^\beta}{\beta}\right)\left[c_1 \cos\left(\frac{\sqrt{4A\lambda(\beta)-B^2}}{2A}\frac{x^\beta}{\beta}\right) + c_2 \sin\left(\frac{\sqrt{4A\lambda(\beta)-B^2}}{2A}\frac{x^\beta}{\beta}\right)\right]. \qquad (19)$$

First boundary condition yields

$$X(0;\beta) = c_1 = 0 \qquad (20)$$

Hence the solution becomes

$$X(x;\beta) = c_2\, exp\left(-\frac{B}{2A}\frac{x^\beta}{\beta}\right) \sin\left(\frac{\sqrt{4A\lambda(\beta)-B^2}}{2A}\frac{x^\beta}{\beta}\right). \qquad (21)$$

Similarly last boundary condition yields

$$X(L) = c_2\, exp\left(-\frac{B}{2A}\frac{L^\beta}{\beta}\right) \sin\left(\frac{\sqrt{4A\lambda(\beta)-B^2}}{2A}\frac{L^\beta}{\beta}\right) = 0 \qquad (22)$$

which indicates that

$$\sin\left(\frac{\sqrt{4A\lambda(\beta)-B^2}}{2A}\frac{L^\beta}{\beta}\right) = 0. \qquad (23)$$

Hence the corresponding eigenvalues become

$$\lambda_n(\beta) = \frac{(2n\pi\beta A)^2 + B^2L^{2\beta}}{4AL^{2\beta}}, \quad 0 < \lambda_1(\beta) < \lambda_2(\beta) < \lambda_3(\beta) \ldots \qquad (24)$$

As a result, the solution to the problem (6)-(7) is concluded in the following form:

$$X_n(x;\beta) = exp\left(-\frac{B}{2A}\frac{x^\beta}{\beta}\right) \sin\left(n\pi\frac{x^\beta}{L^\beta}\right), n = 1,2,3, \ldots \qquad (25)$$

The fractional differential equation of $T$ for each eigenvalue $\lambda_n(\beta)$ is given as follows:

$$\frac{d^\alpha T(t;\alpha,\beta)}{dt^\alpha} = \left(C - \lambda(\beta)\right)T(t;\alpha,\beta) \qquad (26)$$

which has the following solutions

$$T_n(t;\alpha,\beta) = exp\left(\left(C - \frac{(2n\pi\beta A)^2 + B^2L^{2\beta}}{4AL^{2\beta}}\right)\frac{t^\alpha}{\alpha}\right), n = 1,2,3, \ldots \qquad (27)$$

Hence the function corresponding to each eigenvalue $\lambda_n(\beta)$ is defined as

$$u_n(x,t;\alpha,\beta) = exp\left(\left(C - \frac{(2n\pi\beta A)^2 + B^2L^{2\beta}}{4AL^{2\beta}}\right)\frac{t^\alpha}{\alpha}\right) exp\left(-\frac{B}{2A}\frac{x^\beta}{\beta}\right) \sin\left(n\pi\frac{x^\beta}{L^\beta}\right) \qquad (28)$$

which satisfies equation (1) and boundary conditions but not the initial condition. In order to acquire the solution to problem (1)-(3), we construct the following series

$$u(x,t;\alpha,\beta) = \sum_{n=1}^{\infty} d_n exp\left(\left(C - \frac{(2n\pi\beta A)^2 + B^2L^{2\beta}}{4AL^{2\beta}}\right)\frac{t^\alpha}{\alpha}\right) exp\left(-\frac{B}{2A}\frac{x^\beta}{\beta}\right) sin\left(n\pi\frac{x^\beta}{L^\beta}\right) \quad (29)$$

which also satisfies both the fractional equation (1) and boundary condition (2). In order to make this solution satisfy the initial condition (3), we must determine the coefficients $d_n$ accurately. Taking the initial condition (3), we get

$$u(x,0) = \sum_{n=1}^{\infty} d_n\, exp\left(-\frac{B}{2A}\frac{x^\beta}{\beta}\right) sin\left(n\pi\frac{x^\beta}{L^\beta}\right) = f(x;\beta)\, exp\left(-\frac{B}{2A}\frac{x^\beta}{\beta}\right). \quad (30)$$

Utilizing the weighted inner product, the coefficients $d_n$ are computed for $n = 1,2,3, \ldots$ as follows:

$$d_n = \frac{2\beta}{L^\beta}\int_0^L f(x;\beta)\, sin\left(n\pi\frac{x^\beta}{L^\beta}\right)\frac{1}{x^{1-\beta}}dx,\ n = 1,2,3, \ldots \quad (31)$$

where the weighted inner product of two functions $f(x), g(x)$ is defined as follows:

$$\langle f(x), g(x)\rangle = \frac{2\beta}{L^\beta}\int_0^L f(x)g(x)\, exp\left(\frac{B}{A}\frac{x^\beta}{\beta}\right)\frac{1}{x^{1-\beta}}dx.$$

## 4. Illustrative Example

In this section, we present an example to illustrate how to implement the method explained in this study and to acquire the solution via this method.

Firstly, we consider the following initial boundary value problem with integer-order derivatives:

$$u_t(x,t) = u_{xx}(x,t) + u_x(x,t) - u(x,t), \quad (32)$$

$$u(0,t) = 0, u(2,t) = 0, \quad (33)$$

$$u(x,0) = sin\left(\frac{\pi}{2}x\right)\exp\left(-\frac{1}{2}x\right), \quad (34)$$

which has the solution in the following form:

$$u(x,t) = sin\left(\frac{\pi}{2}x\right)\exp\left(-\frac{1}{2}x\right)\exp\left(-\left(\frac{\pi^2}{4}+\frac{5}{4}\right)t\right) \quad (35)$$

where $0 \le x \le 2$, $t \ge 0$.

Now, take the same problem with space and time fractional derivatives:

$$\frac{\partial^\alpha u(x,t;\alpha,\beta)}{\partial t^\alpha} = \frac{\partial^{2\beta}u(x,t;\alpha,\beta)}{\partial x^{2\beta}} + \frac{\partial^\beta u(x,t;\alpha,\beta)}{\partial x^\beta} - u(x,t;\alpha,\beta), \quad (36)$$

$$u(0,t;\alpha,\beta) = u(2,t;\alpha,\beta) = 0, \quad (37)$$

$$u(x,0;\alpha,\beta) = sin\left(\pi\frac{x^\beta}{2^\beta}\right) exp\left(-\frac{1}{2}\frac{x^\beta}{\beta}\right), \quad (38)$$

where $0 < \alpha \le 1, 1 < 2\beta \le 2, 0 \le x \le 2, 0 \le t \le T$.

Carrying out the separation of the variables to (36) yields

$$\frac{d^\alpha T(t;\alpha,\beta)}{dt^\alpha}\frac{1}{T(t;\alpha,\beta)} = \frac{d^{2\beta}X(x;\beta)}{dx^{2\beta}}\frac{1}{X(x;\beta)} + \frac{d^\beta X(x;\beta)}{dx^\beta}\frac{1}{X(x;\beta)} - 1 = -\lambda(\beta) \quad (39)$$

which has the corresponding fractional Sturm-Liouville problem:

$$\frac{d^{2\beta}X(x;\beta)}{dx^{2\beta}} + \frac{d^\beta X(x;\beta)}{dx^\beta} + \lambda(\beta)X(x;\beta) = 0, \quad (40)$$

$$X(0;\beta) = X(2;\beta) = 0. \quad (41)$$

The corresponding eigenvalues $\lambda_n(\beta)$ and solutions $X_n(x;\beta)$ to the problem (40)-(41) for $n = 1,2,3, \ldots$ are acquired in the following form:

$$\lambda_n(\beta) = \frac{(2n\pi\beta)^2 + 2^{2\beta}}{4.2^{2\beta}}, \quad 0 < \lambda_1(\beta) < \lambda_2(\beta) < \lambda_3(\beta) < \cdots \quad (42)$$

$$X_n(x;\beta) = exp\left(-\frac{1}{2}\frac{x^\beta}{\beta}\right) sin\left(n\pi\frac{x^\beta}{2^\beta}\right). \quad (43)$$

The fractional differential equation of $T$ for each eigenvalue $\lambda_n(\beta)$ is given as follows:

$$\frac{d^\alpha T(t;\alpha,\beta)}{dt^\alpha} = \left(-1 - \frac{(2n\pi\beta)^2 + 2^{2\beta}}{4.2^{2\beta}}\right)T(t;\alpha,\beta). \quad (44)$$

The solution of which becomes

$$T_n(t;\alpha,\beta) = exp\left(\left(-1 - \frac{(2n\pi\beta)^2 + 2^{2\beta}}{4.2^{2\beta}}\right)\frac{t^\alpha}{\alpha}\right), \quad n = 1,2,3, \ldots \quad (45)$$

Therefore, the specific solutions to problem (36)-(38) for $n = 1,2,3, \ldots$ are in the following form:

$$u_n(x,t;\alpha,\beta) = exp\left(\left(-1 - \frac{(2n\pi\beta)^2 + 2^{2\beta}}{4.2^{2\beta}}\right)\frac{t^\alpha}{\alpha}\right) exp\left(-\frac{1}{2}\frac{x^\beta}{\beta}\right) sin\left(n\pi\frac{x^\beta}{2^\beta}\right). \quad (46)$$

As a result, the general solution to problem (36)-(38) becomes

$$u(x,t;\alpha,\beta) = \sum_{n=1}^{\infty} d_n exp\left(\left(-1 - \frac{(2n\pi\beta)^2 + 2^{2\beta}}{4.2^{2\beta}}\right)\frac{t^\alpha}{\alpha}\right) exp\left(-\frac{1}{2}\frac{x^\beta}{\beta}\right) sin\left(n\pi\frac{x^\beta}{2^\beta}\right). \quad (47)$$

In order to determine the unknown coefficients $d_n$, we plug $t = 0$ into the general solution (47) and proceed taking the initial condition (38) into account:

$$u(x,0) = \sum_{n=1}^{\infty} d_n exp\left(-\frac{1}{2}\frac{x^\beta}{\beta}\right) sin\left(n\pi\frac{x^\beta}{2^\beta}\right) = sin\left(\pi\frac{x^\beta}{2^\beta}\right) exp\left(-\frac{1}{2}\frac{x^\beta}{\beta}\right). \quad (48)$$

Through the inner product, the coefficients $d_n$ for $n = 1, 2, 3, \ldots$ are determined as

$$d_n = \frac{2\beta}{2^\beta}\int_0^2 sin\left(\pi\frac{x^\beta}{2^\beta}\right) exp\left(-\frac{1}{2}\frac{x^\beta}{\beta}\right) exp\left(-\frac{1}{2}\frac{x^\beta}{\beta}\right) sin\left(n\pi\frac{x^\beta}{2^\beta}\right) exp\left(\frac{x^\beta}{\beta}\right)$$

After rearrangement, we have

$$d_n = \frac{2\beta}{2^\beta}\int_0^2 sin\left(\pi\frac{x^\beta}{2^\beta}\right) sin\left(n\pi\frac{x^\beta}{2^\beta}\right)\frac{1}{x^{1-\beta}}dx.$$

Orthogonality property provides us that $d_n = 0$ for $n \neq 1$ and for $n = 1$, therefore we get

$$d_1 = \frac{2\beta}{2^\beta}\int_0^2 \left(\pi\frac{x^\beta}{2^\beta}\right)\frac{1}{x^{1-\beta}}dx = 1.$$

Thus the general solution to the time-space fractional problem is obtained in the following form:

$$u(x,t;\alpha,\beta) = exp\left(\left(-1 - \frac{(2\pi\beta)^2 + 2^{2\beta}}{4.2^{2\beta}}\right)\frac{t^\alpha}{\alpha}\right) exp\left(-\frac{1}{2}\frac{x^\beta}{\beta}\right) sin\left(\pi\frac{x^\beta}{2^\beta}\right). \quad (49)$$

Notice that as fractional orders $\alpha$ and $\beta$ gets close to 1, the solution (49) to the time-space fractional problem (36)-(38) approaches to the solution (35) of the corresponding initial boundary value problem (32)-(34). This points to the accuracy of the obtained solution.

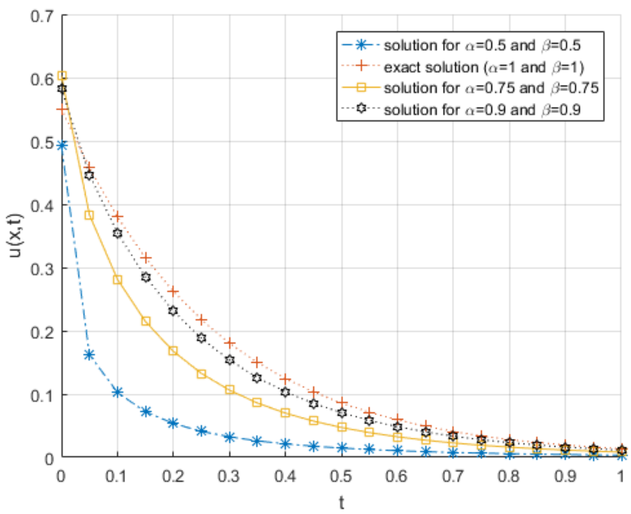


**Figure 1.** The graph of solutions $x = 0.5$ for various values of $\alpha$ and $\beta$.

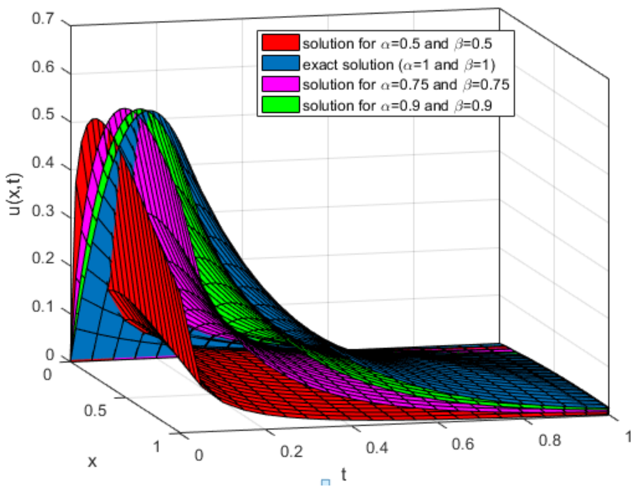


**Figure 2.** 3D graphs of solutions for various values of $\alpha$ and $\beta$.

## 5. Conclusions

In this research, analytic solutions to space-time fractional problems are calculated by means of the separation of variables method and inner product in one dimension. As the application of this method to initial boundary value problems of ordinary derivatives is performed, the corresponding Sturm-Liouville problem is taken into account to determine the eigenvalues of the problem, and then specific solutions are formed. Finally, with the help of the initial condition and orthogonality of the inner product, the general solution to the problem is acquired. An illustrative example is also provided to prove the effectiveness of the method for space-time fractional differential equations.

In future studies, fuzzy space-time fractional problems will be considered as other applications of this method.

## Declaration of Ethical Standards

The authors of this article declare that the materials and methods used in this study do not require ethical committee permission and/or legal-special permissions.

## Conflict of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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