J. Algebra Comb. Discrete Appl.
3(3) • 145–154

Received: 19 September 2015
Accepted: 06 December 2015

Journal of Algebra Combinatorics Discrete Structures and Applications

Enumeration of symmetric (45,12,3) designs with nontrivial automorphisms*

Research Article

Dean Crnković, Doris Dumičić Danilović, Sanja Rukavina

Abstract: We show that there are exactly 4285 symmetric (45,12,3) designs that admit nontrivial automorphisms. Among them there are 1161 self-dual designs and 1562 pairs of mutually dual designs. We describe the full automorphism groups of these designs and analyze their ternary codes. R. Mathon and E. Spence have constructed 1136 symmetric (45,12,3) designs with trivial automorphism group, which means that there are at least 5421 symmetric (45,12,3) designs. Further, we discuss trigeodetic graphs obtained from the symmetric $(45, 12, 3)$ designs. We prove that k-geodetic graphs constructed from mutually non-isomorphic designs are mutually non-isomorphic, hence there are at least 5421 mutually non-isomorphic trigeodetic graphs obtained from symmetric $(45, 12, 3)$ designs.

2010 MSC: 05B05, 20D45, 94B05, 05C38

Keywords: Symmetric design, Linear code, Automorphism group, k-geodetic graph

1. Introduction

The terminology and notation in this paper for designs and codes are as in [2, 3, 6].

One of the main problems in design theory is that of classifying structures with given parameters. Classification of designs has been considered in detail in the monograph [17]. Complete classification of designs with certain parameters has been done just for some designs with relatively small number of points, and in the case of symmetric designs complete classification is done just for a few parameter triples (see [22]). The classification of projective planes of order 9 has been solved in 1991 (see [20]), and Kaski and Östergård classified all biplanes with k=11 in 2008 (see [18]). Hence, the parameter triple (45,12,3) is the next for symmetric designs of order 9 to be classified. Since the complete classification of symmetric (45,12,3) designs seems to be out of reach with the current techniques and computers, only

* *This work has been fully supported by Croatian Science Foundation under the project 1637.*
Dean Crnković (Corresponding Author), Doris Dumičić Danilović, Sanja Rukavina; Department of Mathematics, University of Rijeka, Radmile Matejčić 2, 51000 Rijeka, Croatia (email: deanc@math.uniri.hr, ddumicic@math.uniri.hr, sanjar@math.uniri.hr).

partial classification of such designs, with certain constrains, is possible. In this paper we manage to classify all symmetric (45,12,3) designs with nontrivial automorphisms.

The first symmetric (45,12,3) design was constructed in [1], and further two symmetric (45,12,3) designs were constructed in [23] as (45,12,3) difference sets. Later on, Kölmel [19] and Ćepulić [5] have independently constructed symmetric (45,12,3) designs having an automorphism of order 5. In his doctoral dissertation [19] Kölmel also determined all (45,12,3) designs having a fixed-point-free automorphism of order 3. Finally, Mathon and Spence [25] showed that there are at least 3752 symmetric (45,12,3) designs, 1136 of them having a trivial automorphism group. Furthermore, Coolsaet, De Jager and Spence, established in [7] that there are exactly 78 non-isomorphic strongly regular graphs with parameters (45,12,3,3), meaning that there are exactly 78 symmetric designs having symmetric incidence matrix with zero diagonal. Ternary codes spanned by the adjacency matrices of these strongly regular graphs (i.e. incidence matrices of the corresponding symmetric designs) have been studied in [8]. The symmetric (45,12,3) design admitting a primitive action of the group $PSp(4,3)$ is described in [10] and [12].

In this paper we give the classification of all symmetric (45,12,3) designs having a nontrivial automorphism group. We show that there exist exactly 4285 symmetric (45,12,3) designs that admit nontrivial automorphisms, which means that there are at least 5421 symmetric (45,12,3) designs. Furthermore, we discuss trigeodetic graphs obtained from the symmetric $(45, 12, 3)$ designs and prove that mutually non-isomorphic designs produce mutualy non-isomorphic k-geodetic graphs.

The paper is organized as follows: after the brief introduction, in Section 2 we give basic information concerning the construction method, in Section 3 we describe the construction of symmetric (45,12,3) designs with nontrivial automorphisms and give a list of the designs and their full automorphism groups, Section 4 gives information about the codes of the constructed designs, and in Section 5 we discuss trigeodetic graphs obtained from the symmetric $(45, 12, 3)$ designs.

For the construction of designs we have used our own computer programs. For isomorphism testing, and to obtain and analyze the full automorphism groups of the designs we have used [14] and [30]. The codes have been analyzed using Magma [4].

2. Outline of the construction

An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, with point set $\mathcal{P}$, block set $\mathcal{B}$ and incidence $\mathcal{I}$ is a t-(v, k, λ) design, if $|\mathcal{P}| = v$, every block $B \in \mathcal{B}$ is incident with precisely k points, and every t distinct points are together incident with precisely λ blocks. A design is called symmetric if it has the same number of points and blocks. An automorphism of a design $\mathcal{D}$ is a permutation on $\mathcal{P}$ which sends blocks to blocks. The set of all automorphisms of $\mathcal{D}$ forms its full automorphism group denoted by $\mathrm{Aut}(\mathcal{D})$.

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be a symmetric (v, k, λ) design and $G \leq \mathrm{Aut}(\mathcal{D})$. The group action of G produces the same number of point and block orbits (see [21, Theorem 3.3]). We denote that number by t, the point orbits by $\mathcal{P}_1, \ldots, \mathcal{P}_t$, the block orbits by $\mathcal{B}_1, \ldots, \mathcal{B}_t$, and put $|\mathcal{P}_r| = \omega_r$ and $|\mathcal{B}_i| = \Omega_i$. An automorphism group G is said to be semi-standard if, after possibly renumbering orbits, we have $\omega_i = \Omega_i$, for $i = 1, \ldots, t$. We denote by γ_{ir} the number of points of $\mathcal{P}_r$ which are incident with a representative of the block orbit $\mathcal{B}_i$. For these numbers the following equalities hold (see [5, 9, 16]):

$$\sum_{r=1}^{t} \gamma_{ir} = k\,, \tag{1}$$

$$\sum_{r=1}^{t} \frac{\Omega_j}{\omega_r} \gamma_{ir}\gamma_{jr} = \lambda\Omega_j + \delta_{ij} \cdot (k - \lambda)\,. \tag{2}$$

Definition 2.1. *A $(t \times t)$-matrix (γ_{ir}) with entries satisfying conditions (1) and (2) is called an orbit matrix for the parameters (v, k, λ) and orbit lengths distributions $(\omega_1, \ldots, \omega_t)$, $(\Omega_1, \ldots, \Omega_t)$.*

The construction of designs admitting an action of a presumed automorphism group, using orbit matrices, consists of the following two basic steps (see [5, 9, 16]):

1. Construction of orbit matrices for the given automorphism group,
2. Construction of block designs for the orbit matrices obtained in this way. This step is often called an indexing of orbit matrices.

In order to construct the orbit matrices for an action of a presumed automorphism group we have to determine all possibilities for the orbit lengths distributions. The following facts, that one can use in that purpose, can be found in [21].

Theorem 2.2. *An automorphism ρ of a symmetric design fixes an equal number of points and blocks. Moreover, ρ has the same cyclic structure, whether considered as a permutation on points or on blocks.*

Theorem 2.3. *Suppose that a nonidentity automorphism ρ of a nontrivial symmetric (v, k, λ) design fixes f points. Then*

$$f \leq v - 2n \qquad \text{and} \qquad f \leq \frac{\lambda}{k - \sqrt{n}} v,$$

where $n = k - \lambda$ is the order of the design. Moreover, if equality holds in either inequality, ρ must be an involution and every non-fixed block contains exactly λ fixed points.

Theorem 2.4. *Suppose that $\mathcal{D}$ is a nontrivial symmetric (v, k, λ) design, with an involution ρ fixing f points and blocks. If $f \neq 0$, then*

$$f \geq \begin{cases} 1 + \frac{k}{\lambda}, & \text{if } k \text{ and } \lambda \text{ are both even,} \\ 1 + \frac{k-1}{\lambda}, & \text{otherwise.} \end{cases}$$

Suppose that $\mathcal{D}$ is a symmetric (v, k, λ) design with an automorphism ρ of prime order p fixing f points. Then $f \equiv v \pmod p$, and $\langle \rho \rangle$ acts semi-standardly on $\mathcal{D}$. In that case, since the action of $G = \langle \rho \rangle$ is semi-standard, it is sufficient to determine point orbit lengths distribution $(\omega_1, \ldots, \omega_t)$. After determining the orbit lengths distributions we proceed with the construction of orbit matrices and corresponding designs, as described in [9].

3. Classification of symmetric (45,12,3) designs with nontrivial automorphisms

In this section we give the classification of all symmetric (45,12,3) designs that admit nontrivial automorphisms. It is known that if ρ is a nonidentity automorphism of a symmetric (45,12,3) design, then $|\rho| \in \{2, 3, 5, 11\}$ (see [25]). It has been shown in [5, 19, 25] that there are exactly 13 symmetric (45,12,3) designs with an automorphism of order 5, and exactly one symmetric (45,12,3) design with an automorphism of order 11. To complete the classification of symmetric (45,12,3) designs with nontrivial automorphisms, we have to classify all symmetric (45,12,3) designs that admit an automorphism of order 2 or 3.

3.1. Symmetric (45,12,3) designs admitting Z_2 as an automorphism group

Let ρ be an involutory automorphism of a symmetric $(45, 12, 3)$ design fixing f points. Then $5 \leq f \leq 15$ and $f \equiv 1 \pmod 2$, hence $f \in \{5, 7, 9, 11, 13, 15\}$. Up to isomorphism there are 682 orbit structures, that produce 2987 mutually non-isomorphic designs. Information about the number of the orbit structures and the designs are given in Table 1.

Table 1. Symmetric (45,12,3) designs having Z_2 as an automorphism group

number of fixed points	5	7	9	11	13	15
number of orbit structures	233	397	32	4	11	5
number of orbit structures that produce designs	45	271	30	0	7	5
number of designs	603	1898	524	0	225	28

3.2. Symmetric (45,12,3) designs admitting Z_3 as an automorphism group

It was determined in [8] that there are exactly 591 orbit matrices for the group Z_3 acting on symmetric (45,12,3) designs. From these orbit matrices we have obtained up to isomorphism exactly 2108 symmetric (45,12,3) designs that admit an automorphism of order three. Information about the number of the orbit matrices and the constructed designs are presented in Table 2.

Table 2. Symmetric (45,12,3) designs having Z_3 as an automorphism group

number of fixed points	0	3	6	9
number of orbit structures	293	245	49	4
number of orbit structures that produce designs	19	25	24	4
number of designs	244	482	125	1775

3.3. All symmetric (45,12,3) designs admitting a nontrivial automorphism group

Comparing the designs described in subsections 3.1 and 3.2 we conclude that up to isomorphism there are exactly 4280 symmetric (45,12,3) designs that admit an automorphism of order 2 or 3. It is known from [5, 19, 25] that there are exactly 13 symmetric (45,12,3) designs with an automorphism of order 5, and only four of them have the full automorphism group whose order is not divisible by 2 or 3. Further, there is exactly one symmetric (45,12,3) design with an automorphism of order 11, and the full automorphism group of that designs is Z_{11}. That shows that there exist exactly 4285 symmetric (45,12,3) designs with a nontrivial automorphism group. Among them there are 1161 self-dual designs and 1562 pairs of mutually dual designs. Information about these 4285 designs and their full automorphism groups are given in Table 3. Some od the automorphism groups have the same description of the structure, but they are not isomorphic. In that case, nonisomorphic groups with the same structure are listed in separate rows of Table 3 (e.g. two groups of order 324 having the structure $(E_{27} : Z_3) : E_4$). Since Mathon and Spence have constructed 1136 symmetric (45,12,3) designs with a trivial automorphism group (see [25]), we conclude that up to isomorphism there are at least 5421 symmetric (45,12,3) designs.

4. Ternary codes from symmetric (45,12,3) designs

The code $C_F(\mathcal{D})$ of a design $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ over the finite field F is the space spanned by the incidence vectors of the blocks over F. If $\mathcal{Q}$ is any subset of the point set $\mathcal{P}$, then we will denote the incidence vector of $\mathcal{Q}$ by $v^{\mathcal{Q}}$. Thus $C_F(\mathcal{D}) = \langle v^B \mid B \in \mathcal{B} \rangle$, and is a subspace of $F^{\mathcal{P}}$, the full vector space of functions from $\mathcal{P}$ to F. The following theorem, that can be found in [2], shows that the code $C_F(\mathcal{D})$ over a field F of characteristic p is not interesting if p does not divide the order of $\mathcal{D}$. In Theorem 4.2 $\mathrm{rank}_p(\mathcal{D})$ denotes the dimension of $C_F(\mathcal{D})$, and j denotes the all-one vector.

Table 3. Symmetric (45,12,3) designs with nontrivial automorphisms

$\lvert\mathrm{Aut}(D)\rvert$	Structure of $\mathrm{Aut}(D)$	no. designs	$\lvert\mathrm{Aut}(D)\rvert$	Structure of $\mathrm{Aut}(D)$	no. designs
51840	$PSp(4,3):Z_2$	1	45	$Z_{15}\times Z_3$	1
19440	$(E_{81}:SL(2,5)):Z_2$	1	36	$S_3\times S_3$	4
1296	$E_{27}:(S_4\times Z_2)$	1	36	$E_9:Z_4$	1
486	$E_{81}:Z_6$	1	36	$Z_2\times(E_9:Z_2)$	1
486	$E_{81}:S_3$	1	32	$Z_4:Q_8$	1
432	$((S_3\times S_3):Z_2)\times S_3$	2	30	$Z_5\times S_3$	1
360	$(Z_{15}\times Z_3):Z_8$	1	27	E_{27}	11
324	$(E_{27}:Z_3):E_4$	1	27	$Z_9:Z_3$	6
324	$(E_{27}:Z_3):E_4$	1	27	$E_9:Z_3$	4
216	$(Z_3\times S_3\times S_3):Z_2$	3	24	$Z_3\times Q_8$	1
216	$(E_9:Z_4)\times S_3$	2	20	$Z_5:Z_4$	1
216	$S_3\times S_3\times S_3$	2	20	$Z_5:Z_4$	1
192	$(E_4\times Q_8):S_3$	2	18	$Z_3\times S_3$	87
162	$E_{27}:Z_6$	8	18	$Z_6\times Z_3$	4
162	$E_{27}:S_3$	5	18	$E_9:Z_2$	4
162	$E_{27}:S_3$	1	16	QD_{16}	7
162	$S_3\times(E_9:Z_3)$	1	16	$Z_2\times D_8$	2
144	$(E_9:Z_8):Z_2$	2	16	$(Z_4\times Z_2):Z_2$	1
108	$S_3\times(E_9):Z_2)$	4	15	Z_{15}	2
108	$E_{27}:Z_4$	2	12	D_{12}	65
108	$E_{27}:E_4$	1	11	Z_{11}	1
108	$Z_3\times S_3\times S_3$	1	9	E_9	213
81	$E_{27}:Z_3$	4	8	D_8	12
81	$E_{27}:Z_3$	1	8	Q_8	7
64	$(E_4.(Z_4\times Z_2)):Z_2$	1	8	Z_8	4
64	$((Z_2\times Q_8):Z_2):Z_2$	1	8	E_8	2
54	$E_{27}:Z_2$	24	8	$Z_4\times Z_2$	2
54	$E_{27}:Z_2$	9	6	S_3	446
54	$(Z_9:Z_3):Z_2$	6	6	Z_6	104
54	$E_9\times S_3$	6	5	Z_5	4
54	$E_9:Z_6$	3	4	E_4	128
48	$(Z_3\times Q_8):Z_2$	3	4	Z_4	71
48	$Z_2\times S_4$	1	3	Z_3	1051
48	$(Z_4\times Z_4):Z_3$	1	2	Z_2	1931

Theorem 4.1. *Let $\mathcal{D}=(\mathcal{P},\mathcal{B},\mathcal{I})$ be a nontrivial 2-(v,k,λ) design of order n. Let p be a prime and let F be a field of characteristic p, where p does not divide n. Then*

$$\mathrm{rank}_p(\mathcal{D})\geq(v-1)$$

with equality if and only if p divides k; in the case of equality we have that $C_F(\mathcal{D})=\langle j\rangle^{\perp}$ and otherwise $C_F(\mathcal{D})=F^{\mathcal{P}}$.

Since the order of a symmetric (45,12,3) design is 9, we consider only the ternary codes of the constructed designs, i.e. codes over the field of order 3. The ternary codes of the 4285 symmetric

(45,12,3) designs with nontrivial automorphisms are divided in 1005 equivalence classes. In Table 4 we give information about code parameters and orders of automorphism groups of representatives of equivalence classes, where the definitions of automorphisms and equivalence of codes are the same as in Magma [4]. The following theorem states that all the codes obtained are self-orthogonal.

Theorem 4.2. *Let $\mathcal{D}$ be a symmetric (45,12,3) design and $C(\mathcal{D})$ be the ternary code of the design $\mathcal{D}$. Then the code $C(\mathcal{D})$ is self-orthogonal, and $j \in C(\mathcal{D})^{\perp}$.*

Proof. The code $C(\mathcal{D})$ is spanned by the rows of the row-point incidence matrix of $\mathcal{D}$. Since each row of $\mathcal{D}$ has 12 points, and any two blocks intersect in 3 points, the code $C(\mathcal{D})$ is self-orthogonal. It is obvious that $j \in C(\mathcal{D})^{\perp}$, because each row of the design $\mathcal{D}$ consist of 12 points. □

Table 4. Ternary codes of the symmetric (45,12,3) designs with nontrivial automorphisms

Parameters	(|Aut(C)|, no. of inequivalent codes)
$[45, 22, 9]$	(11,1)
$[45, 21, 6]$	(60466176,1), (5832,1), (2592,1), (1944,1), (216,1), (108,1), (24,3), (12,1), (9,8), (8,4), (6,3), (4,10), (3,4), (2,19)
$[45, 20, 12]$	(2,3)
$[45, 20, 9]$	(15,1), (9,8), (6,4), (3,7), (2,2)
$[45, 20, 6]$	(45349632,1), (31104,1), (23328,1), (11664,1), (10368,1), (5832,1), (3888,2), (1944,2), (1152,1),(972,1), (648,2), (324,3), (288,1), (216,1), (162,5), (108,2), (72,6), (54,4), (36,10), (27,3), (24,11), (18,1), (12,34), (9,5), (8,6), (6,7), (5,2), (4,59), (3,1), (2,139)
$[45, 19, 12]$	(64,1), (32,1), (4,1), (2,1)
$[45, 19, 9]$	(162,1), (81,4), (54,2), (27,2), (18,1), (9,6), (4,1), (3,4)
$[45, 19, 6]$	(226748160,1), (52488,1), (23328,1), (17496,1), (11664,1), (5832,3), (4608,1), (1944,1), (1296,1), (972,1), (648,1), (486,2), (324,1), (288,2), (216,2), (108,3), (72,7), (54,4), (36,20), (24,9), (20,1), (18,9), (16,3), (12,37), (9,2), (8,6), (6,32), (4,51), (3,16), (2,132)
$[45, 18, 12]$	(20,1)
$[45, 18, 9]$	(18,2), (9,1), (6,2), (2,2)
$[45, 18, 6]$	(209952,1), (52488,1), (23328,1), (8748,1), (7290,1), (5832,1), (1944,2), (1296,3), (432,1), (324,3), (216,2), (162,2), (108,6), (72,7), (54,4), (48,3), (36,14), (27,1), (24,1), (18,11), (16,1), (12,18), (8,5), (6,22), (4,16), (3,8), (2,34)
$[45, 17, 12]$	(192,2), (48,2)
$[45, 17, 9]$	(360,1), (81,1)
$[45, 17, 6]$	(69984,1), (3888,1), (2916,1), (1944,2), (486,1), (432,1), (324,6), (288,1), (216,1), (162,1), (144,1), (108,3), (54,1), (36,3), (18,1), (16,2), (12,4), (9,1), (8,1), (6,3), (4,1), (2,1)
$[45, 16, 9]$	(486,1), (324,1)
$[45, 16, 6]$	(972,1), (432,1), (216,1), (108,1)
$[45, 15, 12]$	(51840,1)
$[45, 15, 9]$	(19440,1)

In Table 5 we give information about the dual codes of the codes presented in Table 4. According to [15] and [26], the [45,28,8] code has the greatest minimum distance among the known ternary [45,28] codes. Further, the best known ternary [45,30] code has minimum distance 7, hence the [45,30,6] code has minimum distance one less than the best known code.

A linear code whose dual code supports the blocks of a t-design admits one of the simplest decoding algorithms, majority logic decoding (see [28]). If a codeword $x = (x_1, \ldots, x_n) \in C$ is sent over a communication channel, and a vector $y = (y_1, \ldots, y_n)$ is received, for each symbol y_i a set of values

$y_i^{(1)}, \ldots, y_i^{(r_i)}$ of r_i linear functions defined by the blocks of the design are computed, and y_i is decoded as the most frequent among the values $y_i^{(1)}, \ldots, y_i^{(r_i)}$. The following result have been obtained by Rudolph [28].

Theorem 4.3. *If C is a linear $[n,k]$ code such that $C^\perp$ contains a set S of vectors of weight w whose supports are the blocks of a 2-(n,w,λ) design, the code C can correct up to*

$$e = \left\lfloor \frac{r+\lambda-1}{2\lambda} \right\rfloor$$

errors by majority logic decoding, where $r = \lambda\frac{n-1}{w-1}$.

Consequently, the codes listed in Table 5 can correct up to two errors by majority logic decoding.

Table 5. Dual codes of ternary codes of the symmetric (45,12,3) designs with nontrivial automorphisms

Parameters	(|Aut(C)|, no. of inequivalent codes)
$[45,30,6]$	(51840,1), (19440,1)
$[45,29,6]$	(972,1), (486,1), (432,1), (324,1), (216,1), (108,1)
$[45,28,8]$	(48,1)
$[45,28,6]$	(69984,1), (3888,1), (2916,1), (1944,2), (486,1), (432,1), (360,1), (324,6), (288,1), (216,1), (192,2), (162,1), (144,1), (108,3), (81,1), (54,1), (48,1), (36,3), (18,1), (16,2), (12,4), (9,1), (8,1), (6,3), (4,1), (2,1)
$[45,27,6]$	(209952,1), (52488,1), (23328,1), (8748,1), (7290,1), (5832,1), (1944,1), (1296,3), (432,1), (324,3), (216,2), (162,2), (108,6), (72,7), (54,4), (48,3), (36,14), (27,1), (24,1), (20,1), (18,13), (16,1), (12,18), (9,1), (8,5), (6,24), (4,16), (3,8), (2,36)
$[45,26,8]$	(3,2)
$[45,26,6]$	(226748160,1), (52488,1), (23328,1), (17496,1), (11664,1), (5832,1), (4608,1), (1944,1), (1296,1), (972,1), (648,1),(486,2), (324,1), (288,2), (216,2), (162,1), (108,3), (81,4), (72,7), (64,1), (54,6), (36,20), (32,1), (27,2), (24,9), (20,1), (18,10), (16,3), (12,37), (9,8), (8,6), (6,32), (4,53), (3,18), (2,133)
$[45,25,8]$	(15,1), (2,2), (3,1)
$[45,25,6]$	(45349632,1), (31104,1), (23328,1), (11664,1), (10368,1), (5832,1), (3888,2), (1944,2), (1152,1), (972,1), (648,2), (324,3), (288,1), (216,1), (162,5), (108,2), (72,6), (54,4), (36,10), (27,3), (24,11), (18,1), (12,34), (9,13), (8,6), (6,11), (5,2), (4,59), (3,7), (2,142)
$[45,24,6]$	(60466176,1), (5832,1), (2592,1), (1944,1), (216,1), (108,1), (24,3), (12,1), (9,8), (8,4), (6,3), (4,10), (3,4), (2,19)
$[45,23,9]$	(11,1)

5. On k-geodetic graphs from symmetric $(45,12,3)$ designs

In this section we present results concerning 3-geodetic (trigeodetic) graphs. We prove that k-geodetic graphs constructed from mutually non-isomorphic designs are mutually non-isomorphic. For applications of k-geodetic graphs in the topological design of computer networks the reader may consult [13]. For further reading on k-geodetic graphs we refer the reader to [27] and [29].

For every 2-(v,k,λ) design $\mathcal{D}$ with replication number r and b blocks it is possible to construct $k-$connected biregular block $K_v^*(r,k,\lambda)$ (a block is a graph with vertex connectivity > 1) of diameter 4

or 5 with vertex degrees r and k, in which there are at most μ paths of minimum length between any pair of vertices, where

$$\mu = \max\left\{\max\left\{|B_i \cap B_j| : i,j = 1,2,\ldots,b,\ i \neq j\right\},\ \lambda\right\},$$

$B_1, B_2, \ldots, B_b$ being blocks of the design (see [29]).

$K_v^*(r,k,\lambda)$ has $v(r+1)$ vertices and $\frac{vr(k+1)}{2}$ edges. If $\mathcal{D}$ is a symmetric design then $K_v^*(r,k,\lambda)$ is $k-$regular graph in which there are at most λ paths of minimum length between each pair of vertices. Graphs in which every pair of nonadjacent vertices has a unique path of minimum length between them are called geodetic graph, bigeodetic graphs are graphs in which each pair of nonadjacent vertices has at most two paths of minimum length between them and graphs in which each pair of nonadjacent vertices has at most k paths of minimum length between them are called k-geodetic graphs (see [13], [29]).

We follow the construction of $K_v^*(r,k,\lambda)$ from a 2-(v,k,λ) design given in [29]. If $B_i = \{P_{i_1}, \ldots P_{i_k}\}$, $1 \leq i \leq b$, is a block of a design $\mathcal{D}$, then $\{x_{i_1 i}, \ldots, x_{i_k i}\}$ are vertices of the complete graph $(K_k)_i$. Graphs $(K_k)_i, 1 \leq i \leq b$, together with v vertices x_{i0}, $1 \leq i \leq v$, x_{i0} and x_{st} being adjacent if $i = s$, form the graph $K_v^*(r,k,\lambda)$ for the design $\mathcal{D}$.

The adjacency matrix of a graph $K_v^*(r,k,\lambda)$ is given as follows

$$\mathbf{A} = \left[\begin{array}{cccc|c} (\mathbf{J}_k - \mathbf{I}_k) & \mathbf{0}_k & \ldots & \mathbf{0}_k & \mathbf{M}_1 \\ \mathbf{0}_k & (\mathbf{J}_k - \mathbf{I}_k) & \cdots & \mathbf{0}_k & \mathbf{M}_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0}_k & \mathbf{0}_k & \cdots & (\mathbf{J}_k - \mathbf{I}_k) & \mathbf{M}_b \\ \hline \mathbf{M}_1^T & \mathbf{M}_2^T & \ldots & \mathbf{M}_b^T & \mathbf{0}_v \end{array}\right],$$

where $\mathbf{M}_i = [m_{r,s}]$, $1 \leq i \leq b$, are $k \times v$ matrices with $m_{1,i_1} = m_{2,i_2} = \ldots = m_{k,i_k} = 1$ for $B_i = \{P_{i_1}, \ldots P_{i_k}\}$, $1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq v$ and $m_{r,s} = 0$ otherwise, $\mathbf{0}_k$ is the $k \times k$ zero-matrix, $\mathbf{J}_k$ is the $k \times k$ all-one matrix, and $\mathbf{I}_k$ is the $k \times k$ identity matrix. Rows of $\mathbf{M}_i$, $1 \leq i \leq b$, are labeled with $x_{i_1 i}, \ldots, x_{i_k i}$, and columns of $\mathbf{M}_i$, $1 \leq i \leq b$, are labeled with $x_{10}, \ldots, x_{v0}$. The number of columns labeled with x_{s0}, $1 \leq s \leq v$ in which matrices $\mathbf{M}_i$ and $\mathbf{M}_j$ both have an entry 1 is equal to $|B_i \cap B_j|$, since in the column x_{s0} there is an entry 1 in both matrices if and only if $P_s \in B_i \cap B_j$. Moreover, the matrix $\mathbf{M}_i$ is determined by the i^{th} row of the incidence matrix $\mathbf{IM} = [d_{i,s}]$ of the design $\mathcal{D}$. Vice versa, the i^{th} row of the incidence matrix $\mathbf{IM} = [d_{i,s}]$ is determined by the matrix $\mathbf{M}_i$, putting $d_{i,s} = 1$ if there exists a row of $\mathbf{M}_i$ having 1 on the position x_{s0}.

Theorem 5.1. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be 2-(v,k,λ) designs. Then the corresponding graphs $K_v^*(r,k,\lambda)^1$ and $K_v^*(r,k,\lambda)^2$ are isomorphic if and only if the designs $\mathcal{D}_1$ and $\mathcal{D}_2$ are isomorphic.*

Proof. Let $\mathcal{D}_1 = (\mathcal{P}_1, \mathcal{B}_1, I_1)$ and $\mathcal{D}_2 = (\mathcal{P}_2, \mathcal{B}_2, I_2)$ be 2-(v,k,λ) designs and α be an isomorphism from $\mathcal{D}_1$ onto $\mathcal{D}_2$. Then there exists unique isomorphism β between the corresponding graphs $K_v^*(r,k,\lambda)^1$ and $K_v^*(r,k,\lambda)^2$ that satisfy

$$\left(P_s^1\alpha = P_t^2 \wedge B_i^1\alpha = B_j^2\right) \Rightarrow \left((K_k)_i^1\,\beta = (K_k)_j^2 \wedge x_{s0}^1\beta = x_{t0}^2\right),$$

where $1 \leq s \leq v$, $1 \leq i \leq b$.

Conversely, each isomorphism from the graph $K_v^*(r,k,\lambda)^1$ onto $K_v^*(r,k,\lambda)^2$ induces unique isomorphism from the design $\mathcal{D}_1$ onto $\mathcal{D}_2$. To prove this statement it is crusial to show that an isomorphism from $K_v^*(r,k,\lambda)^1$ onto $K_v^*(r,k,\lambda)^2$ maps vertices $\{x_{10}^1, \ldots, x_{v0}^1\}$ of $K_v^*(r,k,\lambda)^1$ onto vertices $\{x_{10}^2, \ldots, x_{v0}^2\}$ of $K_v^*(r,k,\lambda)^2$.

If the designs $\mathcal{D}_1$ and $\mathcal{D}_2$ are not symmetric, then $r \neq k$ and since the vertices $x_{10}^1, \ldots, x_{v0}^1$ and $x_{10}^2, \ldots, x_{v0}^2$ have degree r and the other vertices of $K_v^*(r,k,\lambda)^1$ and $K_v^*(r,k,\lambda)^2$ have degree k, it is clear that an isomorphism from $K_v^*(r,k,\lambda)^1$ onto $K_v^*(r,k,\lambda)^2$ maps the set $\{x_{10}^1, \ldots, x_{v0}^1\}$ onto $\{x_{10}^2, \ldots, x_{v0}^2\}$.

If $\mathcal{D}_1$ and $\mathcal{D}_2$ are symmetric designs then $r = k$. A vertex x^1_{i0} and a vertex adjacent to x^1_{i0} have no common neighbour, while a vertex that do not belong to $\{x^1_{10}, \ldots, x^1_{v0}\}$ has $k-2$ commmon neighbours with any of its neighbour. Similarly, a vertex x^2_{i0} and a vertex adjacent to him have no common neighbour, while a vertex that do not belong to $\{x^2_{10}, \ldots, x^2_{v0}\}$ has $k-2$ commmon neighbours with any of its neighbour. Hence, we conclude that $\{x^1_{10}, \ldots, x^1_{v0}\}$ is mapped onto $\{x^2_{10}, \ldots, x^2_{v0}\}$.

So, an isomorphism from $K^*_v(r,k,\lambda)^1$ onto $K^*_v(r,k,\lambda)^2$ maps $(K_k)^1_i$ onto $(K_k)^2_j$, and $\mathbf{M}^1_i$ onto $\mathbf{M}^2_j$, and it induces unique isomorphism from the design $\mathcal{D}_1$ onto $\mathcal{D}_2$. □

Graphs $K^*_{45}(12,12,3)$ constructed from symmetric (45,12,3) designs are 12-connected and 12-regular graphs of diameter 4 with 585 vertices and 3510 edges. For each pair of nonadjacent vertices there are at most three paths of minimum length between them. From all known triplanes of order nine one can obtain 5421 non-isomorphic graphs $K^*_{45}(12,12,3)$, since non-isomorphic designs produce non-isomorphic trigeodetic graphs. The following theorem, which is proved in [11], shows that Table 3 gives information on automorphism groups of all trigeodetic graphs constructed from the symmetric (45,12,3) designs with nontrivial automorphisms, and that there are at least 1136 trigeodetic graphs $K^*_{45}(12,12,3)$ having trivial group as the full automorphism group.

Theorem 5.2. *Let $\mathcal{D}$ be a 2-(v,k,λ) design. Then the full automorphism group of $\mathcal{D}$ is isomorphic to the full automorphism group of the corresponding k-geodetic graph $K^*_v(r,k,\lambda)$.*

All symmetric (45,12,3) designs admitting nontrivial automorphisms can be found at

http://www.math.uniri.hr/∼sanjar/structures/.

References

[1] R. W. Ahrens, G. Szekeres, On a combinatorial generalization of the 27 lines associated with a cubic surface, J. Austral. Math. Soc. 10 (1969) 485–492.

[2] E. F. Assmus Jr., J. D. Key, Designs and Their Codes, Cambridge University Press, Cambridge, 1992.

[3] T. Beth, D. Jungnickel, H. Lenz, Design Theory, 2nd Edition, Cambridge University Press, Cambridge, 1999.

[4] W. Bosma, J. Cannon, C. Playoust, The Magma algebra system I: The user language, J. Symbolic Comput. 24(3-4) (1997) 235–265.

[5] V. Ćepulić, On symmetric block designs (45,12,3) with automorphisms of order 5, Ars Combin. 37 (1994) 33–48.

[6] C. J. Colbourn, J. F. Dinitz (Eds.), The CRC Handbook of Combinatorial Designs, 2nd Edition, CRC Press, Boca Raton, 2007.

[7] K. Coolsaet, J. Degraer, E. Spence, The strongly regular $(45,12,3,3)$ graphs, Electron. J. Combin. 13(1) (2006) Research Paper 32, 1–9.

[8] D. Crnković, B. G. Rodrigues, S. Rukavina, L. Simčić, Ternary codes from the strongly regular (45,12,3,3) graphs and orbit matrices of 2-(45,12,3) designs, Discrete Math. 312(20) (2012) 3000–3010.

[9] D. Crnković, S. Rukavina, Construction of block designs admitting an abelian automorphism group, Metrika 62(2-3) (2005) 175–183.

[10] D. Crnković, S. Rukavina, On some symmetric (45, 12, 3) and (40, 13, 4) designs, J. Comput. Math. Optim. 1(1) (2005) 55–63.

[11] D. Crnković, S. Rukavina, L. Simčić, On triplanes of order twelve admitting an automorphism of order six and their binary and ternary codes, Util. Math., to appear.

[12] U. Dempwolff, Primitive rank 3 groups on symmetric designs, Des. Codes Cryptogr. 22(2) (2001) 191–207.
[13] C. E. Frasser, k-geodetic Graphs and Their Application to the Topological Design of Computer Networks, Argentinian Workshop in Theoretical Computer Science, 28 JAIIO-WAIT '99 (1999) 187–203.
[14] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.7.2; 2013. (http://www.gap-system.org)
[15] M. Grassl, Bounds on the minimum distance of linear codes and quantum codes, Online available at http://www.codetables.de, Accessed on 2014-03-11.
[16] Z. Janko, Coset enumeration in groups and constructions of symmetric designs, Combinatorics '90 (Gaeta, 1990), Ann. Discrete Math. 52 (1992) 275–277.
[17] P. Kaski, P. R. J. Östergård, Classification Algorithms for Codes and Designs, Springer, Berlin, 2006.
[18] P. Kaski and P. R. J. Östergård, There are exactly five biplanes with $k = 11$, J. Combin. Des. 16(2) (2008) 117–127.
[19] T. Kölmel, Einbettbarkeit symmetrischer (45,12,3) Blockplaene mit fixpunktfrei operierenden Automorphismen, Heidelberg, 1991.
[20] C. W. H. Lam, G. Kolesova, L. H. Thiel, A computer search for finite projective planes of order 9, Discrete Math. 92(1-3) (1991) 187–195.
[21] E. Lander, Symmetric Designs: An Algebraic Approach, Cambridge University Press, Cambridge, 1983.
[22] R. Mathon, A. Rosa, 2-(v, k, λ) Designs of Small Order, in: Handbook of Combinatorial Designs, 2nd ed., C.J. Colbourn, J.H. Dinitz (Editors), Chapman & Hall/CRC, Boca Raton (2007) 25–58.
[23] R. L. MacFarland, A family of difference sets in non-cyclic groups, J. Combin. Theory Ser. A 15 (1973) 1–10.
[24] V. Mandekić-Botteri, On symmetric block designs (45,12,3) with involutory automorphism fixing 15 points, Glas. Mat. Ser. III 36(56)(2) (2001) 193–222.
[25] R. Mathon, E. Spence, On 2-(45,12,3) designs, J. Combin. Des. 4(3) (1996) 155–175.
[26] MinT, Dept. of Mathematics, University of Salzburg, The online database for optimal parameters of (t,m,s)-nets, (t,s)-sequences, orthogonal arrays, linear codes, and OOAs, Online available at http://mint.sbg.ac.at/index.php, Accessed on 2014-03-11.
[27] R. M. Ramos, J. Sicilia, M. T. Ramos, A generalization of geodetic graphs: K-geodetic graphs, Investigacion Operativa 6 (1998) 85–101.
[28] L. Rudolph, A class of majority logic decodable codes, IEEE Trans. Inform. Theory 13(2) (1967) 305–307.
[29] N. Srinivasan, J. Opatrný, V. S. Alagar, Construction of geodetic and bigeodetic blocks of connectivity $k \geq 3$ and their relation to block designs, Ars Combin. 24 (1987) 101–114.
[30] L.H. Soicher, DESIGN - a GAP package, Version 1.6, 23/11/2011. (http://www.gap-system.org/Packages/design.html)