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The covering number of M_{24}

Research Article

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Abstract: A *finite cover* $\mathcal{C}$ of a group G is a finite collection of proper subgroups of G such that G is equal to the union of all of the members of $\mathcal{C}$. Such a cover is called *minimal* if it has the smallest cardinality among all finite covers of G. The *covering number* of G, denoted by $\sigma(G)$, is the number of subgroups in a minimal cover of G. In this paper the covering number of the Mathieu group M_{24} is shown to be 3336.

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1. Introduction

A finite collection $\mathcal{C}$ of proper subgroups of a group G is said to be a *finite cover* of G if $\bigcup_{H\in\mathcal{C}} H = G$. Of course if G is cyclic then G does not admit such a cover, but any group with a finite noncyclic homomorphic image has a finite cover. The *covering number* of such a group G is denoted by $\sigma(G)$, and is defined by $\sigma(G) = min\{|\mathcal{C}| : \mathcal{C} \text{ is a finite cover of } G\}$. Any cover satisfying $|\mathcal{C}| = \sigma(G)$ is called *minimal*.

In [3] J. H. E. Cohn proved that if G is a finite noncyclic supersolvable group then $\sigma(G) = p + 1$, where p is the least prime such that G has more than one subgroup of index p, and conjectured that if G is a finite noncyclic solvable group, then $\sigma(G) = p^a + 1$, where p^a is the order of the smallest chief factor of G with more than one complement in G. This conjecture was proven by Tomkinson in [11], who suggested investigating the covering numbers of simple groups. In [2], R. Bryce, V. Fedri, and L. Serena determined the covering numbers of some linear groups. The covering numbers of the Suzuki groups were investigated by M. S. Lucido in [9].

A. Maróti considers alternating and symmetric groups in [10], wherein it is shown that $\sigma(\mathbb{S}_n) = 2^{n-1}$ if n is odd and not equal to 9, that $\sigma(\mathbb{S}_n) \leq 2^{n-2}$ if n is even, and that if n is not equal to 7 or 9 then $\sigma(\mathbb{A}_n) \geq 2^{n-2}$ with equality if and only if $n \equiv 2 \ (mod\ 4)$. Further results on the covering numbers of small alternating and symmetric groups can be found in [3, 5, 7, 8].

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In [6], P. E. Holmes determined the covering numbers of the Mathieu groups M_{11}, M_{22}, and M_{23}, as well as the Lyons group and the O'Nan group, and gave upper and lower bounds for the covering numbers of the Janko group J_1 and the McLaughlin group. The covering number of M_{12} was determined by L. C. Kappe, D. Nikolova-Popova, and E. Swartz in [8].

The aim of this paper is to show that $\sigma(M_{24}) = 3336$.

2. Preliminaries

Throughout we use standard terminology and notation from group theory. We will write $N \cdot H$ and $N \backslash H$ to denote a split extension of N by H and a general extension of N by H respectively. If π is an element of a permutation group and the disjoint cycle decomposition of π has k_i cycles of length m_i, $1 \le i \le r$, with $m_1 > m_2 > ... > m_r$, we will write the cycle type of π as $m_1^{k_1} m_2^{k_2} ... m_r^{k_r}$.

Let G be a group and $x \in G$. If $\langle x \rangle$ is maximal among cyclic subgroups of G then we call x a *principal element* and $\langle x \rangle$ a *principal subgroup* of G. It is easy to see that a collection $\mathcal{C}$ of proper subgroups of G is a cover if and only if every principal subgroup is contained in a member of $\mathcal{C}$.

If G is a finite noncyclic group and $\mathcal{C}$ is a finite cover of G, then by replacing each subgroup $H \in \mathcal{C}$ with a maximal subgroup M of G such that $H \le M$, we can obtain a cover $\mathcal{C}'$ of G consisting of maximal subgroups with $|\mathcal{C}'| \le |\mathcal{C}|$. So, for the purpose of determining the covering number of such a group it suffices to consider covers consisting solely of maximal subgroups.

3. The Mathieu group M_{24}

In light of the discussion in section 2, we begin with the maximal subgroups and the principal elements of M_{24}. As seen in [4], there are 9 conjugacy classes of maximal subgroups of M_{24}, which we denote by $\mathcal{M}_i$, $1 \le i \le 9$ ordered such that $|\mathcal{M}_1| \le |\mathcal{M}_2| \le ... \le |\mathcal{M}_9|$. The sizes of these conjugacy classes of maximal subgroups are given by $(|\mathcal{M}_1|, ..., |\mathcal{M}_9|) = (24, 276, 759, 1288, 1771, 2024, 3795, 40320, 1457280)$. If $H_i \in \mathcal{M}_i$ for $i = 1, ..., 9$ then the isomorphism types of the H_i are as follows: $H_1 \cong M_{23}$, $H_2 \cong M_{22} \cdot \mathbb{Z}_2$, $H_3 \cong \mathbb{Z}_2^4 \cdot \mathbb{A}_8$, $H_4 \cong M_{12} \cdot \mathbb{Z}_2$, $H_5 \cong \mathbb{Z}_2^6 \cdot (\mathbb{Z}_3 \backslash \mathbb{S}_6)$, $H_6 \cong L_3(4) \cdot \mathbb{S}_3$, $H_7 \cong \mathbb{Z}_2^6 \cdot (L_3(2) \times \mathbb{S}_3)$, $H_8 \cong L_2(23)$, and $H_9 \cong L_2(7)$. Let $X = \{j \in \mathbb{Z} \mid 1 \le j \le 24\}$, and for a positive integer k let $\binom{X}{k}$ denote the set of all subsets of X with cardinality k. We note that H_1, H_2, and H_6 are stabilizers in the actions of M_{24} on X, $\binom{X}{2}$, and $\binom{X}{3}$ respectively.

The principal elements of M_{24} (represented on 24 points) have cycle types $8^2 4^1 2^1 1^2$, $10^2 2^2$, $11^2 1^2$, $12^1 6^1 4^1 2^1$, 12^2, $14^1 7^1 2^1 1^1$, $15^1 5^1 3^1 1^1$, $21^1 3^1$, and $23^1 1^1$. We will denote the sets of principal elements with these cycle types by $\mathcal{T}_1, \ldots, \mathcal{T}_9$ respectively. We remark that $\mathcal{T}_6$, $\mathcal{T}_7$, $\mathcal{T}_8$, and $\mathcal{T}_9$ are each the union of two conjugacy classes of principal elements with the same cycle type, while the remaining $\mathcal{T}_i$ consist of a single conjugacy class of elements. The cardinalities of these sets are given by $(|\mathcal{T}_1|, \ldots, |\mathcal{T}_9|) = (15301440, 12241152, 22256640, 20401920, 20401920, 34974720, 32643072, 23316480, 21288960)$.

We describe the incidence between the sets $\mathcal{T}_1, \ldots, \mathcal{T}_9$ and the classes $\mathcal{M}_1, \ldots, \mathcal{M}_9$ of maximal subgroups with a matrix $A = (a_{i,j})$ where the entry $a_{i,j}$ in row $\mathcal{T}_i$ and column $\mathcal{M}_j$ is the number of elements from $\mathcal{T}_i$ contained in each maximal subgroup from class $\mathcal{M}_j$. The entries of this matrix were computed using the Magma algebra system [1], and are given in Table 1.

Observe that the elements from $\mathcal{T}_1$, $\mathcal{T}_3$, $\mathcal{T}_6$, $\mathcal{T}_7$, and $\mathcal{T}_9$ each fix a point of X and therefore are contained within the subgroups from class $\mathcal{M}_1$. Each element from $\mathcal{T}_8$ has a single cycle of length 3 and is therefore contained within a unique member of class $\mathcal{M}_6$. From table 1 we can see that the subgroups from class $\mathcal{M}_4$ contain elements from each of $\mathcal{T}_2$, $\mathcal{T}_4$, and $\mathcal{T}_5$, and since each of these sets of principal elements consists of a single conjugacy class, every element from $\mathcal{T}_2 \cup \mathcal{T}_4 \cup \mathcal{T}_5$ is contained within some member of $\mathcal{M}_4$. Consequently, $\mathcal{M}_1 \cup \mathcal{M}_4 \cup \mathcal{M}_6$ is a cover of M_{24} by $24 + 1288 + 2024 = 3336$ maximal subgroups, and $\sigma(M_{24}) \le 3336$.

Table 1. The incidence matrix A

$\mathcal{T}_i \backslash \mathcal{M}_j$	$\mathcal{M}_1$	$\mathcal{M}_2$	$\mathcal{M}_3$	$\mathcal{M}_4$	$\mathcal{M}_5$	$\mathcal{M}_6$	$\mathcal{M}_7$	$\mathcal{M}_8$	$\mathcal{M}_9$
$\mathcal{T}_1$	1275120	110880	20160	23760	8640	15120	4032	0	0
$\mathcal{T}_2$	0	88704	0	28512	6912	0	0	0	0
$\mathcal{T}_3$	1854720	80640	0	17280	0	0	0	2760	0
$\mathcal{T}_4$	0	73920	26880	31680	23040	0	5376	0	0
$\mathcal{T}_5$	0	0	0	15840	11520	0	5376	1012	0
$\mathcal{T}_6$	1457280	126720	46080	0	0	17280	9216	0	0
$\mathcal{T}_7$	1360128	0	43008	0	18432	16128	0	0	0
$\mathcal{T}_8$	0	0	0	0	0	11520	6144	0	0
$\mathcal{T}_9$	887040	0	0	0	0	0	0	528	0

Now suppose that $\mathcal{C}$ is a cover of M_{24} which consists of maximal subgroups. For $1 \leq i \leq 9$, let $x_i = |\mathcal{C} \cap \mathcal{M}_i|$. Since the subgroups from class $\mathcal{M}_9$ contain no principal elements, we may assume without loss of generality that $x_9 = 0$. Then since $\mathcal{C}$ is a cover of M_{24} we must have

$$\sum_{j=1}^{8} a_{i,j} x_j \geq |\mathcal{T}_i|, \quad 1 \leq i \leq 9. \tag{1}$$

The reader can verify (by integer linear programming, for example) that if $(x_1, ..., x_8)$ is a tuple of nonnegative integers with $x_j \leq |\mathcal{M}_j|$ for $1 \leq j \leq 8$ which satisfies the system of inequalities given by (1), then $\sum_{j=1}^{8} x_j \geq 3336$. Thus for any such cover $\mathcal{C}$ we have $|\mathcal{C}| \geq 3336$, and so we conclude that $\sigma(M_{24}) = 3336$.

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