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Regular handicap tournaments of high degree

Research Article

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Abstract: A *handicap distance antimagic labeling* of a graph $G = (V, E)$ with n vertices is a bijection $f : V \to \{1, 2, \ldots, n\}$ with the property that $f(x_i) = i$ and the sequence of the weights $w(x_1), w(x_2), \ldots, w(x_n)$ (where $w(x_i) = \sum_{x_j \in N(x_i)} f(x_j)$) forms an increasing arithmetic progression with difference one. A graph G is a *handicap distance antimagic graph* if it allows a handicap distance antimagic labeling. We construct $(n-7)$-regular handicap distance antimagic graphs for every order $n \equiv 2 \pmod 4$ with a few small exceptions. This result complements results by Kovář, Kovářová, and Krajc [P. Kovář, T. Kovářová, B. Krajc, On handicap labeling of regular graphs, manuscript, personal communication, 2016] who found such graphs with regularities smaller than $n - 7$.

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1. Motivation

The study of handicap distance antimagic graphs has been motivated by incomplete round-robin type tournaments with various properties.

A *complete round robin tournament of n teams* is a tournament in which every team plays the remaining $n-1$ teams. When the teams are ranked $1, 2, \ldots, n$ according to their strength, it is apparent that the sum of rankings of all opponents of the i-th ranked team, denoted $w(i)$, is $w(i) = n(n+1)/2 - i$, and the sequence $w(1), w(2), \ldots, w(n)$ is a decreasing arithmetic progression with difference one. Because complete round robin tournaments are generally considered to be fair, a tournament of n teams in which every team plays precisely r opponents, where $r < n - 1$ and the sequence $w(1), w(2), \ldots, w(n)$ is a decreasing arithmetic progression with difference one is called a *fair incomplete round robin tournament*. A disadvantage of such a tournament is that the best team plays the weakest opponents, while the weakest team plays the strongest opponents. This disadvantage is eliminated in *equalized incomplete round robin tournaments* in which the sum of rankings of all opponents of every team is the same. Some results on fair incomplete round robin tournaments can be found in [6] and [3].

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However, if we want to give the weaker teams a better chance of winning, the weakest team should play the weakest opponents, while the strongest one should play the strongest opponents. That is, the sequence $w(1), w(2), \ldots, w(n)$ should be an increasing arithmetic progression. A tournament in which this condition is satisfied, and every team plays $r < n-1$ games, is called a *handicap incomplete round robin tournament.*

The existence of such tournaments with $n \equiv 0 \pmod 4$ is studied by the authors in [D. Froncek and A. Shepanik, Handicap incomplete tournaments of order $n \equiv 0 \pmod 4$, manuscript, personal communication, 2016], Kovář [P. Kovář, On regular handicap graphs, personal communication, June 16, 2016] and Kovářová [T. Kovářová, On regular handicap graphs, personal communication, June 16, 2016]. Kovář, Kovářová, and Krajc [P. Kovář, T. Kovářová, B. Krajc, On handicap labeling of regular graphs, manuscript, personal communication, 2016] found such tournaments for $n \equiv 2 \pmod 4$ and $r \le n-11$ and proved that they can exist only when r is odd and at most $n-7$. We provide a construction of handicap incomplete round robin tournaments for $n \equiv 2 \pmod 4$ and the missing regularity $r = n-7$ with a few small exceptions.

2. Basic notions

By a graph $G = (V, E)$ we mean a finite undirected graph without loops or multiple edges. For graph theoretic terminology we refer to Chartrand and Lesniak [2].

Motivated by properties of magic squares, Vilfred [12] introduced the concept of sigma labelings. The same concept was introduced by Miller et al. [10] under the name 1-vertex magic vertex labeling. Sugeng et al. [11] introduced the term distance magic labeling, which currently seems to be most commonly used. A survey on distance magic graphs was published recently [1]. Many newer results can by found in an extensive survey with much wider focus by Gallian [7].

Definition 2.1. *A* distance magic labeling *of a graph G of order n is a bijection $f : V \to \{1, 2, \ldots, n\}$ with the property that there is a positive integer μ such that $\sum_{y \in N(x)} f(y) = \mu$ for every $x \in V$. The constant μ is called the magic constant of the labeling f. The sum $\sum_{y \in N(x)} f(y)$ is called the weight of the vertex x and is denoted by $w(x)$.*

When we think of the vertices as of teams and identify their labels with their rankings, we can see that a distance magic graph is providing a structure of a fair incomplete tournament described above.

In [4] the first author introduced two closely related concepts, namely the distance antimagic and handicap distance antimagic labelings (which was called an *ordered distance antimagic labeling* in that paper) and showed their relationship to certain types of incomplete round robin tournaments. The term "handicap distance antimagic labeling" was originally coined by Kovářová [T. Kovářová, On regular handicap graphs, personal communication, June 16, 2016].

Definition 2.2. *A* distance d-antimagic labeling *of a graph $G = (V, E)$ with n vertices is a bijection $f : V \to \{1, 2, \ldots, n\}$ with the property that there exists an ordering of the vertices of G such that the sequence of the weights $w(x_1), w(x_2), \ldots, w(x_n)$ forms an arithmetic progression with difference d. When $d = 1$, then f is called just* distance antimagic labeling. *A graph G is a* distance d-antimagic graph *if it allows a distance d-antimagic labeling, and a* distance antimagic graph *when $d = 1$.*

It should be obvious that a graph G is distance magic if and only if its complement $\overline{G}$ is distance antimagic.

In distance antimagic graphs the weight of a vertex is not tied to its own label. All that we require is that the sequence $w(x_1), w(x_2), \ldots, w(x_n)$ forms an arithmetic progression. We now impose an additional condition on the labeling and require that a vertex with a lower label has a lower weight than a vertex with a higher label.

Definition 2.3. *A* handicap distance d-antimagic labeling *of a graph* $G = (V, E)$ *with* n *vertices is a bijection* $f : V \to \{1, 2, \dots, n\}$ *with the property that* $f(x_i) = i$ *and the sequence of the weights* $w(x_1), w(x_2), \dots, w(x_n)$ *forms an increasing arithmetic progression with difference* d. *When* $d = 1$, *the labeling is called just a* handicap distance antimagic labeling *(or a* handicap labeling *for short).*

A graph G *is a* handicap distance d-antimagic graph *if it allows a handicap distance* d*-antimagic labeling, and a* handicap distance antimagic graph *or a* handicap graph *when* $d = 1$.

Again, if we identify each team in a tournament with its ranking, then an r-regular handicap distance d-antimagic graph is nothing else than a model of a handicap incomplete round robin tournament, since the sum of rankings of opponents of team i is its weight $w(i)$ and the sequence of weights is an increasing arithmetic progression.

Our constructions will be based on the properties of magic rectangles, which are a generalization of the magic squares mentioned above.

Definition 2.4. *A* magic rectangle $MR(a, b)$ *is an* $a \times b$ *array whose entries are* $1, 2, \dots, ab$, *each appearing once, with all row sums equal to a constant* ρ *and all column sums equal to a constant* σ.

It is easy to observe that a and b must be either both even or both odd. The following existence result was proved by Harmuth [8, 9] more than 130 years ago.

Theorem 2.5. [8, 9] *A magic rectangle* $MR(a, b)$ *exists if and only if* $a, b > 1$, $ab > 4$, *and* $a \equiv b \pmod 2$.

3. Known results

Kovář, Kovářová, and Krajc [P. Kovář, T. Kovářová, B. Krajc, On handicap labeling of regular graphs, manuscript, personal communication, 2016], Kovář [P. Kovář, On regular handicap graphs, personal communication, June 16, 2016] and Kovářová [T. Kovářová, On regular handicap graphs, personal communication, June 16, 2016] proved the following results.

Theorem 3.1. *Let* G *be an* r*-regular handicap graph on* n *vertices, where* $n \equiv 2 \pmod 4$. *Then* $r \equiv 3 \pmod 4$ *and* $r \leq n - 7$.

Theorem 3.2. *For* $n \equiv 2 \pmod 4$, *there exists an* r*-regular handicap graph if* $3 \leq r \leq n - 11$ *and* $r \equiv 3 \pmod 4$ *except when* $r = 3$ *and* $n \leq 26$.

Their result leaves open the case of $n \equiv 2 \pmod 4$ and $r = n-7$ for $n \geq 14$. In the following section, we prove the existence of such graphs with the exception of $n = 14, 18, 22, 26, 34, 38$, which remain in doubt.

In our constructions, we will also use the following result by Kovář [P. Kovář, On regular handicap graphs, personal communication, June 16, 2016] and Kovářová [T. Kovářová, On regular handicap graphs, personal communication, June 16, 2016].

Theorem 3.3. *For* $n \equiv 0 \pmod 4$ *there exists an* $(n - 7)$*-regular handicap graph whenever* $n \geq 16$.

In [5] the first author made made an observation, which was a special case of the following.

Observation 3.4. *Let* G *be an* r*-regular distance* 2*-antimagic graph with vertices* $x_1, x_2, \dots, x_n$, *labeling* f *and weight function* w *such that* $f(x_i) = i$ *and* $w(x_i) = k - 2i$ *for some constant* k. *Then* $\overline{G}$, *the complement of* G, *is an* $(n - r - 1)$*-regular handicap graph with labeling* f *and weight function* $\overline{w}$ *such that* $\overline{w}(x_i) = n(n + 1)/2 - k + i$. *The converse is obviously also true.*

Proof. Label the vertices of the complete graph K_n so that vertex x_i is labelled i. The sum of labels of all neighbors of x_i is then indeed equal to $n(n + 1)/2 - i$. Every neighbor of x_i contributes its label to either $w(x_i)$ or $\overline{w}(x_i)$. Therefore, we have

$$w(x_i) + \overline{w}(x_i) = n(n+1)/2 - i$$

and

$$\overline{w}(x_i) = n(n+1)/2 - i - w(x_i).$$

Because $w(x_i) = k - 2i$, it follows that

$$\overline{w}(x_i) = n(n+1)/2 - i - (k - 2i) = n(n+1)/2 - k + i.$$

This completes the proof. □

We will use the observation in our constructions and instead of constructing directly $(n-7)$-regular handicap graphs, we will construct 6-regular distance 2-antimagic graphs satisfying assumptions of Observation 3.4. We will call such graphs *genuine distance* 2*-antimagic graphs* and the labeling will be called a *genuine distance* 2*-antimagic labeling*.

The following observation was proved in a more general form in [4]. To avoid introduction of a new notion that would be only used in its special form, we state the observation as follows.

Observation 3.5. [4] *The graph $G = K_a \square K_b$ admits a genuine distance 2-antimagic labeling f such that $f(x) = p$ implies $w(x) = (a+b)(ab+1)/2 - 2p$ for every $x \in V(G)$ whenever there exists a magic rectangle $MR(a,b)$.*

4. New results

Lemma 4.1. *There exists a 6-regular genuine distance 2-antimagic graph on 12 vertices.*

Proof. Let $G = K_2 \square K_6$. By Theorem 2.5 there exist a magic rectangle $MR(2,6)$. The assertion follows directly from Observation 3.5. □

Lemma 4.2. *There exists a 6-regular genuine distance 2-antimagic graph H on 30 vertices.*

Proof. We denote the vertices of H by y_{st} and z_{st} for $1 \le s \le 3$ and $1 \le t \le 5$. The edge set will consist of edges $y_{st}y_{pt}$, $z_{st}z_{pt}$, and $y_{st}z_{sq}$ for every for every $1 \le s \le p \le 3$ and $1 \le t \le 5$ with $t \ne q$.

Let $MR(3,5)$ be a 3×5 magic rectangle with entries m_{st} for $1 \le r \le 3$ and $1 \le s \le 5$. We label the vertices y_{st} by entries m_{st} in the natural way, that is, $f(y_{st}) = m_{st}$ while the vertices z_{st} obtain the labels raised by 15, that is, $f(z_{st}) = m_{st} + 15$. Notice that the vertices y_{st} have the highest weights.

We now rename the vertices so that vertex y_{st} or z_{st} becomes x_i when $w(y_{st}) = 124 - 2i$ or $w(z_{st}) = 124 - 2i$, respectively. One can check that this labeling has the required property and H is a 6-regular genuine distance 2-antimagic graph. □

Now we are ready to present our construction.

Lemma 4.3. *There exists a 6-regular genuine distance 2-antimagic graph G on n vertices for every $n \equiv 2 \pmod 4$ and $n \ge 42$.*

Proof. We use two building blocks, graph H on 30 vertices from the previous Lemma, and a graph J on $m \equiv 0 \pmod 4$ vertices whose existence is guaranteed by Lemma 4.1 and Theorem 3.3. Our graph G will then have $n = m + 30$ vertices.

Let H be the graph on 30 vertices constructed in Lemma 4.2 with vertices $x_1, x_2, \ldots, x_{30}$, genuine distance 2-antimagic labeling f_H and vertex weights w_H satisfying $f_H(x_i) = i$ and $w_H(x_i) = 124 - 2i$.

It follows from Theorem 3.3 and Observation 3.4 that for any $m \equiv 0 \pmod 4$ there exists a genuine distance 2-antimagic labeling graph J with vertices $u_1, u_2, \dots, u_m$, labeling f_J and vertex weights w_J satisfying $f_J(u_j) = j$ and $w_J(u_j) = 4m - 2j$.

We use H and J as components of G and rename the vertices so that x_i becomes v_i for $i = 1, 2, \dots, 15$, u_j becomes v_{j+15} for $j = 1, 2, \dots, m$ and x_i becomes v_{i+m} for $i = 16, 17, \dots, 30$. Then we label all vertices using labeling function f_G as $f_G(v_i) = i$ to obtain the desired genuine distance 2-antimagic labeling.

We can check that for $i = 1, 2, \dots, 15$ we have

$$w_G(v_i) = w_H(x_i) + 4m = 124 - 2i + 4m = 4(m + 30) + 4 - 2i = 4n + 4 - 2i,$$

because v_i has two neighbors in the "lower" part of H (that is, among vertices $v_1, v_2, \dots, v_{15}$) where the labels have not changed, and four neighbors in the "upper" part (among vertices $v_{m+16}, v_{m+17}, \dots, v_{m+30}$), where each label was increased by m. For $i = 16, 17, \dots, m + 15$ we have

$$w_G(v_i) = w_J(u_{i-15}) + 90 = 4m + 4 - 2(i - 15) + 90 = 4(m + 30) + 4 - 2i = 4n + 4 - 2i,$$

because the labels of all six neighbors were increased by 15. Finally, for $i = m + 16, m + 17, \dots, m + 30$ we have

$$w_G(v_i) = w_H(x_{i-m}) + 2m = 124 - 2(i - m) + 2m = 4(m + 30) + 4 - 2i = 4n + 4 - 2i,$$

because v_i has four neighbors in the "lower" part of H where the labels have not changed, and two neighbors in the "upper" part where each label was increased by m. □

Our main result now follows directly from Lemma 4.3 and Observation 3.4.

Theorem 4.4. *For $n \equiv 2 \pmod 4$, there exists an $(n - 7)$-regular handicap graph when $n = 30$ or $n \geq 42$.*

This, together with the result by Kovář [P. Kovář, On regular handicap graphs, personal communication, June 16, 2016] and Kovářová [T. Kovářová, On regular handicap graphs, personal communication, June 16, 2016], gives an almost complete characterization of handicap graphs for $n \equiv 2 \pmod 4$.

Theorem 4.5. *For $n \equiv 2 \pmod 4$, there exists an r-regular handicap graph if and only if $3 \leq r \leq n - 7$ and $r \equiv 3 \pmod 4$ except when $r = 3$ and $n \leq 26$ and possibly when $r = n - 7$ and $n \in \{14, 18, 22, 26, 34, 38\}$.*

Since no magic rectangles of orders $14, 22, 26, 34$, or 38 exist, one has to hope that a computer aided search would help to settle the existence question for these orders. A construction for $n = 18$ similar to that for $n = 30$ may exist, but the authors were unable to find one.

References

[1] S. Arumugam, D. Froncek, N. Kamatchi, Distance magic graphs – a survey, J. Indones. Math. Soc. Special Edition (2011) 1–9.

[2] G. Chartrand, L. Lesniak, Graphs and Digraphs, Chapman and Hall, CRC, Fourth edition, 2005.

[3] D. Froncek, Fair incomplete tournaments with odd number of teams and large number of games, Congr. Numer. 187 (2007) 83–89.

[4] D. Froncek, Handicap distance antimagic graphs and incomplete tournaments, AKCE Int. J. Graphs Comb. 10(2) (2013) 119–127.

[5] D. Froncek, Handicap incomplete tournaments and ordered distance antimagic graphs, Congr. Numer. 217 (2013) 93–99.

[6] D. Froncek, P. Kovář, T. Kovářová, Fair incomplete tournaments, Bull. Inst. Combin. Appl. 48 (2006) 31–33.
[7] J. Gallian, A dynamic survey of graph labeling, Electron. J. Combin. 5 (# DS6) (2015) 43 pp.
[8] T. Harmuth, Ueber magische Quadrate und ähnliche Zahlenfiguren, Arch. Math. Phys. 66 (1881) 286–313.
[9] T. Harmuth, Ueber magische Rechtecke mit ungeraden Seitenzahlen, Arch. Math. Phys. 66 (1881) 413–447.
[10] M. Miller, C. Rodger, R. Simanjuntak, Distance magic labelings of graphs, Australas. J. Combin. 28 (2003) 305–315.
[11] K. A. Sugeng, D. Froncek, M. Miller, J. Ryan, J. Walker, On distance magic labeling of graphs, J. Combin. Math. Combin. Comput. 71 (2009) 39–48.
[12] V. Vilfred, Σ-labelled graph and circulant graphs, Ph.D. Thesis, University of Kerala, Trivandrum, India, 1994.