Düzce University
Journal of Science & Technology

Research Article

Optimal Control for A SEIR Epidemiological Model Under the Effect of Different Incidence Rates

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DOI: 10.29130/dubited.1076222

ABSTRACT

In this study, optimal control problem for a fractional SEIR epidemiological model under the effect of bilinear and saturate incidence rate functions is investigated. These rates play an important role in the realistic modeling of an epidemic by describing the interaction between susceptible and infected individuals of a population. This interaction is highly decisive in whether the disease will turn into a pandemic or not. Therefore, these functions can be defined in different forms depending on the course of the epidemic. The model discussed in this study is defined in terms of Caputo. Dimensional compatibility is guaranteed before posing the optimal control problem. The main objective of the proposed optimal control problem is to minimize the number of infected individuals and the cost of education given to susceptible individuals as a preventive measure. Euler-Lagrange equations corresponding to the optimality conditions of the considered model are first determined by Hamiltonian's formalism. Afterward, the optimal system with right and left fractional Caputo derivatives are solved numerically by the forward-backward sweep method combined with the fractional Euler method. Optimal solutions are interpreted graphically for varying values of the incidence rate coefficients and the fractional parameter. According to the simulation results, it is seen that the education given to susceptible individuals is significantly effective in slowing down the epidemic.

Anahtar Kelimeler: *Optimal control, Hamiltonian formalism, Caputo fractional derivative, SEIR model, Bilinear incidence rate, Saturated incidence rate, Forward-backward sweep method, Fractional Euler method.*

Farklı İnsidans Oranlarının Etkisi Altında Bir SEIR Epidemiyolojik Modelinin Optimal Kontrolü

ÖZET

Bu çalışmada, bilineer ve doymuş insidans hızı fonksiyonlarının etkisi altında bir kesirli SEIR salgın modeli için optimal kontrol problemi incelenmiştir. Bu oranlar, bir popülasyonun duyarlı ve enfekte bireyleri arasındaki etkileşimi tanımlayarak bir salgının gerçekçi modellenmesinde önemli bir rol oynamaktadır. Bu etkileşim, hastalığın pandemiye dönüşüp dönüşmeyeceği konusunda oldukça belirleyicidir. Dolayısıyla bu fonksiyonlar salgının seyrine göre farklı şekillerde tanımlanabilir. Bu çalışmada, bilineer ve doymuş insidans fonksiyonlarının etkileri tartışılmaktadır. İncelenen epidemiyolojik model, Caputo kesirli türevlidir. Optimal kontrol problemini ortaya koymadan önce boyutsal uyumluluk garanti edilir. Önerilen optimal kontrol probleminin temel amacı, enfekte birey sayısını ve duyarlı bireylere önleyici tedbir olarak verilen eğitimin maliyetini en aza indirmektir. Bu amaçla öncelikle ele alınan modelin optimallik koşullarına karşılık gelen Euler-Lagrange denklemleri hesaplanır. Daha sonra sağ ve sol kesirli Caputo türevli optimal sistem, kesirli Euler yöntemi ile birleştirilmiş ileri-geri süpürme yöntemi ile sayısal olarak çözülmüştür. Simülasyon sonuçlarına göre duyarlı bireylere verilen eğitimin salgının yavaşlatılmasında önemli ölçüde etkili olduğu görülmektedir.

Keywords: *Optimal kontrol, Hamilton formülü, Caputo kesirli türevi, SEIR modeli, Bilineer insidans oranı, Doymuş insidans oranı, İleri-geri süpürme yöntemi, Kesirli Euler yöntemi.*

Geliş: 19/02/2022, Düzeltme: 24/05/2022, Kabul: 31/05/2022

I. INTRODUCTION

Predicting the course of epidemic diseases in a population and how to control them are the main research topics of mathematical epidemiology [1-3]. Epidemiological diseases can be quite deadly as experienced from the COVID-19. Therefore, global analysis of epidemics should be done with mathematical models as well as experimental studies. Because making predictions about the course of the epidemic based on the solutions of mathematical models is economical in many respects. For this, both the compartments of the disease and the model should be well defined. That's why models are open to continuous improvement. The dynamics of the epidemiological models may vary depending on the type of disease, the genus of population, and the environment in which they live.

Integer-order systems have always been used to model epidemiological diseases. However, the ability of fractional derivatives to describe the heritability and memory features inherent in epidemiological diseases has made fractional-order models remarkable, especially in recent years. As seen during the COVID-19 epidemic, decreasing infected individuals may take a long periof of time depending on the development of vaccination strategies and treatment methods. In this sense, fractional-order systems, thanks to their nonlocal description, can model the long-tailed damping behavior of the disease quite realistically. Among them, Ertürk and Kumar [4] discussed the numerical existence and uniqueness of the solutions for a fractional COVID-19 model in terms of generalized Caputo derivative, and also obtained numerical solutions by Diethelm's predictor-corrector algortihm. Dokuyucu and Dutta [5] studied on the existence and uniqueness of solutions for an Ebola disease modelled by Caputo-Fabrizio fractional derivative, and calculated numerical solutions by Atangana-Owolabi algorithm based on predictor-corrector method. Naik et al. [6] researched system analysis and numerical solutions of a COVID-19 epidemic model in terms both of Caputo and Atangana-Baleanu derivatives. Akgül et al. [7] also investigated system analysis and numerical solutions for a model in terms of fractal fractional derivative for COVID-19 epidemic. Gao et al. [8] have recetly developed a fractional-order model, taking into account the individuals who were not reported for the COVID-19 outbreak, and proposed a modified predictor-corrector method to solve this model numerically. Veeresha et al. [9] have considered a fractional-order SIR model for childhood disease which is unfortunately still deadly in many undeveloped countries, and solved the system by combining Laplace transform and q-homotopy analysis method. From a similar point of view, Gao and Baskonus [10] have recently developed a fractional-order model representing wave behavior in computer virus propagation and comparatively evaluated the solutions by applying the natural transform method and the variational iterational method.

In mathematical epidemiology, SIR [11,12], SEIR [13,14], SEIQR [15,16] models are the basis of examining the course of epidemics. All other models emerge from the development of these basic models according to the nature of the disease. Among them, the models in which various treatment strategies are adapted as control functions are quite remarkable. Introducing the optimal treatment strategies for an epidemic is as important as mathematical modeling the spread of the disease [17]. These strategies can include single or combined treatment methods, preventive measures such as vaccination, public awareness training, and cost expenditures. All these strategies act as control functions within the model. What is important here is how and for which purpose these control functions are adapted to the model. As is known, the treatment costs of many types of diseases are quite high. In addition, treatment methods can not only be beneficial to patients but also cause devastating effects that can be noticed with a series of experiences. Considering all these, the necessity of researches on mathematical models with control strategies is undeniable, as it is economical, saves time, and prevents unpredictable damage to the patient. Mathematically, optimal control studies proceed through two objectives: determining the cost function and foundation of optimality conditions [18]. In epidemic models, both the determination of the objective function and the adaptation of the optimal control variables to the system can vary greatly according to the course of the disease. Therefore, each optimal control scenario should be analyzed separately. In addition to suggesting an optimal control problem, methods such as treatments or vaccines whose effects are taken into account in some studies are included with a fixed ratio of compartments in the system [19,20]. In other words,

the rate of the population currently vaccinated is taken into account. This is a frequently used perspective, especially when developing a new model fitting the experimental data. This represents the existing situation when the necessary measures have been taken. However, since vaccination is considered as a control function, not a fixed rate in optimal control problems, the answer to the question of what rate should be vaccinated to minimize the number of infected individuals is sought. Apart from all these, examining the chaotic behavior of diseases and bifurcation analysis is a separate and a frequently studied research topic [21-25].

For the epidemiological models, the interaction between the susceptible and infected individuals is given by a functional relation called the incidence rate. This is a critique factor that significantly affects the behavior of the disease. Because what actually makes a disease epidemic or not is the rate of interaction of S and I individuals with each other. This relationship has been revealed and examined with different functionals. Among the most commonly used are: the standard bilinear type βIS [26]; saturated type presented by Anderson and May [27]; specific nonlinear type introduced by Beddington [28] and DeAngelis [29] and modified nonlinear type considered by Gui-Hua Li and Yong-Xin Zhang [30].

In the recent studies on fractional-order systems under the incidence rate effect, there are tuberculosis infection model with nonlinear incidence rate [31], influenza disease modelled with half saturated incidence function [32], and a fractional SEIR model in terms of a nonlinear incidence rate [33], etc. The common considered incidence rate in the epidemiological models is the bilinear function because of its computational convenience. Using the bilinear incidence rate makes more sense for modeling only the early phases of an epidemic. But, for example, since saturation rate includes behavioral changes and population effects of infected individuals due to the parameters in its definition, it can avoid unlimited exposure of susceptible individuals to infected individuals when appropriate parameters are selected. Or, in complex models where epidemics become more chaotic, incidence rate functions with higher nonlinearity may be more appropriate. In other words, the choice of this function is completely related to the disease to which the model corresponds.

Despite the importance of analyzing all aspects of fractional-order systems with incidence rate effect, the researches for their optimal control unfortunately only tend back to the last few years. For example, Shi et al. [34] have presented the system analysis and the optimal control design for a fractional tuberculosis model with saturated incidence rate. Even the optimal control of the integer-order model of this disease studied by Baba et al. [35] is rather new. Zarin et al. [36] have discussed the dynamics and optimal control of virüs spreading among the rabies under the effect of convex incidance rate. Similarly, Khan et al. [37] have studied on the existence and uniqueness of the solutions, stability analysis, and optimal control of a fractional COVID-19 model with a convex incidence ratio. Boukhouima et al. [38] have researched the optimal control problem and stability analysis of a fractional-order model with general incidence rate effect for HIV-AIDS model.

The present work examining the optimal control problem is based on the model discussed by Khan et al. [33]. Research on systems in the literature progress under different perspectives. A significant number of them, such as [33], are interested in the systems for determining equilibrium points and performing stability analysis of them at these points. However, another issue that is as important as analyzing the dynamic properties of systems is the enhancement of control strategies. This is very important in terms of mathematically investigating the effects of vital preventive or treatment measures in controlling epidemics that threaten public health significantly. Each treatment method acts on the model as a control variable. As mentioned above, developing various control strategies on models saves time and treatment costs by developing predictions about the course of disease corresponding to the model. All this constitutes the motivation for the present study. On the other hand, the effect of the incidence rate on the optimal control problem constitutes the second motivation of this study. The nonlinearity of the incidence rate function can be considered in many types as it represents the future status of a disease in a population. For example, the course of the COVID 19 outbreak in the UK and India is not the same due to many factors, and this needs to be represented by different incidence rates while modeling the disease. The aim of our study is not to develop a new model that represents a set of experimental results. The incidence rates in our model have already been discussed in [33] in terms of the stability analysis of the system. In here, we purpose to introduce a control strategy for this uncontrolled model and make a comparison by investigating the

effect of the same incidence rates on it. In optimal control problems, the optimality conditions represented by the Euler-Lagrange equations are obtained with either Lagrangian or Hamiltonian functionals. Hamilton's formalism is applied for the current work. We apply the fractional Euler method to obtain the numerical solution of the optimal system. For this, numerical algorithms have been performed in the MATLAB program. The results are compared for the controlled and uncontrolled systems according to the incidence rates and fractional-order parameters.

The outline of the study is as follows. In the 2nd section, basic definitions and theorems are reminded. Problem formulation and numerical solutions are included in Section 3. Consequently in Section 4, the numerical results are held and evaluated, and a brief foresight is given for the future studies on this subject.

II. PRELIMINARIES

Definition 2.1. [39] Let $\psi \in C[a,b]$ and $\alpha \in [m-1,m)$ $(m \in \mathbb{N})$, the left and right sided Caputo fractional derivatives are as follows:

$$ {}_{a}^{C}D_{t}^{\alpha}\psi(t) = \frac{1}{\Gamma(m-\alpha)}\int_{a}^{t}(t-\tau)^{m-\alpha-1}\left(\frac{d}{dt}\right)^{m}\psi(\tau)d\tau, \tag{1.1} $$

$$ {}_{t}^{C}D_{b}^{\alpha}\psi(t) = \frac{1}{\Gamma(m-\alpha)}\int_{t}^{b}(\tau-t)^{m-\alpha-1}\left(-\frac{d}{dt}\right)^{m}\psi(\tau)d\tau, \tag{1.2} $$

where $\Gamma(\alpha)$ is the Euler's gamma function.

The fact that the Caputo derivative of the constant is zero and the integral transformation of this derivative requires physically interpretable integer-order initial conditions has made it widely used in engineering applications such as control theory. The main difference that separates the fractional optimal control problem from the classical one is that both right-handed and left-handed fractional derivatives are included in the optimal system. This makes the numerical techniques for the fractional optimal systems more difficult than the classical ones. The notion of optimal control for a fractional-order dynamical system described with Riemann-Liouville derivative was first introduced by Agrawal [40]. Optimality conditions for a dynamical system in terms of Caputo fractional derivative was also firstly presented by Agrawal [41].

The main purpose of an optimal control problem (OCP) for a fractional-order system is to find the optimal control function $u(t)$ optimizing the following performance index

$$ J(u) = \int_{0}^{1} p(x,u,t)dt, \tag{1.3} $$

subjected to the fractional-order dynamical system

$$ {}_{0}^{C}D_{t}^{\alpha}x(t) = q(x(t),u(t),t), \tag{1.4} $$

and the initial condition

$$ x(0) = x_0, \tag{1.5} $$

where $x(t)$ and $u(t)$ denote the state and control functions, p and q are known functions. Thus, the necessary optimality conditions obtained by the Hamiltonian formalism are as follows:

$$ {}_{0}^{C}D_{t}^{\alpha}x = \frac{\partial H}{\partial \lambda} \ (\text{state equation}), \tag{1.6} $$

$$ {}_{t}^{C}D_{1}^{\alpha}\lambda = \frac{\partial H}{\partial x} \ (\text{co - state equation}), \tag{1.7} $$

$$ \frac{\partial H}{\partial u} = 0. \ (\text{control equation}) \tag{1.8} $$

The optimal system (1.6)-(1.8) is determined by the fundamental relations of variational calculus and the integration by parts formula for fractional operators. In addition to the given initial condition $x(0)$, the final value of co-state (Lagrange/adjoint) variable $\lambda(1)$ known as transversality condition is

needed to get the solution of the optimality conditions (1.6)-(1.8). The calculation of this value is inevitable in the solution process.

Unlike the classical theory, the optimality conditions arising in the FOCPs are constructed in terms of left and right sided fractional derivatives, regardless of the type of fractional operator. This requires introducing the numerical methods to solve the optimal system. The most commonly used numerical methods for FOCPs can be briefly given as follows:

- Diethelm's Predictor-Corrector method [42], which is applied in combination with the Forward-Backward sweep algorithm.
- The Fractional Euler Method (FEM) [43], which has been applied by changing the direction of the fractional derivative in optimality conditions for the control function from right to left. This method does not require the calculation of corrector terms. This feature provides computational ease in the development of algorithm. For this computational simplicity, we prefer to apply this method in the current study.
- Numerical approximations with various families of orthogonal polynomials [44].

It should also be noted that there are not numerous numerical methods for FOCPs.

The incidence rate function $F(S,I)$ represents the interaction concentration of susceptible and infected individuals. This function is continuously differentiable and satisfies the following properties [45]:

1. For $\forall I \geq 0,\ F(0,I)=0,$
2. For $\forall S>0$ and $\forall I \geq 0,\ F_S(S,I)>0;$ i.e. F is monotonously increasing due to $S,$
3. For $\forall S,I \geq 0,\ F_I(S,I)\leq 0;$ i.e. F is monotonously non-increasing due to $I.$

Table 1. *Some important incidence rate functions in the literature*

Types of $F(S,I)$	***Description***	***Reference Studies***
Bilinear	βSI	[26]
Saturated	$\dfrac{\beta SI}{1+a_1 I}$	[35,46]
Beddington- DeAngelis	$\dfrac{\beta SI}{1+a_1 S+a_2 I}$	[28,29]
Convex	$\beta SI(1+\delta I)$	[36]
Harmonic mean type	$\dfrac{\beta SI}{S+I}$	[47]
Crowley–Martin	$\dfrac{\beta SI}{(1+\alpha S)(1+\gamma I)}$	[48]
Hattaf et. al. (A specific type)	$\dfrac{\beta SI}{1+\delta_1 S+\delta_2 I+\delta_3 SI}$	[49]

III. FORMULATION OF THE PROBLEM

In the present study, our purpose is to introduce the optimal control strategy for a fractional SEIR model under the effects of two separate incidence rate functions. The considered model without control strategy is as follows [33]:

$$\begin{cases} {}_0^C D_t^\alpha S(t) = \Lambda - F(S,I) + \mu_1 I + \theta R, \\ {}_0^C D_t^\alpha E(t) = F(S,I) - (d+\tau)E, \\ {}_0^C D_t^\alpha I(t) = \tau E - (d+\mu_1+\mu_2+\delta)I, \\ {}_0^C D_t^\alpha R(t) = \mu_2 I - (d+\theta)R. \end{cases} \tag{1.9}$$

The descriptions of system coefficients are listed in *Table 2*. The parameter values in [33] are used to compare the effect of the control function on the system (1.9).

Table 2. *Description of parameter for the system* (1.9)

Paramaters	*Description*
Λ :	*The constant recruitment rate for the susceptible individuals.*
d :	*The rate of natural death*
τ :	*The rate of transition from exposed to infected individuals*
μ_1 :	*The rate of transition from infected to the susceptible individuals*
μ_2 :	*The recovery rate of infected individuals*
δ :	*Mortality rate in the infected individuals*
θ :	*The rate of immunized individuals in the recovered compartment*
$F(S,I)$	*The incidence rate function, which denotes the rate at which the suspect becomes infected and joins in the exposed individuals.*

First, let us construct the OCP on the model (1.9), which aims both of minimizing the number of infected individuals and the cost of training. It is our assumption so that it can be introduced for the future researches. For example, it may be the cost of vaccination or treatment methodologies. We are inspired by study [50] while constructing the OCP. The determination of OCP is not unique because it depends on the objective function.

Thus, our objective functional (performance index) is

$$\min \to J(u) = \int_0^{t_f} \left[I(t) + \frac{1}{2}\omega u^2(t) \right] dt, \tag{1.10}$$

where the control function $u(t)$ means the rate of susceptible individuals being educated at any time t, ω is the positive weight which can be chosen to offset the cost due to the importance of the target feature, and t_f denotes the final time of the control. The admissible set of control functions is defined as $U = \{u(t) : 0 \le u \le 1,\ t \in [0, t_f]\}$. Before formulating the optimal control problem, we ensure the dimension consistency in the model [51]. Thus, the improved model to which the control parameter is adapted is as follows:

$$\begin{cases} {}_0^C D_t^\alpha S(t) = \Lambda^\alpha - F^\alpha(S,I) + \mu_1{}^\alpha I + \theta^\alpha R - uS, \\ {}_0^C D_t^\alpha E(t) = F^\alpha(S,I) - (d^\alpha + \tau^\alpha)E, \\ {}_0^C D_t^\alpha I(t) = \tau^\alpha E - (d^\alpha + \mu_1{}^\alpha + \mu_2{}^\alpha + \delta^\alpha)I, \\ {}_0^C D_t^\alpha R(t) = \mu_2{}^\alpha I - (d^\alpha + \theta^\alpha)R + uS. \end{cases} \tag{1.11}$$

To determine the optimality conditions (1.6)-(1.8), which are calculated by Pontryagin's maximum principle, we define the Hamiltonian function which is a functional used to solve an optimal control problem for a fractional-order dynamical system [18]:

$$\begin{aligned} H = I + \frac{1}{2}\omega u^2(t) + \lambda_1 \{\Lambda^\alpha - F^\alpha(S,I) + \mu_1{}^\alpha I + \theta^\alpha R - uS\} + \lambda_2 \{F^\alpha(S,I) - (d^\alpha + \tau^\alpha)E\} \\ + \lambda_3 \{\tau^\alpha E - (d^\alpha + \mu_1{}^\alpha + \mu_2{}^\alpha + \delta^\alpha)I\} + \lambda_4 \{\mu_2{}^\alpha I - (d^\alpha + \theta^\alpha)R + uS\}, \end{aligned} \tag{1.12}$$

where $\lambda_i (i=1,2,3,4)$ are the co-state variables. Hamiltonian function describes the relationship between the state and control variables. This is done with the help of the Lagrange multiplier, which has no physical meaning but only serves as a tool. Thus, Hamiltonian establishes the relation between the dynamical system and the objective function.

Using the Hamiltonian function (1.12) and the optimality condition (1.7), the adjoint system is handled in the following form:

$$\begin{cases} {}_t^C D_{t_f}^{\alpha} \lambda_1(t) = -\lambda_1(t)\left(\dfrac{\partial F^{\alpha}}{\partial S} + u(t)\right) + \lambda_2(t)\dfrac{\partial F^{\alpha}}{\partial S} + \lambda_4(t)u(t), \\ {}_t^C D_{t_f}^{\alpha} \lambda_2(t) = -\lambda_2(t)\left(d^{\alpha} + \tau^{\alpha}\right) + \lambda_3(t)\tau^{\alpha}, \\ {}_t^C D_{t_f}^{\alpha} \lambda_3(t) = -\lambda_1(t)\left(S\dfrac{\partial F^{\alpha}}{\partial I} - {\mu_1}^{\alpha}\right) + \lambda_2(t)S\dfrac{\partial F^{\alpha}}{\partial I} - \lambda_3(t)\left(d^{\alpha} + {\mu_1}^{\alpha} + {\mu_2}^{\alpha} + \delta^{\alpha}\right) + \lambda_4(t){\mu_2}^{\alpha} + 1, \\ {}_t^C D_{t_f}^{\alpha} \lambda_4(t) = \lambda_1(t)\theta^{\alpha} - \lambda_4(t)\left(d^{\alpha} + \theta^{\alpha}\right). \end{cases} \tag{1.13}$$

Before creating the numerical algorithm, we consider converting the right-handed derivatives in (1.13) to the left-handed derivatives. For this, we use the following property.

Lemma 3.1. [42] For $\alpha \in (0,1]$, the relation between the right and left-sided Caputo fractional derivatives of a given function $\varphi(t)$ is given by

$$ {}_t^C D_{t_f}^{\alpha} \varphi(t) = {}_0^C D_t^{\alpha} \varphi(t_f - t). \tag{1.14}$$

Then, we can rewrite the adjoint system (1.13) as below

$$\begin{cases} {}_0^C D_t^{\alpha} \lambda_1(t_f - t) = -\lambda_1(t_f - t)\left(\dfrac{\partial F^{\alpha}}{\partial S} + u(t_f - t)\right) + \lambda_2(t_f - t)\dfrac{\partial F^{\alpha}}{\partial S} + \lambda_4(t_f - t)u(t_f - t), \\ {}_0^C D_t^{\alpha} \lambda_2(t_f - t) = -\lambda_2(t_f - t)\left(d^{\alpha} + \tau^{\alpha}\right) + \lambda_3(t_f - t)\tau^{\alpha}, \\ {}_0^C D_t^{\alpha} \lambda_3(t_f - t) = -\lambda_1(t_f - t)\left(S\dfrac{\partial F^{\alpha}}{\partial I} - {\mu_1}^{\alpha}\right) + \lambda_2(t_f - t)S\dfrac{\partial F^{\alpha}}{\partial I} - \lambda_3(t_f - t)\left(d^{\alpha} + {\mu_1}^{\alpha} + {\mu_2}^{\alpha} + \delta^{\alpha}\right) \\ \qquad\qquad + \lambda_4(t_f - t){\mu_2}^{\alpha} + 1, \\ {}_0^C D_t^{\alpha} \lambda_4(t_f - t) = \lambda_1(t_f - t)\theta - \lambda_4(t_f - t)\left(d^{\alpha} + \theta^{\alpha}\right). \end{cases} \tag{1.15}$$

Thus, 8 nonlinear equations constituting the state (1.11) and co-state (1.15) systems are in terms of left-sided fractional derivatives. It means that rearranged form of adjoint system brings computational ease while coding the algorithm.

Finally, the optimality condition (1.8) gives the optimal control function:

$$u(t) = \min\left\{\max\left\{\frac{\left(\lambda_1(t) - \lambda_4(t)\right)S(t)}{\omega}, 0\right\}, 1\right\}. \tag{1.16}$$

In the numerical simulations, we illustrate all the comparative results for the bilinear and saturated incidence functions. We discuss existing incidance functions in order to make various comparisons with the uncontrolled version of the model [33]. In the next section, however, we avoid computational repetition by using a general expression for incidence rate function F^{α} for the sake of simplicity while giving the algorithm. In fact, F^{α} refers to $\beta^{\alpha} SI$ for the bilinear and $\dfrac{\beta^{\alpha} SI}{1 + {a_1}^{\alpha} I}$ for the saturated incidence functions when obtaining the simulations. As previously stated, there are many different

options for incidence rate functions depending on the problem description. It means that the present model or any epidemic model can be analyzed for other type incidence rate functions.

IV. NUMERICAL RESULTS AND DISCUSSION

A. Numerical Algorithm

As mentioned before, there are a limited number of methods used to solve fractional OCPs. Among them, the Forward-Backward Sweep Method (FSBM) combined with Fractional Euler Method (FEM) is used in the present study because it is a direct method without perturbation, does not require additional constraints or assumptions and also a correction algorithm. FSBM ensures that the optimal system depending on the state and co-state variables is divided into two separate systems and one is solved by using the last results of the other until convergence is achieved. We implement the algorithm using MATLAB software. The steps of the method adapted to the system are as follows:

***Step 1*:** Define the initial conditions $S(t_0)=S_{t_0}, E(t_0)=E_{t_0}, I(t_0)=I_{t_0}, R(t_0)=R_{t_0}$ and the system parameters.

***Step 2*:** Divide the time interval $\left[t_0, t_f\right]$ into subintervals of length $h=\frac{t_f-t_0}{N}$. Hence denote each time node with $t_n=t_0+nh,\ n=0,1,...,N.$

***Step 3*:** Compute the control function $u(t_n)=\frac{\left(\lambda_1(t_n)-\lambda_4(t_n)\right)S(t_n)}{\omega}$, $n=0,1,2,...,N.$ In this step, initial value of control function can be computed by the initial condition $\left(S_{t_0}, E_{t_0}, I_{t_0}, R_{t_0}\right)$ and also the transversality conditions for $\lambda_i(t),\ i=1,2,3,4.$

***Step 4*:** Solve the state functions by applying FEM as follows:

$$\begin{cases} S(t_n)=S_{t_0}+\frac{h^\alpha}{\Gamma(\alpha+1)}\sum_{n=0}^{m-1}c_{n,m}\left[\Lambda^\alpha-F^\alpha\left(S(t_n),I(t_n)\right)+\mu_1{}^\alpha I(t_n)+\theta^\alpha R(t_n)-uS(t_n)\right], \\ E(t_n)=E_{t_0}+\frac{h^\alpha}{\Gamma(\alpha+1)}\sum_{n=0}^{m-1}c_{n,m}\left[F^\alpha\left(S(t_n),I(t_n)\right)-\left(d^\alpha+\tau^\alpha\right)E(t_n)\right], \\ I(t_n)=I_{t_0}+\frac{h^\alpha}{\Gamma(\alpha+1)}\sum_{n=0}^{m-1}c_{n,m}\left[\tau^\alpha E(t_n)-\left(d^\alpha+\mu_1{}^\alpha+\mu_2{}^\alpha+\delta^\alpha\right)I(t_n)\right], \\ R(t_n)=R_{t_0}+\frac{h^\alpha}{\Gamma(\alpha+1)}\sum_{n=0}^{m-1}c_{n,m}\left[\mu_2{}^\alpha I(t_n)-\left(d^\alpha+\theta^\alpha\right)R(t_n)+uS(t_n)\right], \end{cases} \tag{1.17}$$

where $n=0,1,...,N,$ and $m=1,2,...,N.$ Also, the coefficient $c_{n,m}$ is described by

$$c_{n,m}=(m-n)^\alpha-(m-n-1)^\alpha. \tag{1.18}$$

***Step 5*:** Solve the co-state functions by applying FEM as follows:

$$
\begin{cases}
\lambda_1\left(t_f - t_{N-m-1}\right) = \dfrac{h^\alpha}{\Gamma(\alpha+1)}\displaystyle\sum_{n=0}^{m} c_{n,m+1}\Bigg[-\lambda_1\left(t_f - t_{N-n}\right)\left(\dfrac{\partial F^\alpha}{\partial S}\left(t_f - t_{N-n}\right) + u\left(t_f - t_{N-n}\right)\right) \\
\qquad\qquad +\lambda_2\left(t_f - t_{N-n}\right)\dfrac{\partial F^\alpha}{\partial S}\left(t_f - t_{N-n}\right) + \lambda_4\left(t_f - t_{N-n}\right)u\left(t_f - t_{N-n}\right)\Bigg], \\
\lambda_2\left(t_f - t_{N-m-1}\right) = \dfrac{h^\alpha}{\Gamma(\alpha+1)}\displaystyle\sum_{n=0}^{m} c_{n,m+1}\Big[-\lambda_2\left(t_f - t_{N-n}\right)\left(d^\alpha + \tau^\alpha\right) + \lambda_3\left(t_f - t_{N-n}\right)\tau^\alpha\Big], \\
\lambda_3\left(t_f - t_{N-m-1}\right) = \dfrac{h^\alpha}{\Gamma(\alpha+1)}\displaystyle\sum_{n=0}^{m} c_{n,m+1}\Bigg[-\lambda_1\left(t_f - t_{N-n}\right)\left(S\left(t_f - t_{N-n}\right)\dfrac{\partial F^\alpha}{\partial I}\left(t_f - t_{N-n}\right) - {\mu_1}^\alpha\right) \\
\qquad\qquad + \lambda_2\left(t_f - t_{N-n}\right)S\left(t_f - t_{N-n}\right)\dfrac{\partial F^\alpha}{\partial I}\left(t_f - t_{N-n}\right) \\
\qquad\qquad -\lambda_3\left(t_f - t_{N-n}\right)\left(d^\alpha + {\mu_1}^\alpha + {\mu_2}^\alpha + \delta^\alpha\right) + \lambda_4\left(t_f - t_{N-n}\right){\mu_2}^\alpha + 1\Big], \\
\lambda_4\left(t_f - t_{N-m-1}\right) = \dfrac{h^\alpha}{\Gamma(\alpha+1)}\displaystyle\sum_{n=0}^{m} c_{n,m+1}\Big[\lambda_1\left(t_f - t_{N-n}\right)\theta^\alpha - \lambda_4\left(t_f - t_{N-n}\right)\left(d^\alpha + \theta^\alpha\right)\Big],
\end{cases}
\tag{1.19}
$$

where the $c_{n,m+1}$ is given in the Step 4.

***Step 6*:** Using the iterative values of state and co-state functions obtained in Step 4 and Step 5, update the values of control function calculated in Step 3.

***Step 7*:** Set the tolerance values for the errors of state and co-state functions. Then, end the calculation if the difference between consecutive values is close enough to remain within these tolerance ranges. The results thus obtained are optimal solutions.

B. Simulation Results

As we mentioned earlier, we build the OCP on the fractional SEIR model whose stability analysis was studied by Khan et al. [33]. Therefore, we use the parameter assumptions of this study to better interpret the effect of control on the system. The parameter values used are as follows: $\Lambda = 0.8,\ a_1 = 0.1,\ \mu_1 = 0.02,\ \mu_2 = 0.02,\ \beta = 0.00004,\ d = 0.001,\ \tau = 0.02,\ \delta = 0.004.$ To make a realistic comparison of the control effect, the initial condition $\left(S_{t_0}, E_{t_0}, I_{t_0}, R_{t_0}\right) = (100, 60, 10, 0)$ in [33] is considered. The importance of the initial population is undeniable when developing a new model based on the results of an experimental study. However, this is outside the scope of the present study. The initial time is $t_0 = 0.$ The duration time of the control's effectiveness on the system is taken to be $t_f = 100$ days. The reproduction number R_0, which is the threshold parameter for the disease to turn into an pandemic, significantly determines the effectiveness of the control on the model. This parameter has already been determined as $R_0 = \dfrac{\Lambda\beta\tau}{d(d+\tau)(d+\mu_1+\mu_2+\delta)}$ by Khan et al. [33] for the stability analysis of the system. Depending on this parameter, there are three cases for the course of an epidemic disease: If $R_0 < 1$, the disease disappears after a short time; If $R_0 = 1$, the disease is endemic; If $R_0 > 1$, the disease turns into a pandemic. These probabilities are emerged due to the problem parameters defining the R_0 number. For example, the values $\beta = 0.004$ and $\beta = 0.00004$ are used in the numerical simulations. These are constants chosen not arbitrarily but to give $R_0 > 1$ and $R_0 < 1$ respectively.

In our problem, we consider the control function $u(t)$ as a preventive measure. Therefore, we assume control as education given to the susceptible population about the disease. The numerical

results obtained should be examined under this assumption. It means that the numerical results and their interpretations will be completely different for different control scenarios.

Firstly, in Figures 1 and 2, we examine the dependence of the system with and without control on the β parameter. In Figures 1 and 2, we arbitrarily chose the fractional parameter $\alpha = 0.9$. For the value of $\beta = 0.004$, that is, in the case of a pandemic, there is a faster decrease in the number of susceptible individuals under the effect of bilinear incidence rate compared to the saturated incidence rate in the uncontrolled system. In other words, for the uncontrolled model, susceptible individuals experience compartment changes by being affected by the disease more rapidly in the bilinear interaction. While susceptible individuals approach almost zero on the 100^{th} day in bilinear interaction, it is more than 50 in the saturated effect. Similarly, while the number of exposed individuals under the bilinear effect exceeds 100, this number remains around 80 in the saturated effect. For both of the bilinear and saturated cases, because in the early stages of the disease, it is not immediately possible to provide education to individuals about the disease and the vaccine has not yet been developed. For this reason, an increase in the number of infected individuals is observed until a certain time in both controlled and uncontrolled cases. In these figures, for the uncontrolled system and in the case of a pandemic (for $\beta = 0.004$), the number of infected in bilinear interaction increases to over 40 on the 100^{th} day, while it remains between 30 and 35 in saturated interaction. It means that the transmission rate of the disease represented by the discussed model is higher in the case of the bilinear effect compared to the saturation. In other words, susceptible individuals become infected more rapidly in the bilinear effect. This may be the reason why the bilinear effect is mostly considered in COVID-19 disease models.

We see from the Figures 1 and 2 that in a pandemic case (for $\beta = 0.004$), for the controlled system, there is a rapid decrease for the susceptible individuals in the 1^{st} week of disease for both incidence and saturated incidence rate effects. However, in the controlled case, unlike the uncontrolled, most of the susceptible individuals pass into the recovered compartment, while a small number of individuals pass into the exposed or infected individual compartments. Again, in the case of a pandemic (for $\beta = 0.004$), the decrease in the number of infected individuals in the controlled system is also quite remarkable. This means that our control strategy has been quite effective in reducing the number of infected individuals. If we remember the performance index (1.10), we aim to minimize two factors: the number of infected individuals and the cost of education. Again, in the case of a pandemic, the decrease in the number of individuals exposed to the disease, thanks to the education as a preventive, shows the effectiveness of the measure taken. Since people cannot predict how deadly the disease is before it turns into an epidemic, even if they are educated about the disease, the effect of training is not noticeable. In other words, in the absence of a pandemic, the effect of the training given is very low. From the graphics, this is easily recognizable in the case $\beta = 0.00004$. Also in this case for both of the controlled and uncontrolled system, infected individuals slowly increase for a short time and then spontaneously decrease and then disappear in accordance with reality. But, in this case for controlled system, the number of susceptible individuals is significantly reduced and they pass into the recovered compartment. This shows that in the absence of a pandemic, although the education given to susceptible individuals does not cause a significant decrease in the number of infected individuals, it causes a decrease in the number of susceptible individuals and increases the number of recovered individuals. This shows that individuals act cautiously and try to protect themselves before the disease turns into a pandemic.

Examining the effect of fractional derivative on epidemiological models is at least as important as other system parameters. Because the behavior of the disease at a certain time can be modeled with any value of the fractional parameter, which does not always have to be an integer value. For example, non-integer order systems are encountered in modeling the behavior of epidemics that take a very long time to fade. For this reason, determining the optimum value of the fractional parameter is very important and a research topic in itself, especially in studies with real data. In Figures 3 and 4, we illustrate the fractional dependency for the controlled and uncontrolled cases. In Figures 3 and 4, we show the fractional parameter dependence of the controlled and uncontrolled systems for both incidence rates. In Figure 3, in the case of bilinear effect for the uncontrolled system, the reduction in the number of susceptible individuals is faster for the $\alpha = 0.9$ value compared to

$\alpha = 1$. On the other hand, in the uncontrolled system for both $\alpha = 1$ and $\alpha = 0.9$, the number of susceptible individuals decreases below 20 with the bilinear incidence rate on the 100th day, while this value remains 60 in the case of the saturated effect. Under the effect of both incidence rates in the controlled system, the reduction in the number of both susceptible and infected individuals is faster in the case of $\alpha = 1$ than in the case of $\alpha = 0.9$. Although it is not included in the graphics, it is obvious that a rapid decrease can be observed in susceptible and infected individuals as the alpha value decreases.

Figures 5 and 6 demonstrate the β parameter dependence of the controlled system, which is the main goal of this study. The reproduction number R_0 is an important threshold parameter in whether a disease turns into an epidemic or not. The purpose of the present study is not to determine this number. This number has already been calculated based on the problem parameters. If $R_0 \leq 1$, the disease dies out or is an endemic; if $R_0 > 1$, it reduces to a pandemic. Considering this comparison, system analysis can be done by determining the threshold value for each of the system parameters that define R_0. That's why comparing R_0 to 1 means comparing β to 0.000059062.

In other words, $R_0 > 1$ actually means $\beta > 0.000059062$. For this reason, we make a comparison in the graphics by choosing values according to this threshold value of β. In the Figures 5 and 6, for increasing values of β greater than 0.000059062, a notable increase in the number of infected individuals can be seen as expected for both incidence rates. However, as we have previously interpreted for Figures 1 and 2, this increase is quite low when there is control, compared to the uncontrolled system. In other words, even if the interaction rate of susceptible and infected individuals is greater than 0.000059062, that is, if there is an pandemic, the training given to susceptible individuals has been quite effective. The most noticeable effect of control is visibly seen when the number of infected individuals increases rapidly in the case of a pandemic. Again as expected, as the β value decreases, the number of individuals exposed and treated decreases gradually as the disease begins to disappear. In other words, education also indirectly contributes to the reduction of treatment costs. As a result, whether there is a pandemic or not, education provides a significant financial advantage by raising awareness among susceptible individuals.

In Figure 7, we give the dependence of the controlled system with the saturated effect to the a_1 parameter. The increase in a_1 causes a sudden increase in the number of susceptible individuals from the 90th day. For $a_1 \geq 1$, the number of infected individuals is very close to each other and shows decreasing behavior.

Considering the controlled model, we expect the effect of the control to last longer for beta values where the epidemiological disease turns into a pandemic. In Figures 8 and 9, we arbitrarily choose $\alpha = 0.9$. Even in the case of $\beta < 0.000059062$, the effect of the control decreases after the 80th day under the bilinear effect, while control begins to decrease after the 70th day under the saturation effect. In other words, in the case where the disease is modeled with bilinear effect, the training period is longer to see the desired effect of the training on the population. This, of course, can increase the training costs somewhat. However, the cost of education to be given during the endemic process can be disregarded as it will undoubtedly be lower than the costs of disease treatment.
In the case of a pandemic (for $\beta > 0.000059062$), the training period for both is almost close to each other. As seen in the COVID-19 pandemic, continued awareness through education is necessary to prevent the spread of the disease. This is the realistic interpretation of the Figures 8 and 9 for $\beta > 0.000059062$.

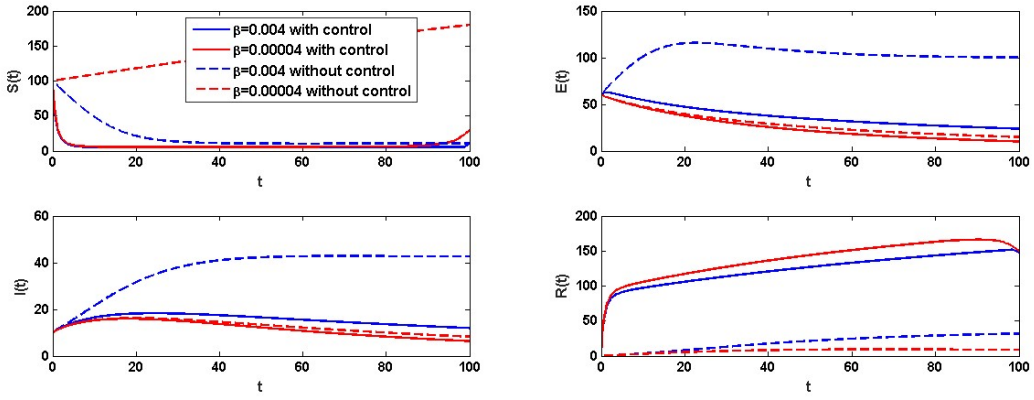

Figure 1. *Behaviour of system* (S,E,I,R) *with and without control due to* ***Bilinear*** *incidence function:* $\alpha = 0.90.$

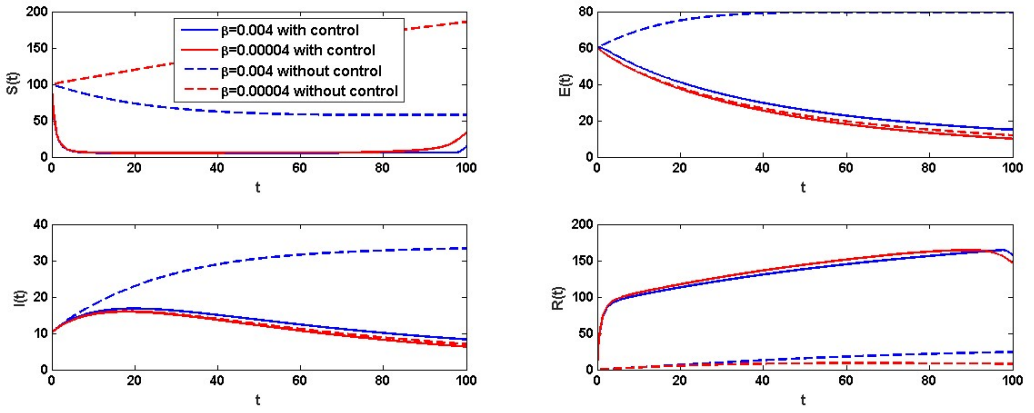

Figure 2. *Behaviour of system* (S,E,I,R) *with and without control due to* ***Saturated*** *incidence function:* $\alpha = 0.90,\ a_1 = 0.1.$

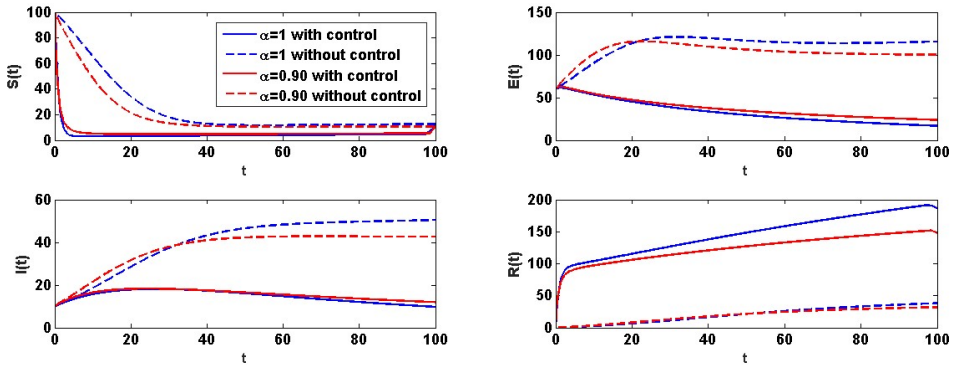

Figure 3. *Dependence of the system* (S,E,I,R) *with and without control on the fractional parameter* α *under the effect of* ***Bilinear*** *incidence function:* $\beta = 0.004.$

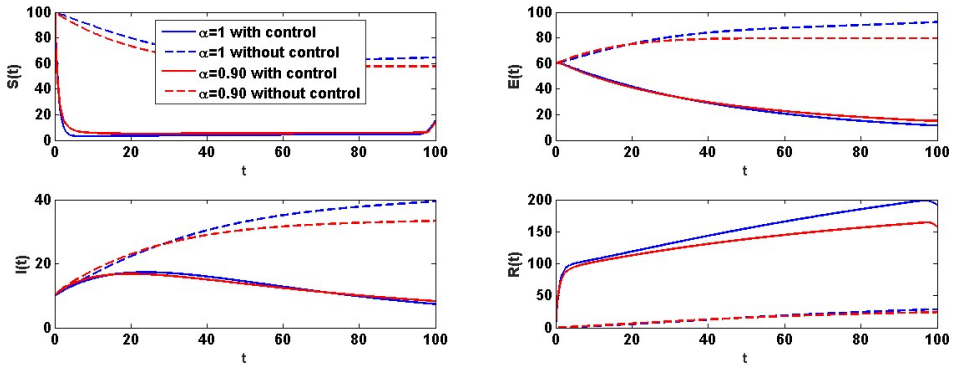

Figure 4. *Dependence of the system* (S,E,I,R) *with and without control on the* ***fractional*** *parameter* α *under the effect of* ***Saturated*** *incidence function:* $\beta = 0.004,\ a_1 = 0.1.$

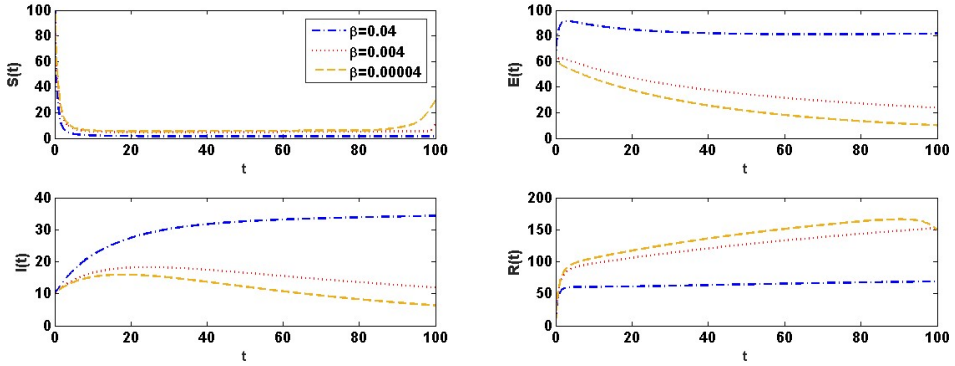

Figure 5. *Dependence of the* ***controlled*** *system* (S,E,I,R) *on the* β *under the effect of* ***Bilinear*** *incidence function:* $\alpha = 0.90.$

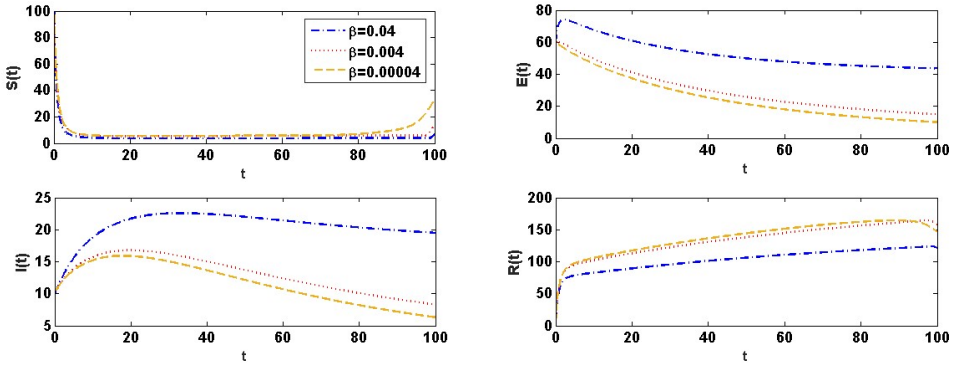

Figure 6. *Dependence of the* ***controlled system*** (S,E,I,R) *on the* β *parameter under the effect of* ***Saturated*** *incidence function:* $\alpha = 0.90,\ \mathrm{a}_1 = 0.1.$

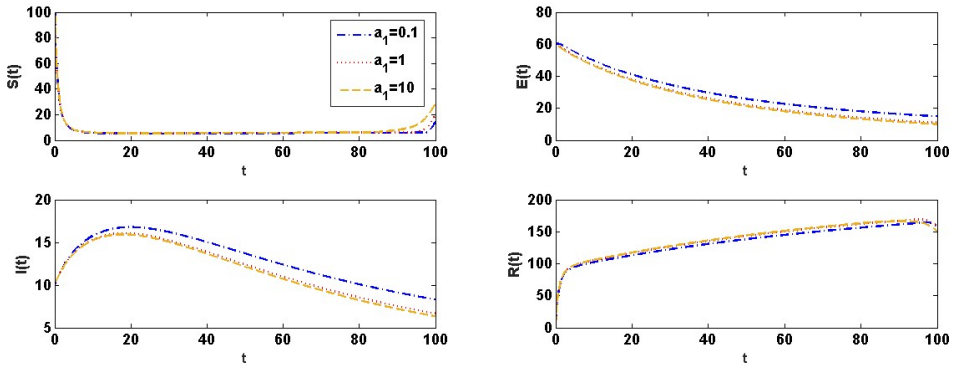

Figure 7. *Dependence of the controlled system* (S,E,I,R) *on the* a_1 ***incidence*** *parameter under the effect of* ***Saturated*** *incidence function:* $\alpha = 0.90,\ \beta$=0.004.

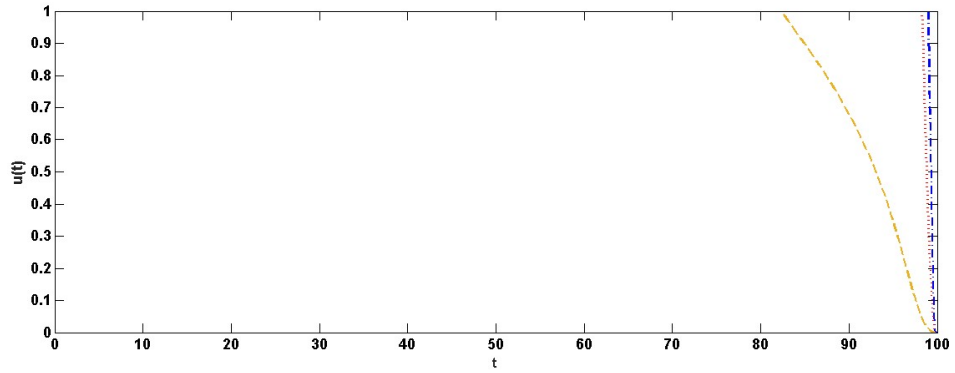

Figure 8. *Dependence of the* ***control function*** *on the* β ***incidence*** *parameter under the effect of* ***Bilinear*** *incidence function:* $\alpha = 0.90$.

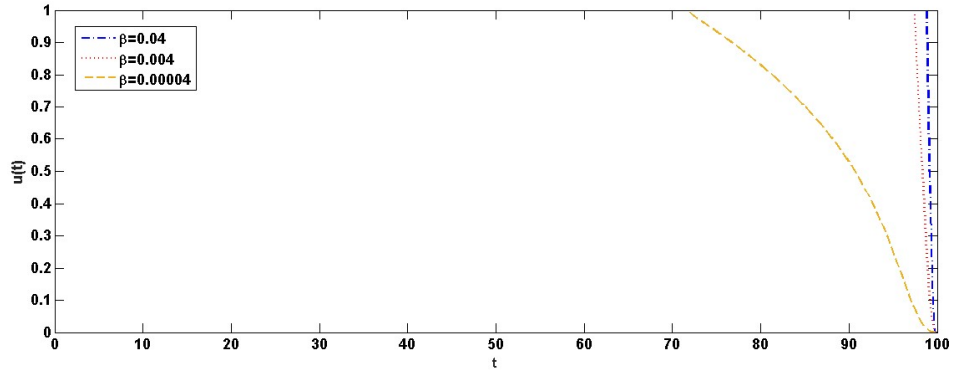

Figure 9. *Dependence of the* ***control function*** *on the* β ***incidence*** *parameter under the effect of* ***Saturated*** *incidence function:* $\alpha = 0.90,\ a_1 = 0.1$.

V. CONCLUSIONS

This study provides optimal control of a fractional-order epidemiological SEIR model whose stability analyzes were introduced by Khan et al. [33]. The main aim is to minimize the number of infected individuals and the cost of training given to susceptible individuals. Optimal control efficacy has been discussed by considering bilinear and saturated incidence rates. The results have revealed the positive impact of education on the population into the endemic and pandemic processes of disease. Since the education given to susceptible individuals has increased the level of awareness, it has caused a significant decrease in the number of infected individuals as aimed. In general, the bilinear incidence

function has shown a more aggressive effect on the number of infected individuals. However, even in this case, the desired reduction in infected individuals has been observed with the effect of education. As a result, educating susceptible individuals in an epidemic is highly effective. In this study, the control effect has been considered as training only. However, it is planned to examine the discussed model by considering different control strategies and incidence rates in future studies.

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