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Valuation rings and modules

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Abstract: The purpose of this paper is to compare and investigate relations between valuation rings and valuation modules.

Keywords: Multiplication module, valuation ring, valuation module.

1 Introduction

Throughout this paper, $\mathscr{R}$ denotes an integral domain, with quotient field K, $T = \mathscr{R} - \{0\}$ and M is a unitary $\mathscr{R}$-module. An $\mathscr{R}$-module M is called a multiplication $\mathscr{R}$-module, if for each submodule N of M, there exists an ideal $I$of $\mathscr{R}$ such that $N = IM$.(For more information about multiplication modules, see[2,4]). An integral domain $\mathscr{R}$ is called a valuation ring, if for each $x \in K = \mathscr{R} - \{0\}, x \in R$or $x^{-1} \in \mathscr{R}$. In [3], valuation modules in case module is torsion-free investigated. Morover in [1], nofinitely generated submodules of faithful multiplication valuation modules is investigated.

2 Valuation Rings

Definition 2. 1. A subring $\mathscr{R}$ of a field K is called a valuation ring of K if for every $\alpha \in K, \alpha \neq 0$, either $\alpha \in R$ or $\alpha^{-1} \in R$.

Example 2. 1.

1) Any field of $K$is a valuation ring of K.
2) Let p be a fixed prime. Let $R \subset Q$, the field of rationals, be defined by

$$R = \left\{ p^r \frac{m}{n} | r \geq 0, (p,m) = (p,n) = (m,n) = 1 \right\}.$$

Then $\mathscr{R}$ is a valuation ring of $\mathbb{Q}$.

Proposition 2. 1. Let V be a valuation ring of K. Then

1. K is the qoutient field of V.
2. Any subring of K containing V is a valuation ring of K.

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3. V is a local ring.
4. V is integrally closed.

Proposition 2. 2. The ideals of a valuation ring are totally ordered by inclusion. Conversely if the ideals of domain V with quotient field K are totally ordered by inclusion, then V is a valuation ring of K.

Corollary 2.1. If V is a valuation ring of K and P is a prime ideal of V,then V_p and $\frac{V}{P}$ are valuation ring.

Corollary 2.2. Any Noetherian valuation ring is a principal ideal domain.

Corollary 2.3. Let V be a Noetherian valuation ring. Then there exists an irredusible element $p \in V$ such that every ideal of V is of the type $I = (p^m), m \geq 1$ and $\cap_{m=1}^{\infty}(p^m) = 0$.

3 Valuation Modules

Let R be an integral domain with quotient field K and M a torsionfree $\mathscr{R}$-module. For $y = \frac{r}{s} \in K$ and $x \in M$, we say that $yx \in M$ if there exists $m \in M$ such that $rx = sm$.

Lemma 3. 1. Let R be an integral domain with quotient field K and M a torsionfree $\mathscr{R}$-module. Then the following conditions are equivalent:

1) For all $y \in K$ and all $x \in M$, $yx \in M$or $y^{-1}M \subseteq M$;

2) For all $y \in K$, $yM \subseteq M$or $y^{-1}M \subseteq M$.

Definition 3.1. Let R be an integral domain with quotient field K. A torsionfree $\mathscr{R}$-module M is called valuation module (VM) if one of the condition of Lemma 3.1 holds.

Example 3.1.

1) Any vector space is a valuation module.

2) Let $\mathscr{R}$ be a domain. $\mathscr{R}$ is a valuation ring if and only if $\mathscr{R}$ is a valuation $\mathscr{R}$-module.

3) Let $R = Z$ and p be a prime integer number. If

$$M = \left\{ p^n \frac{a}{b} | a, b, n \in Z, b \neq 0, n \geq 1 \ , \ (p,a) = (p,b) = (a,b) = 1 \right\}$$

then M is a valuation module.

4) Z is not a valuation Z-module.

An $\mathscr{R}$-module M is said to be integrally closed whenever $y^n m_n + \cdots + y m_1 + m_0 = 0$ for some $n \in N, y \in K$ and $m_i \in M$, then $y m_n \in M$.

Lemma 3.2. Any valuation module is integrally closed.

Proposition 3.1. Let K be the quotient field of a domain $\mathscr{R}$ and Ma torsionfree $\mathscr{R}$-module. Let S be the set, ordered by inclusion, of all nonempty subsets of M. Then the following conditions are equivalent:

1) M is a valuation module;

2) $S' = \{(N : M) \,|N \in S\}$ is totally ordered;

3) For $U = \{rM|r \in R\}$ the subset of S, U' is totally ordered.

Corollary 3.1. Let $\mathscr{R}$ be a domain and M a torsionfree $\mathscr{R}$-module. Then M is a valuation module if and only if for any submodules N, L of M,$(N : M) \subseteq (L : M)$ or $(L : M) \subseteq (N : M)$.

Corollary 3.2. Let $\mathscr{R}$ be a domain and M a faithful multiplication $\mathscr{R}$-module. Then M is a valuation module if and only if for any two submodules N, L of $M, N \subseteq L$ or $L \subseteq N$.

Remark 3.1. R^2 is a valuation $\mathscr{R}$-module, but not a multiplication $\mathscr{R}$-module. Note that $R \oplus (0) \nsubseteq (0) \oplus R$ and $(0) \oplus R \nsubseteq (0) \oplus R$.

Note that $\mathscr{R}$ does not have non-zero maximal submodules as an $\mathscr{R}$-module. Any vector space is a VM, but an infinite dimensional vector space has infinite number of maximal submodules. So it is not necessary that each valuation module has a (unique) maximal submodule.

Theorem 3.1. Let M be a valuation $\mathscr{R}$-module. Then the following statements are true.

1) For any submodule N of M, such that $\frac{M}{N}$ is a torsionfree $\mathscr{R}$-module, $\frac{M}{N}$ is a (VM).

2) If $M$is finitely generated, then for each $p \in Spec(R)$, M_p is a valuation R_p-module.

3) If M' is a torsionfree $\mathscr{R}$-module and $\varphi : M \longrightarrow M'$ is an epimorphism, then M' is a valuation module too.

The following give the relations between valuation rings and valuation modules.

Lemma 3.3. Let $\mathscr{R}$ be a valuation ring and M a torsionfree $\mathscr{R}$-module. Then M is a valuation $\mathscr{R}$-module.

Lemma 3.4. If M is a multiplication valuation $\mathscr{R}$-module, then M is finitely generated and $\mathscr{R}$ is a valuation ring.

Lemma 3.5. Let $\mathscr{R}$ be a valuation domain. Then every finitely generated torsion-free $\mathscr{R}$-module is free.

Lemma 3.6. Let $\mathscr{R}$ be a domain. Then $\mathscr{R}$ is a valuation ring if and only if every free $\mathscr{R}$-module is a valuation module.

Corollary 3.3. Let M be a multiplication valuation module over an integral domain $\mathscr{R}$. Then M is isomorphic to $\mathscr{R}$.

An element u of an $\mathscr{R}$-module M is said to be unit provided that u is not contained in any maximal submodule of M. In a multiplication $\mathscr{R}$-module M, $u \in M$ is unit if and only if $M = Ru$.

Theorem 3.2. Let $\mathscr{R}$ be a local ring (not necessarily an integral domain) with unique principal maximal ideal $I = (p)$ and M a multiplication $\mathscr{R}$-module such that

$$\cap_{n=1}^{\infty} (p^n) M = (0).$$

Then the only proper submodules of $M$are (0) and $(p^m) M$, for some $m \geq 1$. Furthermore, if M is faithful, then either $p$is nilpotent or M is a valuation module.

Theorem 3.3. Let M be a finitely generated module over an integrally closed ring $\mathscr{R}$. If M is a valuation module, then M is a free $\mathscr{R}$- module and $\mathscr{R}$ is a valuation ring.

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