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Some characterizations of constant breadth spacelike curves in Minkowski 4-space E_1^4

Huseyin Kocayiğit and Zennure Çiçek

Celal Bayar University, Faculty of Sciences, Department of Mathematics, Manisa, Turkey

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Abstract: In this study, the differential equation characterizations of constant breadth spacelike curves are given in the Minkowski 4-space E_1^4. Furthermore, a criterion for a spacelike curve to be a curve of constant breadth in E_1^4 is introduced. As an example, the obtained results are applied to the case that the curvatures k_1, k_2, k_3 are constants and are discussed.

Keywords: Constant breadth curve; spacelike curve; Frenet frame.

1 Introduction

Euler introduced the constant breadth curves in 1778 [6]. He considered these special curves in the plane. Later, many mathematicians have studied these curves. Struik published a brief review of the most important publications on this subject [18]. Also, Ball [1], Barbier [2], Blaschke [3,4] and Mellish [12] investigated the properties of plane curves of constant breadth. Fujiwara obtained a space curve of constant breadth by taking a closed curve whose normal plane at a point P has only one more point Q in common with the curve, and for which the distance $d(P,Q)$ is constant [7]. He also defined and studied constant breadth surfaces. Later, Smakal studied the constant breadth space curves [17]. Furthermore, Blaschke considered the notion of curve of constant breadth on the sphere [4]. Moreover, Reuleaux studied the curves of constant breadth and gave the method related to these curves for the kinematics of machinery [14]. Then, constant breadth curves had an importance for engineering sciences and Tanaka used the constant breadth curves in the kinematics design of Com follower systems [19].

Moreover, Köse has presented some concepts for constant breadth space curves in Euclidean 3-space in [10] and Sezer has obtained the differential equations characterizing constant breadth space curves and introduced a criterion for these curves [16]. Constant breadth curves were investigated by Mağden and Köse in Euclidean 4-space [11]. Moreover, constant breath curves have been studied in Minkowski space. Kazaz, Önder and Kocayiğit have studied spacelike curves of constant breadth in Minkowski 4-space [8]. Önder, Kocayiğit and Candan have obtained and studied the differential equations characterizing constant breadth curves in Minkowski 3-space [13]. Furthermore, Kocayiğit and Önder have showed that constant breadth spacelike curves are helices, normal curves and spherical curves in some special cases in Minkowski 3-space [9].

In this study, we give differential equations characterizing spacelike curves of constant breadth in the Minkowski 4-space E_1^4. Furthermore, we give a criterion characterizing these curves in E_1^4.

* Corresponding author e-mail: huseyin.kocayigit@cbu.edu.tr; zennurecicek@hotmail.com

2 Differential Equations Characterizing Constant Breadth Spacelike Curves in E_1^4

Let (C) be a unit speed regular spacelike curve in the Minkowski 4-space E_1^4 with parametrization $\alpha : I \subset IR \to E_1^4$. Denote by $\{\mathbf{T},\ \mathbf{N},\ \mathbf{B},\mathbf{E}\}$ the moving Frenet frame along the spacelike curve (C) in E_1^4. Then, we can give following Frenet formulae,

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \\ \mathbf{E}' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -\varepsilon_1 k_1 & 0 & k_2 & 0 \\ 0 & \varepsilon_2 k_2 & 0 & k_3 \\ 0 & 0 & \varepsilon_1 k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \\ \mathbf{E} \end{bmatrix}$$

where k_1, k_2 and k_3 express the first, second and third curvatures of the curve (C), respectively and $\mathbf{T},\ \mathbf{N},\ \mathbf{B},\mathbf{E}$ express tangent, principal normal, first binormal and second binormal, respectively and they satisfy the following equalities [20].

$$\begin{aligned} & g(\mathbf{T},\mathbf{T}) = 1,\ g(\mathbf{N},\mathbf{N}) = \varepsilon_1, g(\mathbf{E},\mathbf{E}) = \varepsilon_2, g(\mathbf{B},\mathbf{B}) = -\varepsilon_1\varepsilon_2\,, \\ & g(\mathbf{T},\mathbf{N}) = g(\mathbf{T},\mathbf{B}) = g(\mathbf{T},\mathbf{E}) = g(\mathbf{N},\mathbf{B}) = 0 \\ & \varepsilon_1 = \varepsilon_2 = \mp 1 \end{aligned}$$

where $\langle , \rangle$ is the Lorentzian inner product defined by

$$\langle \mathbf{a}, \mathbf{b} \rangle = -a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4,$$

here $\mathbf{a} = (a_1, a_2, a_3, a_4)$, $\mathbf{b} = (b_1, b_2, b_3, b_4)$ are the vectors in E_1^4.

Definition 2.1. Let (C) be a unit speed regular spacelike curve in E_1^4, and let $\alpha(s)$ position vector of the curve (C). If the curve (C) has parallel tangents $\mathbf{T}$ and $\mathbf{T}^*$ in opposite direction at the opposite points $\alpha(s)$ and $\alpha^*(s)$ of the curve and the distance between opposite points is always constant then the curve (C) is named a spacelike curve of constant breadth in E_1^4. Furthermore, a pair of spacelike curves (C) and (C^*), for which the tangent vectors at the corresponding points are in opposite directions and parallel, and the distance between corresponding points is always constant, is called a spacelike curve pair of constant breadth in E_1^4.

We suppose that (C) and (C^*) be a pair of unit speed spacelike curves in E_1^4 with position vectors $\alpha(s)$ and $\alpha^*(s^*)$, where s and s^* are arc length parameters of the curves, respectively, and let (C) and (C^*) have parallel tangents in opposite directions at the opposite points. Then the curve (C^*) may be represented by the equation

$$\alpha^*(s) = \alpha(s) + m_1(s)\mathbf{T}(s) + m_2(s)\mathbf{N}(s) + m_3(s)\mathbf{B}(s) + m_4(s)\mathbf{E}(s) \tag{1}$$

where $m_i(s)$, $(1 \le i \le 4)$ are differentiable functions of s. Differentiating Eq.(1) with respect to s and using the Frenet formulae we gain

$$\begin{aligned} \frac{d\alpha^*}{ds} = \mathbf{T}^* \frac{ds^*}{ds} = & \left(1 + \frac{dm_1}{ds} - \varepsilon_1 m_2 k_1\right)\mathbf{T} + \left(m_1 k_1 + \frac{dm_2}{ds} + \varepsilon_2 m_3 k_2\right)\mathbf{N} \\ & + \left(m_2 k_2 + \frac{dm_3}{ds} + \varepsilon_1 k_3 m_4\right)\mathbf{B} + \left(m_3 k_3 + \frac{dm_4}{ds}\right)E \end{aligned}$$

Since $\mathbf{T} = -\mathbf{T}^*$ at the corresponding points of (C) and (C^*), we have

$$\begin{cases} 1 + \frac{dm_1}{ds} - \varepsilon_1 m_2 k_1 = -\frac{ds^*}{ds}, \\ m_1 k_1 + \frac{dm_2}{ds} + \varepsilon_2 m_3 k_2 = 0, \\ m_2 k_2 + \frac{dm_3}{ds} + \varepsilon_1 m_4 k_3 = 0 \\ m_3 k_3 + \frac{dm_4}{ds} = 0 \end{cases} \tag{2}$$

We know that the curvature of the curve (C) is $\lim(\Delta\varphi/\Delta s) = (d\varphi/ds) = k_1(s)$, where $\varphi = \int_0^s k_1(s)ds$ is the angle between tangent vectors of the curve (C) and a given fixed direction at the point $\alpha(s)$. Then from (2) we have the following system

$$m_1' = \varepsilon_1 m_2 - f(\phi),\ m_2' = -m_3\varepsilon_2\rho k_2 - m_1,\ m_3' = -m_4\varepsilon_1\rho k_3 - m_2\rho k_2,\ m_4\prime = -m_3\rho k_3. \tag{3}$$

Here and after we will use $(')$ to show the differentiation with respect to φ. In (3), $f(\varphi) = \rho + \rho^*$ and, $\rho = \frac{1}{k_1}$ and $\rho^* = \frac{1}{k_1^*}$ are the radius of curvatures at the points $\alpha(s)$ and $\alpha^*(s^*)$, respectively. From (3) eliminating $m_2,\ m_3,\ m_4$ and their derivatives we obtain following differential equation

$$\begin{array}{l} \frac{d}{d\varphi}\left[\frac{\varepsilon_2}{\rho k_3}\frac{d}{d\varphi}\left[\frac{1}{\rho k_2}\left(\varepsilon_1\frac{d^2m_1}{d\varphi^2}+m_1\right)\right]-\frac{k_2}{k_3}\frac{dm_1}{d\varphi}\right]-\varepsilon_2\frac{k_3}{k_2}\left(\varepsilon_1\frac{d^2m_1}{d\varphi^2}+m_1\right) \\ +\frac{d}{d\varphi}\left[\frac{\varepsilon_2}{\rho k_3}\frac{d}{d\varphi}\left(\frac{1}{\rho k_2}\varepsilon_1\frac{df}{d\varphi}\right)-\frac{k_2}{k_3}f\right]-\varepsilon_2\frac{k_3}{k_2}\varepsilon_1\frac{df}{d\varphi}=0 \end{array} \tag{4}$$

Then, the following theorem can be given.

Theorem 2.1. The general differential equation characterizing constant breadth spacelike curves in E_1^4 is given by (4). Let now consider the system (3) again. The distance d between the opposite points is constant. Then we can write following equality.

$$d^2 = \|\mathbf{d}\|^2 = \|\alpha^* - \alpha\|^2 = m_1^2 + \varepsilon_1 m_2^2 - \varepsilon_1\varepsilon_2 m_3^2 + \varepsilon_2 m_4^2 = const. \tag{5}$$

In addition, the system (3) can be written as follows:

$$f(\varphi) = \varepsilon_1 m_2,\ m_2' = -m_3\varepsilon_2\rho k_2,\ m_3' = -\varepsilon_1 m_4\rho k_3 - m_2\rho k_2,\ m_4' = -m_3\rho k_3,\ m_1 = 0, \tag{6}$$

or

$$m_1' = \varepsilon_1 m_2,\ m_2' = -m_1 - m_3\varepsilon_2\rho k_2,\ m_3' = -m_4\varepsilon_1\rho k_3 - m_2\rho k_2,\ m_4\prime = -m_3\rho k_3, \tag{7}$$

such that, the system (7) describes the curve (1).

Let us consider the system (7) with special chosen $m_1 = const$. Here, eliminating first $m_1,\ m_2,\ m_3$ and their derivatives, and then $m_1,\ m_2,\ m_4$ and their derivatives, respectively, we gain following linear differential equations of second order

$$\begin{cases} (\rho k_3)m_4\prime\prime - (\rho k_3)' m_4' - \varepsilon_1(\rho k_3)^3 m_4 = 0,\ \rho k_2 \neq 0, \\ (\rho k_3)m_3\prime\prime - (\rho k_3)\prime m_3' - \varepsilon_1(\rho k_3)^3 m_3 = 0,\ \rho k_3 \neq 0. \end{cases} \tag{8}$$

By changing the variable φ of the form $\xi = \int_0^\varphi \rho(t)k_3(t)dt$, Equations given by (8) can be transformed the differential equations with constant coefficients as follows:

$$\frac{d^2m_4}{d\xi^2} - \varepsilon_1 m_4 = 0 \quad \text{and} \quad \frac{d^2m_3}{d\xi^2} - \varepsilon_1 m_3 = 0 \tag{9}$$

respectively. If $\varepsilon_1 = 1$. Then, the general solutions of differential equations (9) are

$$\begin{cases} m_3 = A\cosh\left(\int_0^\varphi \rho k_3 dt\right) + B\sinh\left(\int_0^\varphi \rho k_3 dt\right), \\ m_4 = C\cosh\left(\int_0^\varphi \rho k_3 dt\right) + D\sinh\left(\int_0^\varphi \rho k_3 dt\right) \end{cases} \tag{10}$$

respectively, where A,B,C and D are real constants. Substituting (10) into (7), we have $A=-D,\ B=-C$. Thus, the solution set of the system (7), in the form

$$\left\{\begin{array}{l} m_1=c=const.,\ m_2=0, \\ m_3=A\cosh\int_0^{\varphi}\rho k_3dt+B\sinh\int_0^{\varphi}\rho k_3dt, \\ m_4=-B\cosh\int_0^{\varphi}\rho k_3dt-A\sinh\int_0^{\varphi}\rho k_3dt \end{array}\right\} \tag{11}$$

$d^2=\|\alpha^*-\alpha\|^2=const.$,so from (11) the breadth of the curve is $k^2=c^2-\varepsilon_2A^2+\varepsilon_2B^2$. If $\varepsilon_1=-1$. Then, the general solutions of differential equations (9) are

$$\left\{\begin{array}{l} m_3=A\cos\left(\int_0^{\varphi}\rho k_3dt\right)+B\sin\left(\int_0^{\varphi}\rho k_3dt\right), \\ m_4=C\cos\left(\int_0^{\varphi}\rho k_3dt\right)+D\sin\left(\int_0^{\varphi}\rho k_3dt\right) \end{array}\right. \tag{12}$$

respectively, where A,B,C and D are real constants. Substituting (12) into (7), we gain $A=D,\ B=C$. Thus, the solution set of the system (7) can be written as

$$\left\{\begin{array}{l} m_1=c=const.,\ m_2=0, \\ m_3=A\cos\int_0^{\varphi}\rho k_3dt+B\sin\int_0^{\varphi}\rho k_3dt, \\ m_4=B\cos\int_0^{\varphi}\rho k_3dt-A\sin\int_0^{\varphi}\rho k_3dt \end{array}\right\} \tag{13}$$

$d^2=\|\alpha^*-\alpha\|^2=const.$,so from (13) the breadth of the curve is $k^2=c^2+\varepsilon_2A^2+\varepsilon_2B^2$.

Now, let us return to the system (6) with $m_1=0$. By changing the variable φ of the form $u=\int_0^{\varphi}\mu(t)\,dt$, $\mu=\rho k_3$ and eliminating $m_1,\ m_2,\ m_4$ and their derivatives we obtain the following linear differential equation

$$\frac{d^2m_3}{du^2}-\varepsilon_1m_3=-\frac{d}{du}\left(\frac{k_2}{k_3}m_2\right), \tag{14}$$

If $\varepsilon_1=1$, then we have following solution

$$m_3=A_1\cosh\int_0^{\varphi}\rho k_3dt+B_1\sinh\int_0^{\varphi}\rho k_3dt-\int_0^{\varphi}\cosh\left[u(\varphi)-u(t)\right]\rho k_2f(t)dt. \tag{15}$$

If $\varepsilon=-1$, then we have following solution

$$m_3=A_1\cos\int_0^{\varphi}\rho k_3dt+B_1\sin\int_0^{\varphi}\rho k_3dt+\int_0^{\varphi}\cos\left[u(\varphi)-u(t)\right]\rho k_2f(t)dt \tag{16}$$

Then, the general solution of the system (6) is

$$\begin{cases} m_1 = 0, \\ m_2 = \varepsilon_1 f(\varphi), (\varepsilon_1 = \mp 1) \\ m_3 = A_1 \cosh \int_0^{\varphi} \rho k_3 dt + B_1 \sinh \int_0^{\varphi} \rho k_3 dt - \int_0^{\varphi} \cosh [u(\varphi) - u(t)] \rho k_2 f(t) dt, (\varepsilon_1 = 1) \\ m_3 = A_1 \cos \int_0^{\varphi} \rho k_3 dt + B_1 \sin \int_0^{\varphi} \rho k_3 dt + \int_0^{\varphi} \cos [u(\varphi) - u(t)] \rho k_2 f(t) dt, (\varepsilon_1 = -1) \\ m_4 = -B_1 \cosh \int_0^{\varphi} \rho k_3 dt - A_1 \sinh \int_0^{\varphi} \rho k_3 dt + \int_0^{\varphi} \sinh [u(\varphi) - u(t)] \rho k_2 f(t) dt, (\varepsilon_1 = 1) \\ m_4 = B_1 \cos \int_0^{\varphi} \rho k_3 dt - A_1 \sin \int_0^{\varphi} \rho k_3 dt - \int_0^{\varphi} \sin [u(\varphi) - u(t)] \rho k_2 f(t) dt \, (\varepsilon_1 = -1), \end{cases} \tag{17}$$

which determines the constant breadth spacelike curve in (1) where A_1, B_1 are real constants.

Furthermore, in this case, i.e.,$m_1 = 0$, from (4) we can write following differential equation

$$\frac{d}{d\varphi} \left[\frac{\varepsilon_2}{\rho k_3} \frac{d}{d\varphi} \left(\frac{1}{\rho k_2} \frac{df}{d\varphi} \right) - \frac{k_2}{k_3} f \right] - \varepsilon_1 \varepsilon_2 \frac{k_3}{k_2} \frac{df}{d\varphi} = 0. \tag{18}$$

By changing the variable φ of the form $w = \int_0^{\varphi} \rho k_2 d\varphi$, (18) becomes

$$\frac{d}{dw} \left[\frac{k_2}{k_3} \varepsilon_1 \left(\frac{d^2 f}{dw^2} - \varepsilon_2 f \right) \right] - \frac{k_3}{k_2} \frac{df}{dw} = 0, \tag{19}$$

which also determines the constant breadth curve in (1).

On the other hand let us consider a unit speed simple closed spacelike curve (C) in E_1^4 for which the normal plane of every point P on the curve meets the curve of a single opposite point Q other than P. Then, we can give following theorem related to spacelike curves of constant breadth in E_1^4.

Theorem 2.2. Let (C) be a closed spacelike curve in E_1^4 having parallel tangents in opposite directions at the opposite points of the curve. The chord joining the opposite points of (C) is a double-normal if and only if (C) is a constant breadth spacelike curve in E_1^4.

Proof: Let the vector $\mathbf{d} = \alpha^* - \alpha = m_1 \mathbf{T} + m_2 \mathbf{N} + m_3 \mathbf{B} + m_4 \mathbf{E}$ be a double-normal of (C) where m_1, m_2, m_3 and m_4 are the functions of s, the arc length parameter of the curve. Then we get $\langle \mathbf{d}, \mathbf{T}^* \rangle = -\langle \mathbf{d}, \mathbf{T} \rangle = m_1 = 0$. Thus from (2) we have

$$m_2 \frac{dm_2}{ds} + m_3 \frac{dm_3}{ds} + m_4 \frac{dm_4}{ds} = 0. \tag{20}$$

It follows that $\varepsilon_1 m_2^2 - \varepsilon_1 \varepsilon_2 m_3^2 + \varepsilon_2 m_4^2 = const.$

Conversely, if the curve (C) is a constant breadth spacelike curve in E_1^4 then $\|\mathbf{d}\|^2 = m_1^2 + \varepsilon_1 m_2^2 - \varepsilon_1 \varepsilon_2 m_3^2 + \varepsilon_2 m_4^2 = const.$ Then as shown, $m_1 = 0$. This means that $\mathbf{d}$ is perpendicular to $\mathbf{T}$ and $\mathbf{T}^*$. So, $\mathbf{d}$ is double normal of the curve (C).

A simple closed spacelike curve having parallel tangents in opposite directions at opposite points can be expressed by the system (17). In this state, a pair of opposite points of the curve is $(\alpha^*(\varphi), \alpha(\varphi))$ for φ, where $0 \leq \varphi \leq 2\pi$. Since the curve (C) is a simple closed spacelike curve we get $\alpha^*(0)=\alpha^*(2\pi)$. Hence from (14) we have

$$\int_0^{2\pi} \rho k_3 dt = 2n\pi, \ \ (n\in \mathrm{Z}). \tag{21}$$

From the equality $ds=\rho d\varphi$, this formula can be written as $\int_C k_3 ds = 2n\pi, \ \ (n\in \mathrm{Z})$. This says that the third integral curvature of the curve (C) is zero. So, we can write following corollary.

Corollary 2.1. The total third curvature of a simple closed spacelike curve (C) of constant breadth is $2n\pi, \ \ (n\in \mathrm{Z})$.

Furthermore, if we take $\frac{k_2}{k_3}=a=constant$, then from (16) we have

$$\frac{d^3 f}{dw^3} - K\frac{df}{dw} = 0,. \tag{22}$$

where $K=\varepsilon_2+\frac{1}{\varepsilon_1 a^2}$. If we assume $K\neq \pm 1$, the general solution of (19) is

$$f = A_2 \sinh \int_0^{\varphi} K\rho k_2 dt + B_2 \cosh \int_0^{\varphi} K\rho k_2 dt + C_1, \tag{23}$$

where A_2, B_2 and C_1 are real constants. Since the curve (C) is a simple closed spacelike curve, i.e., $\alpha^*(0)=\alpha^*(2\pi)$, from (23) it follows,

$$\int_0^{\varphi} K\rho k_2 dt = 2n\pi, \ \ (n\in \mathrm{Z}). \tag{24}$$

Using the equality $ds=\rho d\varphi$, this formula may be given as $\int_C k_2 ds = 2\frac{n}{K}\pi, \ \ (K,n\in \mathrm{Z})$. This says that the second integral curvature of the curve (C) is $2\frac{n}{K}\pi, \ \ (K,n\in \mathrm{Z})$. Then, we can give the following corollary.

Corollary 2.2. The total second curvature of a simple closed spacelike curve (C) of constant breadth with $a=k_2/k_3=constant$ is $2\frac{n}{K}\pi$, where $n\in \mathrm{Z}$ and $K=\varepsilon_2+\frac{1}{\varepsilon_1 a^2}$.

3 A Criterion for Constant Breadth Spacelike Curves in E_1^4

Let us suppose that the curve (C) is a constant breadth spacelike curve in E_1^4 and $\alpha(s)$ denotes the position vector of the curve. If (C) is a closed curve, then the position vector $\alpha(s)$ must be a periodic function of period $\omega=2\pi$, where ω is the total length of (C). Then the curvatures $k_1(s)$, $k_2(s)$ and $k_3(s)$ are also periodic of the same period. However, periodicity of the curvatures and closeness of the curve are not sufficient to guarantee that a spacelike space curve is a constant breadth curve in E_1^4. That is, if a spacelike curve is closed curve (periodic), it may be the constant breadth curve or not. Therefore, to guarantee that a spacelike curve is a constant breadth curve, we may use the system (7) characterizing a constant breadth spacelike curve and follow the similar way given in [5].

For this purpose, first we condider the following Frenet formulas at a generic point on the curve (C),

$$\frac{dT}{ds}=k_1\mathbf{N}, \ \frac{d\mathbf{N}}{ds}=-\varepsilon_1 k_1 T + k_2\mathbf{B}, \ \frac{d\mathbf{B}}{ds}=\varepsilon_2 k_2\mathbf{N}+k_3 E, \ \frac{dE}{ds}=-k_3\varepsilon_1 B. \tag{25}$$

By expressing the formula (22) in terms of φ and allowing for $\frac{d\varphi}{ds} = k_1 = \frac{1}{\rho}$ we have

$$\frac{dT}{d\varphi} = \mathbf{N},\ \frac{d\mathbf{N}}{d\varphi} = -\varepsilon_1 T + \rho k_2 \mathbf{B},\ \frac{d\mathbf{B}}{d\varphi} = \varepsilon_2 \rho k_2 \mathbf{N} + \rho k_3 E,\ \frac{dE}{d\varphi} = -\rho k_3 B. \tag{26}$$

Moreover we can write the Frenet vectors $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, $\mathbf{E}$ in the coordinate forms as

$$T = \sum_{i=1}^{4} t_i \mathbf{e}_i,\ N = \sum_{i=1}^{4} n_i \mathbf{e}_i,\ B = \sum_{i=1}^{4} b_i e_i,\ \mathbf{E} = \sum_{i=0}^{4} \hat{e}_i e_i. \tag{27}$$

$\{\mathbf{T}, \mathbf{N}, \mathbf{B}, \mathbf{E}\}$is the orthonormal base in E_1^4. So, substituting (27) and their derivatives into (26), we get the systems of linear differential equations

$$\begin{cases} \frac{dt_1}{d\varphi} = n_1, \quad \frac{dt_2}{d\varphi} = n_2, \quad \frac{dt_3}{d\varphi} = n_3, \frac{dt_4}{d\varphi} = n_4 \\ \frac{dn_1}{d\varphi} = -\varepsilon_1 t_1 + \rho k_2 b_1,\ \frac{dn_2}{d\varphi} = -\varepsilon_1 t_2 + \rho k_2 b_2,\ \frac{dn_3}{d\varphi} = -\varepsilon_1 t_3 + \rho k_2 b_3, \frac{dn_4}{d\varphi} = -\varepsilon_1 t_4 + \rho k_2 b_4 \\ \frac{db_1}{d\varphi} = \rho k_3 \hat{e}_1 + \varepsilon_2 \rho k_2 n_1,\ \frac{db_2}{d\varphi} = \rho k_3 \hat{e}_2 + \varepsilon_2 \rho k_2 n_2,\ \frac{db_3}{d\varphi} = \rho k_3 \hat{e}_3 + \varepsilon_2 \rho k_2 n_3, \frac{db_4}{d\varphi} = \rho k_3 \hat{e}_4 + \varepsilon_2 \rho k_2 n_4 \\ \frac{d\varepsilon_1}{d\varphi} = -\rho k_3 b_1, \frac{d\varepsilon_2}{d\varphi} = -\rho k_3 b_2, \frac{d\varepsilon_3}{d\varphi} = -\rho k_3 b_3, \frac{d\varepsilon_4}{d\varphi} = -\rho k_3 b_4 \end{cases} \tag{28}$$

From (28), we find that $\{t_1, n_1, b_1, \hat{e}_1\}$, $\{t_2, n_2, b_2, \hat{e}_2\}$, $\{t_3, n_3, b_3, \hat{e}_3\}$, $\{t_4, n_4, b_4, \hat{e}_4\}$ are four independent solutions of the following system of differential equations:

$$\frac{d\psi_1}{d\varphi} = \psi_2,\ \frac{d\psi_2}{d\varphi} = -\psi_1 + \rho k_2 \psi_3,\ \frac{d\psi_3}{d\varphi} = \rho k_3 \psi_4 - \rho k_2 \psi_2, \frac{d\psi_4}{d\varphi} = -\rho k_3 \psi_3. \tag{29}$$

If the curve (C) is the constant breadth spacelike curve, then the systems (7) and (29) must be the same system. So, we observe that $\psi_1 = m_1,\ \psi_2 = m_2,\ \psi_3 = m_3,\ \psi_4 = m_4$. For brevity, we can express (7) or (29) in the form

$$\frac{d\psi}{d\varphi} = A(\varphi)\psi, \tag{30}$$

where

$$\psi = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix},\ A(\varphi) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\varepsilon_1 & 0 & \rho k_2 & 0 \\ 0 & \varepsilon_2 \rho k_2 & 0 & \rho k_3 \\ 0 & 0 & \varepsilon_1 \rho k_3 & 0 \end{bmatrix}.$$

Obviously, (30) is a special case of the general linear differential equations shortened to the form

$$\begin{cases} \frac{d\psi}{dt} = A(t)\psi, \\ \varphi = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{bmatrix},\ A(t) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, (4 \le n) \end{cases} \tag{31}$$

where $a_{ij}(t)$ are assumed to be periodic and continuous of period ω (See [5,15]). Let the initial conditions be $\psi_i(0) = x_i,\ (i = 1,2,...,n)$. Let us take $x = [x_1, x_2, ..., x_n]^T$ and

$$\psi(t,x) = [m_1(t,x), m_2(t,x), \ldots m_n(t,x)]^T.$$

Then the equation (31) may be written in the form $\frac{d\psi}{dt}=A(t)\,\psi,\quad \psi(0)=x$ as is well known from [5], the solution $\psi(t,x)$ of this equation is periodic of period ω, if

$$\int_0^{\omega} A(\xi)\,\psi(\xi,x)\,d\xi=0,$$

and

$$\begin{cases} \psi(t,x) & =\{E+M(t)\}x,\ \ (E=unit\ matrix), \\ M(t) & =IA(t)+I^{(2)}A(t)+\cdots+I^{(n)}A(t)+\cdots, \\ (IA)(t) & \equiv I^{(I)}A(t)=\int_0^t A(\xi)\,d\xi \\ \left(I^{(n)}A\right)(t)=\int_0^t A(\xi)\left(I^{(n-1)}A\right)(\xi)\,d\xi,\ \ n>1. \end{cases} \tag{32}$$

Moreover, the following theorem can be given in [5]:

Theorem 3.1. The equations $\frac{d\psi}{dt}=A(t)\,\psi$ possess a non-vanishing periodic solution of period ω, if and only if $\det(M(\omega))=0$. In particular, in order that the equations $\frac{d\psi}{dt}=A(t)\,\psi$ possess n linearly independent periodic solutions of period ω, the necessary and sufficient condition is that $M(\omega)$ be a zero matrix.

Now, let us apply this theorem to the system (30). If $M(\omega)=0$, there exist the unit vector functions $\mathbf{T},\mathbf{N},\mathbf{B},\mathbf{E}$ of period ω, such that each set of functions $\{t_i,n_i,b_i,\varepsilon_i\},\ \ (i=1,2,3,4)$ form a solution of the equation (30) corresponding to the initial conditions (A_i,B_i,C_i,D_i). The curve (C) can be described as follows

$$\alpha(s)=\int_0^s \mathbf{T}(s)ds \quad or \quad \alpha(\varphi)=\int_0^{\varphi}\rho(\varphi)\mathbf{T}(\varphi)d\varphi.$$

Here, in order to find $\mathbf{T}$, we can use the equation

$$\begin{bmatrix} t_i \\ n_i \\ b_i \\ \varepsilon_i \end{bmatrix}=\{E+M(\varphi)\}\begin{bmatrix} A_i \\ B_i \\ C_i \\ D_i \end{bmatrix},\ \ (i=1,2,3,4), \tag{33}$$

which is established by (29). If we take the initial conditions as $t_i(0)=A_i,\ n_i(0)=B_i,$ $b_i(0)=C_i,\ \varepsilon_i=D_i,\ \ (i=1,2,3,4)$ such that (A_1,A_2,A_3,A_4), (B_1,B_2,B_3,B_4), (C_1,C_2,C_3,C_4), (D_1,D_2,D_3,D_4) form an orthonormal frame, then from (30) we gain

$$t_i=(1+m_{11})A_i+m_{12}B_i+m_{13}C_i,\ m_{14}D_i;\quad (i=1,2,3,4). \tag{34}$$

As the spacelike curve (C) is a constant breadth curve, which is also periodic of period ω, it is clear that

$$\int_0^{\omega}\rho t_i d\varphi=0. \tag{35}$$

Hence, form (34) and (35), we have

$$A_i \int_0^{\omega} \rho\,(1+m_{11})\,d\varphi + B_i \int_0^{\omega} \rho m_{12} d\varphi + C_i \int_0^{\omega} \rho m_{13} d\varphi + D_i \int_0^{\omega} \rho m_{14} d\varphi = 0; \quad (i = 1,2,3,4).$$

Since the coefficient determinant $\Delta \neq 0$ in this system, we have the equalities as

$$\int_0^{\omega} \rho\,(1+m_{11})\,d\varphi = 0 = \int_0^{\omega} \rho m_{12} d\varphi = \int_0^{\omega} \rho m_{13} d\varphi = \int_0^{\omega} \rho m_{14} d\varphi, \tag{36}$$

which are the conditions for a spacelike curve to be constant breadth curve in E_1^4. Here, we can take the period $\omega = 2\pi$ because of $0 \leq \varphi \leq 2\pi$. Thus we obtain the following corollary.

Corollary 3.1. Let (C) be a regular curve in E_1^4 such that $\rho(\varphi) > 0, k_2(\varphi)$ and $k_3(\varphi)$ are continuous periodic functions of period ω. Then (C) is a constant breadth spacelike curve and also periodic of period ω, if and only if

$$M(\omega) = 0, \quad \int_0^{\omega} \rho\,(1+m_{11})\,d\varphi = 0 = \int_0^{\omega} \rho m_{12} d\varphi = \int_0^{\omega} \rho m_{13} d\varphi = \int_0^{\omega} \rho m_{14} d\varphi, \tag{37}$$

holds, where

$$\begin{cases} M(t) = IA(t) + I^{(2)}A(t) + \cdots + I^{(n)}A(t) + \cdots, \\ A(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\varepsilon_1 & 0 & \rho k_2 & 0 \\ 0 & \varepsilon_2 \rho k_2 & 0 & \rho k_3 \\ 0 & 0 & \varepsilon_1 \rho k_3 & 0 \end{bmatrix} \end{cases} \tag{38}$$

and $m_{ij}(t)$ are the entries of the matrix$M(t)$. From (32) and (38), the matrix $M(t)$ can be constructed and each m_{ij} involves infinitely many integrations. Thus, we can give the conditions (37) in the following forms:

$$\begin{cases} \int_0^{\omega} \rho(\varphi)d\varphi - \int_0^{\omega}\int_0^{r}\int_0^{s} \varepsilon_1 \rho(\varphi) dsdtd\varphi + \int_0^{\omega}\int_0^{\phi}\int_0^{p}\int_0^{r}\int_0^{s} \rho(\varphi)\varepsilon_1 [\varepsilon_1 - \varepsilon_2 \lambda(p)\lambda(s)]\, dtdsdrdpd\varphi - \cdots = 0 \\ \int_0^{\omega}\int_0^{s} \rho(\varphi)dtd\varphi - \int_0^{\omega}\int_0^{p}\int_0^{r}\int_0^{s} \rho(\varphi)[\varepsilon_1 - \varepsilon_2 \lambda(t)\lambda(s)] dtdsdrd\varphi + \cdots = 0 \\ \int_0^{\omega}\int_0^{r}\int_0^{s} \rho(\varphi)\lambda(t)\, dtdsd\varphi \\ \qquad - \int_0^{\omega}\int_0^{\phi}\int_0^{p}\int_0^{r}\int_0^{s} \rho(\varphi)\,[\varepsilon_1 \lambda(t) - \varepsilon_2 \lambda(p)\{\lambda(t)\lambda(s) + \varepsilon_1 \mu(t)\mu(s)\}]\, dtdsdrdpd\varphi + \cdots = 0 \\ \int_0^{\omega}\int_0^{p}\int_0^{r}\int_0^{s} \rho(\varphi)\lambda(s)\mu(t)\, dtdsdrd\varphi \\ \qquad - \int_0^{\omega}\int_0^{q}\int_0^{\phi}\int_0^{p}\int_0^{r}\int_0^{s} \rho(\varphi)\lambda(p)\mu(t)[\varepsilon_1 - \varepsilon_2 \lambda(t)\lambda(s) - \varepsilon_1 \mu(t)\mu(s)]\, dtdsdpd\phi d\varphi dq + \ldots = 0 \end{cases} \tag{39}$$

where $\lambda(\xi) = \rho(\xi) k_2(\xi), \mu(\xi) = \rho(\xi)k_3(\xi)$.

Example 3.1. Let us consider the special case $\rho = const.$, $k_2 = const.$ and $k_3 = const.$ In this case, from (36), we have

$$\begin{cases} \omega - \varepsilon_1 \frac{\omega^3}{3!} + \varepsilon_1 \left(\varepsilon_1 - \varepsilon_2 \rho^2 k_2^2\right) \frac{\omega^5}{5!} - \varepsilon_1 \left(\varepsilon_1 - \varepsilon_2 \rho^2 k_2^2\right)^2 \frac{\omega^7}{7!} \ldots = 0 \\ \frac{\omega^2}{2!} - \left(\varepsilon_1 - \varepsilon_2 \rho^2 k_2^2\right) \frac{\omega^4}{4!} + \left(\varepsilon_1 - \varepsilon_2 \rho^2 k_2^2\right)^2 \frac{\omega^6}{6!} - \ldots = 0 \\ k_2 \left[\frac{\omega^3}{3!} - \left(\varepsilon_1 - \varepsilon_2 \rho^2 k_2^2 - \varepsilon_1 \rho^2 k_3^2\right) \frac{\omega^5}{5!} + \left(\varepsilon_1 - \varepsilon_2 \rho^2 k_2^2 - \varepsilon_1 \rho^2 k_3^2\right)^2 \frac{\omega^7}{7!} - \ldots = 0\right] \\ k_2 k_3 \left[\frac{\omega^4}{4!} - \left(\varepsilon_1 - \varepsilon_2 \rho^2 k_2^2 - \varepsilon_1 \rho^2 k_3^2\right) \frac{\omega^6}{6!} + \ldots = 0\right] \end{cases} \tag{40}$$

or

$$\begin{cases} \varepsilon_1 \varepsilon_2 \rho^2 k_2^2 \left(1 - \varepsilon_1 \varepsilon_2 \rho^2 k_2^2\right)^{\frac{1}{2}} \omega + \sin\left[\left(1 - \varepsilon_1 \varepsilon_2 \rho^2 k_2^2\right)^{\frac{1}{2}} \omega\right] = 0, \\ \cos\left[\left(\varepsilon_1 - \varepsilon_2 \rho^2 k_2^2\right)^{\frac{1}{2}} \omega\right] = 1 \quad or \quad \left(\varepsilon_1 - \varepsilon_2 \rho^2 k_2^2\right)^{\frac{1}{2}} \omega = 2k\pi, \quad (k \in Z), \\ k_2 \left[\left(\varepsilon_1 - \varepsilon_2 \rho^2 k_2^2 - \varepsilon_1 \rho^2 k_3^2\right)^{\frac{1}{2}} \omega - \sin\left[\left(\varepsilon_1 - \varepsilon_2 \rho^2 k_2^2 - \varepsilon_1 \rho^2 k_3^2\right)^{\frac{1}{2}} \omega\right]\right] = 0, \\ k_2 k_3 \left[\left[\left(\varepsilon_1 - \varepsilon_2 \rho^2 k_2^2 - \varepsilon_1 \rho^2 k_3^2\right)^{1/2}\right]^2 \frac{\omega^2}{2} - \sin^2\left[\left(\frac{\left(\varepsilon_1 - \varepsilon_2 \rho^2 k_2^2 - \varepsilon_1 \rho^2 k_3^2\right)^{\frac{1}{2}} \omega}{2}\right)\right]\right] \end{cases} \tag{41}$$

where $\omega = 2k\pi \; if \; \varepsilon_1 = 1$and $\omega = -2k\pi \; if \; \varepsilon_1 = -1$. It is seen that all of the equalities (40) or (41) are satisfied simultaneously, if and only if $\rho k_2 = 0$, $\rho k_3 = 0$, that is, $\rho = const. > 0$ and k_2, $k_3 = 0$. Therefore, only ones with $\rho = const. > 0$ and k_2, $k_3 = 0$ of the curves with $\rho = const. > 0$ and $k_2, k_3 = const.$ are curves of constant breadth, which are spacelike circles inE_1^4.

Now let us construct the relation characterizing these circles. Since $\rho k_2 = 0$, $\rho k_3 = 0$, system (7) becomes

$$m_1' = \varepsilon_1 m_2, \; m_2' = -m_1, \; m_3' = 0, \; m_4\prime = 0. \tag{42}$$

If $\varepsilon_1 = 1, \varepsilon_2 = \mp 1$ the general solution of (39) is

$$\begin{aligned} m_1 &= A\cos\varphi + B\sin\varphi \\ m_2 &= B\cos\varphi - A\sin\varphi \\ m_3 &= K \\ m_4 &= L \end{aligned} \tag{43}$$

Where A, B, K, L are arbitrary constants.

Consequently, replacing (43) into (1), we gain the equation

$$\alpha^*(\varphi) = \alpha(\varphi) + (A\cos\varphi + B\sin\varphi)\mathbf{T} + (-A\sin\varphi + B\cos\varphi)\mathbf{N} + K\mathbf{B} + L\mathbf{E},$$

which gives the constant distance $d = \|\alpha^* - \alpha\| = \left(\left|A^2 + B^2 \pm K^2 \mp L^2\right|\right)^{\frac{1}{2}}$. In this case, a pair of opposite points of the curve is $(\alpha^*(\varphi), \alpha(\varphi))$ for φ in$0 \le \varphi \le 2\pi$.

Since $\rho k_3 = 0, \; \rho k_2 = 0$ system (7) becomes

$$m_1' = \varepsilon_1 m_2, \; m_2' = -m_1, \; m_3' = 0, \; m_4\prime = 0$$

If $\varepsilon_1 = -1, \varepsilon_2 = +1$ the general solution of (39) is

$$\begin{aligned} m_1 &= A\cosh\varphi + B\sinh\varphi \\ m_2 &= -B\cosh\varphi - A\sinh\varphi \\ m_3 &= K \\ m_4 &= L \end{aligned} \tag{44}$$

Where A, B, E and F are arbitrary constants.

Consequently, replacing (44) into (1), we obtain the equation

$$\alpha^*(\varphi) = \alpha(\varphi) + (A\cosh\varphi + B\sinh\varphi)\,\mathbf{T} + (-A\sinh\varphi - B\cosh\varphi)\,\mathbf{N} + K\mathbf{B} + L\mathbf{E},$$

which represents the Spacelike circles with the diameter $d = \|\alpha^* - \alpha\| = \left(\left|A^2 - B^2 + K^2 + L^2\right|\right)^{\frac{1}{2}}$. In this case, a pair of opposite points of the curve is $(\alpha^*(\varphi), \alpha(\varphi))$ for φ in$0 \leq \varphi \leq 2\pi$.

4 4. Conclusion

The characterizations and determinations of the special curves or curve pairs are important in the curve theory. A differential equation or a system of differential equations with respect to the curvatures can determinate the special curves or curve pairs. In this paper, the differential equations characterizing the constant breadth spacelike curves are studied in E_1^4. Moreover, a criterion for a spacelike space curve to be the curve of constant breadth in E_1^4 is given.

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