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Strong uniform consistency rates of conditional quantiles for time series data in the single functional index model

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Abstract: The main objective of this paper is to estimate non-parametrically the quantiles of a conditional distribution when the sample is considered as an α-mixing sequence. First of all, a kernel type estimator for the conditional cumulative distribution function (*cond-cdf*) is introduced. Afterwards, we give an estimation of the quantiles by inverting this estimated *cond-cdf*, the asymptotic properties are stated when the observations are linked with a single-index structure. The pointwise almost complete convergence and the uniform almost complete convergence (with rate) of the kernel estimate of this model are established. This approach can be applied in time series analysis. For that, the whole observed time series has to be split into a set of functional data, and the functional conditional quantile approach can be employed both in foreseeing and building confidence prediction bands.

Keywords: Conditional quantile, conditional cumulative distribution, derivatives of conditional cumulative distribution, functional random variable, kernel estimator, nonparametric estimation, strong mixing processes.

1 Introduction

Estimating quantiles of any distribution is a substantial part of Statistics, it guaranties to build confidence ranges deriving many applications in numerous fields, chemistry, geophysics, medicine, meteorology,.... Furthermore, Statistics for functional random variables become progressively important, the latest literature in this domain presents the great potential of these functional statistical methods. The most famous case of functional random variable corresponds to the situation when we observe random curve on different statistical units. Such data are called *Functional Data*. Numerous multivariate statistical technics, mainly parametric in the functional model terminology, have been extended to functional data and good analysis on this area can be found in Ramsay and Silverman ([23] and [24]) or Bosq [5]. Lately, nonparametric methods considering functional variables have been grown with very interesting practical motivations dealing with environmetrics, (see Damon and Guillas [9], Fern´andez et al. *et al*. [10], Aneiros et al. [1]), chemometrics (see Ferraty and Vieu [14]), meteorological sciences (see Besse *et al*. [3], Hall and Heckman [22]), speech recognization problem (see Ferraty and Vieu [15]), radar range profile (see Hall et al. [21], Dabo-Niang *et al*. [8]), medical data (see Gasser *et al*. [20]). Moreover, forecasting techniques cover a big part of the statistical problems. Because a continuous time series can be seen as a sequence of dependent functional random variables, the above mentioned functional methodology can be used for time-series forecasting (see for instance Ferraty *et al*., [11], for a functional forecasting approach of time-series based on conditional expectation estimation). This article suggests to bring together the three former statistical aspects in order to derive a method for estimating conditional quantiles in situation when the data are both dependent and of functional nature. In particular, we focus on the nonparametric estimation of the conditional

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quantiles of a real random variable given a functional random variable under mixing assumption. We start by estimating the conditional distribution by means of a kernel estimator and we derive an estimate of the conditional quantiles (see Section 2). From a theoretical point of view, a crucial problem is linked with the so-called *curse of dimensionality*. Actually, in a nonparametric context, it is well known that the rate of convergence decreases with the dimension of the space in which the conditional variable is valued. But here, the conditional variable takes its values in an infinite dimensional space. One way to override this problem is to consider some concentration hypotheses acting on the distribution of the functional variable which allows to obtain asymptotic properties of our kernel estimates (see Section 3). This approach is used to derive a new method to forecast time series.

2 The model and the estimates

2.1 The functional nonparametric framework.

Consider a random pair (X,Y) where Y is valued in $\mathbb{R}$ and X is valued in some infinite dimensional Hilbertian space with scalar product $< .,. >$. Let $(X_i,Y_i)_{i=1\cdots n}$ be the statistical sample of pairs which are identically distributed like (X,Y), but not necessarily independent. Henceforward, X is called functional random variable *f.r.v.* Let x be fixed in $\mathscr{H}$ and let $F(\theta,\ y,\ x)$ be the conditional cumulative distribution function (*cond-cdf*) of Y given $<\theta,X>=<\theta,x>$, specifically:

$$\forall y\in\mathbb{R}\,,\quad F(\theta,\ y,\ x)\ =\mathbb{P}(Y\ \leq\ y|\ <X,\theta>=<x,\theta>).$$

Saying that, we are implicitly assuming the existence of a regular version for the conditional distribution of Y given $<\theta,X>$. Now, let t_γ be the γ-order quantile of the distribution of Y given $<\theta,X>=<\theta,x>$. From the *cond-cdf* $F(\theta,\ .,\ x)$, the general definition of the γ-order quantile is given as:

$$t_\theta(\gamma)\ =\ inf\{t\ \in\mathbb{R}\ :\ F\ (\theta,\ t,\ x)\ \geq\ \gamma\},\ \forall\gamma\ \in\ (0,\ 1).$$

In order to simplify our framework and to focus on the main interest of our paper (the functional feature of $<\theta,X>$), we assume that $F(\theta,\ .,\ x)$is strictly increasing and continuous in a neighborhood of t_γ . This is insuring that the conditional quantile t_γ is uniquely defined by:

$$t_\theta(\gamma)\ =\ F^{-1}(\theta,\ \gamma,\ x). \tag{1}$$

Next, in all what follows, we assume only smoothness restrictions for the *cond-cdf* $F\ (\theta,\ .,\ x)$through nonparametric modelling (Section 2.4). We suppose also that $(X_i,Y_i)_{i\in\mathbb{N}}$ is an α-mixing sequence, which is one among the most general mixing structures. The α-mixing condition together with the functional approach allow to deal with continuous time processes (see Section 4 for instance).

In our infinite dimensional purpose, we use the terminology *functional nonparametric*, where the word *functional* referees to the infinite dimensionality of the data and where the word *nonparametric* referees to the infinite dimensionality of the model. Such *functional nonparametric* statistics is also called doubly *infinite dimensional* (see Ferraty and Vieu [16], for more details). We also use the terminology *operatorial statistics* since the target object to be estimated (the *cond-cdf* $F(\theta,\ .,\ x)$) can be viewed as a nonlinear operator.

2.2 The estimators

The kernel estimator $\widehat{F}(\theta,\ .,\ x)$ of $F(\theta,\ .,\ x)$is presented as follows:

$$\widehat{F}\,(\theta,\ y,\ x)=\frac{\sum_{i=1}^{n}K\left(h_K^{-1}\,(<x-X_i,\theta>)\right)H(h_H^{-1}(y-Y_i))}{\sum_{i=1}^{n}K\left(h_K^{-1}\,(<x-X_i,\theta>)\right)}, \tag{2}$$

where K is a kernel function, $\mathscr{H}$ a cumulative distribution function and $h_K = h_{K,n}$ (resp. $h_H = h_{H,n}$) a sequence of positive real numbers. Note that using similar ideas, Roussas [26] introduced some related estimates but in the special case when X is real, while Samanta [27] produced previous asymptotic study. As a by-product of (2.1) and (2.2), it is easy to derive an estimator $\hat{t}_\gamma$ of t_γ:

$$\widehat{t_\theta}(\gamma) = \widehat{F}^{-1}(\theta, \gamma\,,\, x). \tag{3}$$

Such an estimator is unique as soon as $\mathscr{H}$ is an increasing continuous function. Such an approach has been largely used in the case where the variable X is of finite dimention (see *e.g* Whang and Zhao, [28], Cai [7], Zhou and Lianng [29] or Gannoun *et. al* [19]).

2.3 Assumptions on the functional variable

Let N_x be a fixed neighborhood of x and let $B(x,h)$ be the ball of center x and radius h, namely $B_\theta(x,h) = \{f \in \mathscr{H} / 0 < \ |< x - f, \theta >| < h\}$. Then, let's consider the following hypotheses:

(H0) $\forall h > 0,\ \mathbb{P}(X \in B_\theta(x,h)) \ = \phi_{\theta,x}(h) > 0,$

(H1) $(X_i, Y_i)_{i\in\mathbb{N}}$ is an α-mixing sequence whose the coefficients of mixture verify:

$$\exists a > 0, \exists c > 0 \ : \ \forall n \in \mathbb{N}, \quad \alpha(n) \le cn^{-a}.$$

(H2) $0 < sup_{i\neq j}\mathbb{P}((X_i, X_j) \in B_\theta(x,h) \times B_\theta(x,h)) = \mathscr{O}\left(\frac{(\phi_{\theta,x})^{\frac{(a+1)}{a}}}{n^{\frac{1}{a}}}\right).$

(H0) can be interpreted as a concentration hypothesis acting on the distribution of the *f.r.v.* X, while (H2) concerns the behavior of the join distribution of the pairs (X_i, X_j).Indeed, this hypothesis is equivalent to assume that, for n large enough

$$sup_{i\neq j}\frac{\mathbb{P}((X_i, X_j) \in B_\theta(x,h) \times B_\theta(x,h))}{\mathbb{P}(X \in B_\theta(x,h))} \le C\left(\frac{\phi_{\theta,x}}{n}\right)^{\frac{1}{a}}.$$

This is one way to control the local asymptotic ratio between the joint distribution and its margin. Remark that the upper bound increases with a. In order words, more the dependence is strong, more restrictive is (H2). The hypothesis (H1) specifies the asymptotic behavior of the α-mixing coefficients.

2.4 The nonparametric model

As usually in nonparametric estimation, we suppose that the *cond-cdf* $F(\theta,\ .,\ x)$verifies some smoothness constraints. Let b_1 and b_2 be two positive numbers; such that:

(H3) $\forall (x_1, x_2) \in N_x \times N_x,\ \forall (y_1, y_2) \in S^2_{\mathbb{R}},\ |F(\theta,\ y_1, x_1) - F(\theta,\ y_2, x_2)| \le C_{\theta,x}\left(\|x_1 - x_2\|^{b_1} + \|y_1 - y_2\|^{b_2}\right),$

(H4) $F(\theta,\ .,\ x)$is j- times continuously differential in some neighborhood of $t_\theta\ (\gamma)$,

(H5) $\forall (x_1, x_2) \in N_x \times N_x,\ \forall (y_1, y_2) \in S^2_{\mathbb{R}},$

$$\left|F^{(j)}(\theta,\ y_1, x_1) - F^{(j)}(\theta,\ y_2, x_2)\right| \le C_{\theta,x}\left(\|x_1 - x_2\|^{b_1} + \|y_1 - y_2\|^{b_2}\right),$$

where, for any positive integer l, $F^{(l)}(\theta,\ z, x)$ denotes its lth derivative (*i.e.* $\left.\frac{\partial^l F(\theta,\ y, x)}{\partial y^l}\right|_{y=z}$).

Let's note that (H3) is used for the prove of the almost complete convergence of $\widehat{t}_\theta(\gamma)$ whereas (H4) and (H5) are needed to establish the rate of convergence.

3 Asymptotic study

This part of paper is devoted, to the theoretical analysis, we start it by giving the almost complete convergence (*a.co.*) of the estimate conditional quantile $\widehat{t}_\theta(\gamma)$. After that, we will focus on the rate of convergence. Concerning the notations, as soon as possible, $C$and C' will denote generic constants. Moreover, from now on, h_H(resp. h_K) is a sequence which tends to 0 with n.

3.1 Pointwise almost complete convergence

Let's begin with the statement of an almost complete convergence property. To this end, we need some assumptions concerning the kernel estimator $\widehat{F}(\theta, ., x)$:

(H6) The restriction of H to the set $\{u \in \mathbb{R}, H(u) \in (0,1)\}$ is a strictly increasing function,

(H7) $\forall (y_1, y_2) \in \mathbb{R}^2$, $|H(y_1) - H(y_2)| \leq C|y_1 - y_2|$ and $\int |t|^{b_2} H^{(1)}(t)\, dt < \infty$,
where, for all $l \in \mathbb{N}^*$, $H^{(l)}(t) = \left.\frac{d^l H(y)}{dy^l}\right|_{y=t}$,

(H8) K is a positive bounded function with support $[-1,1]$ such that $\forall u \in (0,1)\ 0 < K(u)$,

(H9) $\frac{\log n}{n\phi_{\theta,x}(h_k)} \underset{n\to\infty}{\to} 0$.

(H10) $(X_i, Y_i)$for $i = 1, \ldots, n$ are strongly mixing with arithmetic coefficient of order $a > 1$and $\exists \beta > 2$ such that

(i) $s_{n,l}^{-(a+1)} = o\left(n^{-\beta}\right)$ $for\ l = 0,1,2;$

(ii) $s_{n,k}^{-(a+1)} = o\left(n^{-\beta}\right)$ $for\ k = 3,4,5,6,7;$

Remark 3.1.

- (H7) insures the existence of $\widehat{t}_\theta(\gamma)$, while (H6) insures its unicity.
- (H0)-(H5) and (H8) are standard assumptions for the distribution conditional estimation in single functional index model, which have been adopted by Bouchentouf *et al*. [4] for i.i.d case.
- (H9) is a technical condition for our results.
- (H10) is similar to that appeared in Ferraty and Vieu [18], it shows the influence of covariance on the convergence rate. Here, $s_{n,l}$ and $s_{n,k}$ will be defined bellow.

Theorem 3.1. Put $s_n = max\{s_{n,0}; s_{n,1}\}$, and suppose that either (H10)-(i) is satisfied together with hypotheses (H0)-(H3) and (H6)-(H9), thus we have:

$$\widehat{t}_\theta(\gamma) - t_\theta(\gamma) \underset{n\to\infty}{\to} 0,\ \ a.co. \tag{4}$$

Proof of Theorem 3.1. The proof is based on the pointwise convergence of $\widehat{F}(\theta,\,.,\,x)$ at $t_\theta(\gamma)$:

$$F(\theta,\,t_\theta(\gamma),\,x)-\widehat{F}(\theta,\,t_\theta(\gamma),\,x)\underset{n\to\infty}{\to}0,\ a.co. \tag{5}$$

Where the proof of the latter follows directly from Lemmas 3.1 and 3.2 which will be given below.

First of all, let's note that because of (H6) and (H7), $\widehat{F}(\theta,\,.,\,x)$is a continuous and strictly increasing function. So, we have:

$$\forall\varepsilon>0,\ \exists\delta(\varepsilon)>0,\forall y,\ \left|\widehat{F}(\theta,\,y,\,x)-\widehat{F}(\theta,\,t_\theta(\gamma),\,x)\right|\le\delta(\varepsilon)\Rightarrow|y-\,t_\theta(\gamma)|\le\varepsilon.$$

This leads us to write

$$\begin{aligned}\forall\varepsilon>0,\ \exists\delta(\varepsilon)>0,\mathbb{P}\left(\left|\widehat{t}_\theta(\gamma)-t_\theta(\gamma)\right|>\varepsilon\right)\ &\le\mathbb{P}\left(\left|\widehat{F}\left(\theta,\,\widehat{t}_\theta(\gamma),\,x\right)-\widehat{F}(\theta,\,t_\theta(\gamma),\,x)\right|\ge\delta(\varepsilon)\right)\\&=\mathbb{P}\left(\left|F(\theta,\,t_\theta(\gamma),\,x)-\widehat{F}(\theta,\,t_\theta(\gamma),\,x)\right|\ge\delta(\varepsilon)\right),\end{aligned}$$

since (3) is implying that $\widehat{F}\left(\theta,\,\widehat{t}_\theta(\gamma),\,x\right)=\gamma=F(\theta,\,t_\theta(\gamma),\,x)$.

Consider now, for $i=1,\cdots,n$ the following notations:

$$K_i(\theta,x)=K\left(h_K^{-1}(<x-X_i,\theta>)\right),\ \ H_i(\,t_\theta(\gamma))=H\left(h_H^{-1}(\,t_\theta(\gamma)-Y_i)\right),$$

$$\widehat{F}_N(\theta,\,t_\theta(\gamma),\,x)=\frac{1}{n\mathbb{E}(K_1(\theta,x))}\sum_{i=1}^{n}K_i(\theta,x)H_i(\,t_\theta(\gamma))\ \text{ and }\ \widehat{F}_D(\theta,\,x)=\frac{1}{n\mathbb{E}(K_1(\theta,x))}\sum_{i=1}^{n}K_i(\theta,x)$$

By using the following decomposition,

$$\begin{aligned}\widehat{F}(\theta,\,t_\theta(\gamma),\,x)-F(\theta,\,t_\theta(\gamma),\,x)=&\frac{1}{\widehat{F}_D(\theta,\,x)}\left\{\widehat{F}_N(\theta,\,t_\theta(\gamma),\,x)-\mathbb{E}\widehat{F}_N(\theta,\,t_\theta(\gamma),\,x)\right\}\\&-\frac{1}{\widehat{F}_D(\theta,\,x)}\left\{F(\theta,\,t_\theta(\gamma),\,x)-\mathbb{E}\widehat{F}_N(\theta,\,t_\theta(\gamma),\,x)\right\}\\&+\frac{F(\theta,\,t_\theta(\gamma),\,x)}{\widehat{F}_D(\theta,\,x)}\left\{\mathbb{E}\widehat{F}_D(\theta,\,x)-\widehat{F}_D(\theta,\,x)\right\},\end{aligned} \tag{6}$$

In what follows, let's denote

$$s_{n,0}^2=\sum_{i=1}^{n}\sum_{j=1}^{n}\left|Cov(\Delta_i(x,\theta),\Delta_j(x,\theta))\right|,$$

$$s_{n,1}^2=\sum_{i=1}^{n}\sum_{j=1}^{n}\left|Cov(\Delta_i(x,\theta)H_i(\,t_\theta(\gamma)),\Delta_j(x,\theta)H_j(\,t_\theta(\gamma)))\right|,$$

where $\Delta_i(x,\theta)=\frac{K\left(h_K^{-1}(<x-X_i,\theta>)\right)}{EK_1(\theta,x)}$. Let now present the following lemmas.

Lemma 3.1. Under the conditions of Theorem (H0)-(H3) and (H6)-(H9), we have

$$\left|F(\theta,\,t_\theta(\gamma),\,x)-\mathbb{E}\widehat{F}_N(\theta,\,t_\theta(\gamma),\,x)\right|=\mathscr{O}\left(h_K^{b_1}\right)+\mathscr{O}\left(h_H^{b_2}\right). \tag{7}$$

Lemma 3.2. Under the assumptions of Theorem 3.1, we have:

(i) $\widehat{F}_D(\theta,\, x) - \mathbb{E}\widehat{F}_D(\theta,\, x) = \mathscr{O}_{a.co}\left(\frac{\sqrt{s_{n,0}^2 \log n}}{n}\right)$,

(ii) $\widehat{F}_N(\theta,\, t_\theta(\gamma)\,,\, x) - \mathbb{E}\widehat{F}_N(\theta,\, t_\theta(\gamma)\,,\, x) = \mathscr{O}_{a.co}\left(\frac{\sqrt{s_{n,1}^2 \log n}}{n}\right)$. The proof of these two lemmas will be done in the same manner as it was given in [13], (since they are a special case of the Lemmas 3.2 and 3.3), the reader may also refer to Ferraty and Vieu [18].It suffices to replace $\widehat{F}(\,t_\gamma|x)$ (resp. $F\left(\,t_\gamma|x)\right)$) by $\widehat{F}(\theta,\, t_\theta(\gamma)\,,\, x)$ (resp. $F(\theta,\, t_\theta(\gamma)\,,\, x)$), and $\widehat{F}_D(x)$, (resp. $F_D(x)$) by $\widehat{F}_D(\theta,\, x)$ (resp. $F_D(\theta,\, x)$) with $d(x_1,x_2) = <x_1 - x_2, \theta>$.

3.2 Pointwise almost complete rate of convergence.

In this section we study the rate of convergence of our conditional quantile estimator $\widehat{t}_\gamma$. Because this kind of results is stronger than the previous one, we have to introduce some additional assumptions. As it is usual in conditional quantiles estimation, the rate of convergence can be linked with the flatness of the *cond-cdf* $F(\,.|x)$around the conditional quantile $t_\theta(\gamma)$. This is one reason why we introduced hypotheses (H4) and (H5). But a complementary way to take into account this local shape constrain is to suppose that:

(H11) $\exists j>0, \forall l,\ 0 \le l \le j,\ \ F^{(l)}(\theta,\, t_\theta(\gamma), x) = 0$ and $\left|F^{(j)}(\theta,\, t_\theta(\gamma), x)\right| > 0.$

Because we focus on the local behavior of $F(\theta,.,x)$around $t_\theta(\gamma)$ via its derivatives that leads us to consider the successive derivatives of $\widehat{F}(\theta,\, .,\, x)$ and subsequently some assumptions on the successive derivatives of the cumulative kernel H:

(H12) The support of $H^{(1)}$is compact and $\forall l \ge j, H^{(l)}$ exists and is bounded.

(H13) $\forall i \ne i'$,the conditional density of $\left(Y_i, Y_{i'}\right)$ given $\left(<X_i,\theta>, <X_{i'},\theta>\right)$ is continuous at $(t_\theta(\gamma),\, t_\theta(\gamma))$.

Theorem 3.2. Put $s_n = max\{s_{n,0}; s_{n,1}\}$ and assume that either (H10)-(i) is satisfied together with hypotheses (H0)-(H9) and (H11)-(H13), we have

$$\hat{t}_\theta(\gamma) - t_\theta(\gamma) = \mathscr{O}\left(\left(h_K^{b_1} + h_H^{b_2}\right)^{\frac{1}{j}}\right) + \mathscr{O}_{a.co}\left(\left(\frac{s_n^2 \log n}{n^2}\right)^{\frac{1}{2j}}\right). \tag{8}$$

Proof of Theorem 3.2. The proof is based on the Taylor expansion of $\widehat{F}(\theta,\, .,\, x)$ at $t_\theta(\gamma)$ and on the use of (H10):

$$\begin{aligned}\widehat{F}(t_\theta(\gamma)\,,\, x) - \widehat{F}\left(\theta,\widehat{t}_\theta(\gamma)\,,\, x\right) &= \sum_{l=1}^{j-1} \frac{(t_\theta(\gamma) - \hat{t}_\theta(\gamma))^l}{l!} \widehat{F}^{(l)}(t_\theta(\gamma)\,,\, x) + \frac{(t_\theta(\gamma) - \hat{t}_\theta(\gamma))^j}{j!} \widehat{F}^{(j)}(t_\theta^*|\, x)\,,\\ &= \sum_{l=1}^{j-1} \frac{\left(t_\theta(\gamma) - \widehat{t}_\theta(\gamma)\right)^l}{l!} \left(\widehat{F}^{(l)}(t_\theta(\gamma)\,,\, x) - F^{(l)}(t_\theta(\gamma)\,,\, x)\right) + \frac{\left(t_\theta(\gamma) - \widehat{t}_\theta(\gamma)\right)^j}{j!} \widehat{F}^{(j)}(t_\theta^*|\, x)\,,\end{aligned}$$

where, for all $y \in R$,

$$\widehat{F}^{(j)}(\theta, y,\, x) = \frac{h_H^{-j} \sum_{i=1}^n K\left(h_K^{-1}(<x - X_i, \theta>)\right) H^{(j)}(h_H^{-1}(y - Y_i))}{\sum_{i=1}^n K\left(h_K^{-1}(<x - X_i, \theta>)\right)}$$

and where $min\left(t_\theta(\gamma), \widehat{t}_\theta(\gamma)\right) < t_\theta^* < max\left(t_\theta(\gamma), \widehat{t}_\theta(\gamma)\right)$. Suppose now that we have the following result.

Lemma 3.3. Put $s_n^* = max\{s_{n,0}; s_{n,2}\}$ and assume that either (H10)-(ii) is satisfied together with under conditions (H0)-(H8) and (H11)-(H12) and if

$$\lim_{n\to\infty} \frac{\log n}{n h_H^{2j-1} \phi_{\theta,x}(h_K)} = 0,$$

then we have:

$$\widehat{F}^{(j)}(\theta,\, t_\theta(\gamma),x) - F^{(j)}(\theta,\, t_\theta(\gamma),x)| = \mathscr{O}\left(h_K^{b_1} + h_H^{b_2}\right) + \mathscr{O}_{a.co}\left(\frac{\sqrt{s_n^{*2}\log n}}{n^2}\right)$$

where

$$s_{n,2}^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\left|Cov(h_K^{-l}\Delta_i(x,\theta)H_i^{(l)}(\,t_\theta(\gamma)), h_H^{-l}\Delta_j(x,\theta)H_j^{(l)}(\,t_\theta(\gamma)))\right|$$

Because of Theorem 3.1, Lemma 3.3 and (H10), we have:

$$\widehat{F}^{(j)}(\theta, t_\theta^*,\, x) \underset{n\to\infty}{\to} F^{(j)}(\theta,\, t_\theta(\gamma),x) > 0,\ a.co.$$

then we derive

$$\begin{aligned}(t_\theta(\gamma) - \hat{t}_\theta(\gamma))^j =& \mathscr{O}\left(\widehat{F}(\theta, t_\theta(\gamma),\, x) - F(\theta, t_\theta(\gamma),\, x)\right) \\ &+ \mathscr{O}\left(\sum_{l=1}^{j-1}(t_\theta(\gamma) - \hat{t}_\theta(\gamma))^l\left(\widehat{F}^{(l)}(\theta,\, t_\theta(\gamma),x) - F^{(l)}(\theta,\, t_\theta(\gamma),x)\right)\right),\ a.co.\end{aligned} \tag{9}$$

Now, comparing the convergence rates given in Lemmas 3.2 and 3.3, we get

$$(t_\theta(\gamma) - \hat{t}_\theta(\gamma))^j = O\left(\widehat{F}(t_\theta(\gamma),\, x) - F(\theta, t_\theta(\gamma),\, x)\right) a.co.$$

Thus, Lemmas 3.1 and 3.2 allow us to get the claimed result. The proof of Lemma 3.3 will be given in the same manner as it was done in Ferraty *et al* [13] (they are a special case of the Lemmas 3.5). It suffices to replace $\widehat{F}^{(j)}\left(t_\gamma \mid x\right)$ $(resp.\ F^{(j)}\left(t_\gamma \mid x\right)$) by $\widehat{F}^{(j)}(\theta,\, t_\theta(\gamma),x)$ (resp. $F^{(j)}(\theta,\, t_\theta(\gamma),x)$), with $d(x_1,x_2) =< x_1 - x_2, \theta >$. The proof of these latter will be given briefly in the appendix.

4 Uniform almost complete convergence and rate of convergence

In this section we derive the uniform version of Theorem 3.1 and Theorem 3.2. The study of the uniform consistency is a crucial tool for studying the asymptotic properties of all estimates of the functional index if is unknown. In the multivariate case, the uniform consistency is a standard extension of the pointwise one, nevertheless, in the studied case, it requires some additional tools and topological conditions (see Ferraty *et al.* [12]). Consequently, coupled with the conditions introduced antecedently, we need the following ones. Firstly, consider

$$S_{\mathscr{H}} \subset \bigcup_{k=1}^{d_n^{S_{\mathscr{H}}}} B_\theta(x_k, r_n) \text{ and } \Theta_{\mathscr{H}} \subset \bigcup_{m=1}^{d_n^{\Theta_{\mathscr{H}}}} B_\theta(\theta_m, r_n), \tag{10}$$

with x_k (resp. θ_j) $\in \mathscr{H}$ and $r_n, d_n^{S_{\mathscr{H}}}, d_n^{\Theta_{\mathscr{H}}}$ are sequences of positive real numbers which tend to infinity as $n$goes to infinity and suppose that ,$d_n^{S_{\mathscr{H}}}, d_n^{\Theta_{\mathscr{H}}}$ are the minimal numbers of open balls with radius r_n in $\mathscr{H}$, which are required to cover $S_{\mathscr{H}}$ and $\Theta_{\mathscr{H}}$.

4.1 Conditional quantile distribution estimation

In this subpart we propose to study the uniform almost complete convergence of our estimator (2.3), to this end, we need to state the following assumptions.

(A1) There exists a differentiable function $\phi(.)$such that $\forall x\in S_{\mathscr{H}}$ and $\forall\theta\in\Theta_{\mathscr{H}}$,

$$0<C\phi(h)\le\phi_{\theta,x}(h)\le C^{'}\phi(h)<\infty \text{ and } \exists\eta_0>0,\ \ \forall\eta<\ \eta_0,\ \ \phi^{'}(\eta)<C.$$

(A2) $\forall(y_1,y_2)\in S_{\mathbb{R}}\times S_{\mathbb{R}}$, $\forall(x_1,x_2)\in S_{\mathscr{H}}\times S_{\mathscr{H}}$,and $\forall\theta\in\Theta_H$,

$$|F(\theta,\ y_1,x_1)-F(\theta,\ y_2,x_2)|\le C\left(\|x_1-x_2\|^{b_1}+\|y_1-y_2\|^{b_2}\right).$$

(A3) The kernel K satisfy (H3) and Lipschitz's condition holds

$$|K(x)-K(y)|\le C\|x-y\|.$$

(A4) For some $\nu\in(0,1)$, $\lim_{n\to\infty}n^{\nu}h_H=\infty$, and for $r_n=\mathscr{O}\left(\frac{\log n}{n}\right)$, the sequences ,$d_n^{S_{\mathscr{H}}}$ and $d_n^{\Theta_{\mathscr{H}}}$ satisfiy:

$$\begin{cases}(i) & \frac{(\log n)^2}{n\phi(h_K)}<\log d_n^{S_{\mathscr{H}}}+\log d_n^{\Theta_{\mathscr{H}}}<\frac{n\phi(h_K)}{\log n},\\ (ii) & \sum_{n=1}^{\infty}n^{\frac{1}{2b_2}}\left(d_n^{S_{\mathscr{H}}}d_n^{\Theta_{\mathscr{H}}}\right)^{1-\beta}<\infty \ \text{ for some } \beta>1,\\ (iii) & n\phi(h_K)=\mathscr{O}\left((\log n)^2\right).\end{cases}$$

(A5) $\forall(y_1,y_2)\in S_{\mathbb{R}}\times S_{\mathbb{R}}$, $\forall(x_1,x_2)\in S_{\mathscr{H}}\times S_{\mathscr{H}}$, and $\forall\theta\in\Theta_{\mathscr{H}}$,

$$\left|F^{(j)}(\theta,\ y_1,x_1)-F^{(j)}(\theta,\ y_2,x_2)\right|\le C\left(\|x_1-x_2\|^{b_1}+\|y_1-y_2\|^{b_2}\right).$$

(A6) For some $\nu\in(0,1)$, $\lim_{n\to\infty}n^{\nu}h_H=\infty$, and for $r_n=\ O\left(\frac{\log n}{n}\right)$, the sequences ,$d_n^{S_{\mathscr{H}}}$ and $d_n^{\Theta_{\mathscr{H}}}$satisfy:

$$\begin{cases}(i) & \frac{(\log n)^2}{nh_H^{2j-1}\phi(h_K)}<\log,d_n^{S_{\mathscr{H}}}+\log d_n^{\Theta_H}<\frac{nh_H^{2j-1}\phi(h_K)}{\log n},\\ (ii) & nh_H^{2j-1}\phi(h_K)=O\left((\log n)^2\right).\end{cases}$$

And let

$$s_{n,3}^2=\sum_{i=1}^{n}\sum_{j=1}^{n}\left|Cov(\Lambda_i,\Lambda_j)\right|,\quad s_{n,4}^2=\sum_{i=1}^{n}\sum_{j=1}^{n}\left|Cov(\Omega_i,\Omega_j)\right|,$$

$$s_{n,5}^2=\sum_{i=1}^{n}\sum_{j=1}^{n}\left|Cov\left(\Delta_i\left(x_{k(x)},\theta_{m(\theta)}\right),\Delta_j\left(x_{k(x)},\theta_{m(\theta)}\right)\right)\right|,\quad s_{n,6}^2=\sum_{i=1}^{n}\sum_{j=1}^{n}\left|Cov(\Gamma_i,\Gamma_j)\right|,$$

$$s_{n,7}^2=\sum_{i=1}^{n}\sum_{j=1}^{n}\left|Cov\left(\Gamma_i^{(l)},\Gamma_j^{(l)}\right)\right|,$$

where

$$\Delta_i(x,\theta)=\frac{1}{h_K\phi(h_K)}1_{B_\theta(x,h)\cup B_\theta\left(x_{k(x)},h\right)}(X_i),$$

$$\Omega_i(x,\theta)=\frac{1}{h_K\phi(h_K)}1_{B_\theta\left(x_{k(x)},h\right)\cup B_{\theta_{m(\theta)}}\left(x_{k(x)},h\right)}(X_i),$$

$$\Delta_i\left(x_{k(x)},\theta_{m(\theta)}\right)=\frac{K\left(h_K^{-1}\left(<x_{k(x)}-X_i,\theta_{m(\theta)}>\right)\right)}{EK\left(h_K^{-1}\left(<x_{k(x)}-X_i,\theta_{m(\theta)}>\right)\right)},$$

$$\begin{aligned}\Gamma_i=&\frac{K\left(h_K^{-1}\left(<x_{k(x)}-X_i,\theta_{m(\theta)}>\right)\right)}{\mathbb{E}K\left(h_K^{-1}\left(<x_{k(x)}-X_i,\theta_{m(\theta)}>\right)\right)}H\left(h_H^{-1}\left(t_y-Y_i\right)\right)\\&-\mathbb{E}\left(\frac{K\left(h_K^{-1}\left(\langle x_{k(x)}-X_i,\theta_{m(\theta)}\rangle\right)\right)}{\mathbb{E}K\left(h_K^{-1}\left(\langle x_{k(x)}-X_i,\theta_{m(\theta)}\rangle\right)\right)}H\left(h_H^{-1}(t_y-Y_i)\right)\right)\end{aligned}$$

and

$$\begin{aligned}\Gamma_i^{(l)}=&\frac{1}{h_H^l}\frac{K\left(h_K^{-1}\left(<x_{k(x)}-X_i,\theta_{m(\theta)}>\right)\right)}{\mathbb{E}K\left(h_K^{-1}\left(<x_{k(x)}-X_i,\theta_{m(\theta)}>\right)\right)}H^{(l)}\left(h_H^{-1}\left(t_y-Y_i\right)\right)\\&-\frac{1}{h_H^l}\mathbb{E}\left(\frac{K\left(h_K^{-1}\left(<x_{k(x)}-X_i,\theta_{m(\theta)}>\right)\right)}{\mathbb{E}K\left(h_K^{-1}\left(<x_{k(x)}-X_i,\theta_{m(\theta)}>\right)\right)}H^{(l)}\left(h_H^{-1}(t_y-Y_i)\right)\right).\end{aligned}$$

Theorem 4.1. Put $s_n^{'}=max\left\{s_{n,3};s_{n,4};s_{n,5};s_{n,6}\right\}$, and assume that either (H10)-(ii) is satisfied together with under hypotheses (H0)-(H3) and (H6)-(H9), (A1) and (A3)-(A4), we have

$$sup_{x\in S_F}\left|\widehat{t_\theta}(\gamma)-t_\theta(\gamma)\right|\underset{n\to\infty}{\to}0\ \ a.co. \tag{11}$$

Proof of Theorem 4.1 The proof of the theorem can be completed by using the following results.

Lemma 4.1. Under the conditions (H0)-(H3) and (H6)-(H9), we have

$$sup_{\theta\in\Theta_{\mathscr{H}}}sup_{x\in S_{\mathscr{H}}}sup_{t\in S_{\mathbb{R}}}\left|F\left(\theta,\ t_\theta(\gamma),\ x\right)-\mathbb{E}\widehat{F}_N\left(\theta,\ t_\theta(\gamma),\ x\right)\right|=\mathscr{O}\left(h_K^{b_1}\right)+\mathscr{O}\left(h_H^{b_2}\right). \tag{12}$$

Lemma 4.2. Under the assumptions of Theorem 4.1, we have:

1 $sup_{\theta\in\Theta_{\mathscr{F}}}sup_{x\in S_{\mathscr{F}}}\left|\widehat{F}_D(\theta,\ x)-\mathbb{E}\widehat{F}_D(\theta,\ x)\right|=\mathscr{O}_{a.co}\left(\frac{\sqrt{max\left\{s_{n,3}^2;s_{n,4}^2;s_{n,5}^2\right\}\log d_n^{S_{\mathscr{F}}}d_n^{\Theta_{\mathscr{F}}}}}{n}\right),$

2 $sup_{\theta\in\Theta_{\mathscr{H}}}sup_{x\in S_{\mathscr{H}}}sup_{t\in S_R}\widehat{F}_N(\theta,\ t_\theta(\gamma),\ x)-\mathbb{E}\widehat{F}_N(\theta,\ t_\theta(\gamma),\ x)=\mathscr{O}_{a.co}\left(\frac{\sqrt{max\left\{s_{n,3}^2;s_{n,4}^2;s_{n,6}^2\right\}\log d_n^{S_{\mathscr{F}}}d_n^{\Theta_{\mathscr{F}}}}}{n}\right)+\mathscr{O}_{a.co}\left(\sqrt{\frac{\log d_n^{S_{\mathscr{F}}}d_n^{\Theta_{\mathscr{F}}}}{n\phi(h_K)}}\right).$

Theorem 4.2. Under hypotheses (H0)-(H3), (H6)-(H10) and (A1)-(A4), we have

$$sup_{\theta\in\Theta_{\mathscr{F}}}sup_{x\in S_{\mathscr{F}}}\left|\widehat{t_\theta}(\gamma)-t_\theta(\gamma)\right|=\mathscr{O}\left(\left(h_K^{b_1}+h_H^{b_2}\right)^{\frac{1}{j}}\right)+\mathscr{O}_{a.co}\left(\left(\frac{\log d_n^{S_{\mathscr{F}}}d_n^{\Theta_{\mathscr{F}}}}{nh_H^{2j-1}\phi(h_K)}\right)^{\frac{1}{2j}}\right).$$

$$+\mathscr{O}_{a.co}\left(\left(\frac{s_n''^2 \log d_n^{S_\mathscr{F}} d_n^{\Theta_\mathscr{F}}}{n^2}\right)^{\frac{1}{2j}}\right),$$

where $s_n''^2 = max\left\{s_{n,3}; s_{n,4}; s_{n,5}; s_{n,7}\right\}$

Remark 4.1. These results extends Theorem 3 or Theorem 4 given in Bouchentouf *et al.* [4] to the mixing case. The effect of covariance structure for dependence case on the convergence rate is reflected in the last term. Specially, if the functional single-index is fixed, it is easy to prove the following corollary that are similar the one given in Bouchentouf *et al* [4].

Corollary 4.1. Under the conditions of Theorem 4.2, we have

1. $sup_{\theta\in\Theta_\mathscr{H}} sup_{x\in S_\mathscr{H}} sup_{y\in S_\mathbb{R}} \left| \hat{F}(\theta, y, x) - F(\theta, y, x)\right| = \mathscr{O}\left(h_K^{b_1} + h_H^{b_2}\right) + \mathscr{O}_{a.co}\left(\sqrt{\frac{\log d_n^{S_\mathscr{H}} + \log d_n^{\Theta_\mathscr{H}}}{n\phi(h_K)}}\right)$

$$+O_{a.co}\left(\frac{\sqrt{s_n'^{*2} \log d_n^{S_\mathscr{F}} d_n^{\Theta_\mathscr{F}}}}{n}\right).$$

2. $sup_{x\in S_\mathscr{H}} sup_{y\in S_\mathbb{R}} \left| \hat{F}(\theta, y, x) - F(\theta, y, x)\right| = O\left(h_K^{b_1} + h_H^{b_2}\right) + O_{a.co}\left(\sqrt{\frac{\log d_n^{S_\mathscr{H}}}{n\phi(h_K)}}\right) + O_{a.co}\left(\frac{\sqrt{s_n'^{*2} \log d_n^{S_\mathscr{F}}}}{n}\right),$

where $s_n'^{*2} = max\left\{s_{n,3}; s_{n,5}; s_{n,6}\right\}$.

Proof of Theorem 4.2. Obviously, the proofs of these two results, namely Theorem 4.2 and Corollary 4.1 can be deduced from the following intermediate results which are only uniform version of Lemma 3.3.

Lemma 4.3. Put $s_n^* = \max\left\{s_{n,0}; s_{n,2}\right\}$, and assume that either *(H10)-(ii)* is satisfied together with under conditions (H0)-(H8) and (H11)-(H12) and if

$$\lim_{n\to\infty} \frac{\log n}{n h_H^{2j-1} \phi_{\theta,x}(h_K)} = 0,$$

then we have:

$$\sup_{\theta\in\Theta_\mathscr{H}} \sup_{x\in S_\mathscr{H}} \sup_{y\in S_\mathbb{R}} \left|\widehat{F}^{(j)}(\theta, t_\theta(\gamma), x) - F^{(j)}(\theta, t_\theta(\gamma), x)\right| = \mathscr{O}(h_K^{b_1} + h_H^{b_2}) + \mathscr{O}_{a.co.}\left(\frac{\sqrt{s_n^{*2} \log d_n^{S_\mathscr{F}} d_n^{\Theta_\mathscr{F}}}}{n}\right),$$

where $s_{n,2}^2 = \sum_{i=1}^n \sum_{j=1}^n \left|Cov\left(h_H^{-l}\Delta_i(x,\theta)H_i^{(l)}(t_\theta(\gamma)), h_H^{-l}\Delta_j(x,\theta)H_j^{(l)}(t_\theta(\gamma))\right)\right|$.

5 Proofs of Technical Lemmas

In order to highlight the main contribution of our paper (i.e. $\alpha-$ mixing and functional variables) some details are voluntarily omitted.

Proof of Lemma 3.1. The asymptotic behavior of bias term is standard, in the sense that it is not affected by the dependence structure of the data. We have

$$\mathbb{E}\widehat{F}_N(\theta,t_\theta(\gamma),x)-F^x_\theta(t_\theta(\gamma))=\frac{1}{\mathbb{E}K_1(x,\theta)}\mathbb{E}\left(K_1(x,\theta)\left[\mathbb{E}\left(H_1(t_\theta(\gamma))|<X_1,\theta>\right)-F^x_\theta(t_\theta(\gamma))\right]\right). \quad (13)$$

and by noting that

$$\mathbb{E}(H_1(t_\theta(\gamma))|<X_1,\theta>)=\int_{\mathbb{R}}H^{(1)}(t)F(\theta,t_\theta(\gamma)-h_Ht,X_1)dt,$$

we can write, because of (H3) and (H7):

$$|\mathbb{E}(H_1(t_\theta(\gamma))|<X_1,\theta>)-F^x_\theta(t_\theta(\gamma))|\leq C_{x,\theta}\int_{\mathbb{R}}H^{(1)}(t)(h_K^{b_1}+|t|^{b_2}h_H^{b_2})dt.$$

Combining this last result with (13) allows us to achieve the proof.

Proof of Lemma 3.2. Following the ideas used in regression [17], the key fact consists in using a pseudo-exponential inequality taking considering the $\alpha-$ mixing structure. We start by writing,

1. Concerning (i), in fact, it can be found that.

$$\begin{aligned}\widehat{F}_D(\theta,x)-\mathbb{E}\widehat{F}_D(\theta,x)&=\frac{1}{n\mathbb{E}(K_1(\theta,x))}\sum_{i=1}^n K_i(\theta,x)-\frac{1}{n\mathbb{E}(K_1(\theta,x))}\sum_{i=1}^n\mathbb{E}K_i(\theta,x)\\&=\frac{1}{n\mathbb{E}(K_1(\theta,x))}\sum_{i=1}^n K_i(\theta,x)-\mathbb{E}K_i(\theta,x)\\&=\frac{1}{n}\sum_{i=1}^n\Delta_i(\theta,x)-\mathbb{E}\Delta_i(\theta,x)=\frac{1}{n}\sum_{i=1}^n\Psi_i(\theta,x)\end{aligned}$$

where $\Psi_i(\theta,x)=K_i(\theta,x)-\mathbb{E}K_i(\theta,x)$ has zero mean and satisfies

$$|\Psi_i(\theta,x)|\leq C_{x,\theta}/\phi_{\theta,x}(h_K),$$

then it allows us to use directly a dependent version of the Fuk-Nagaev's exponential inequality [18] and obtain

$$\widehat{F}_D(\theta,x)-\mathbb{E}\widehat{F}_D(\theta,x)=\mathscr{O}_{a.co.}\left(\frac{\sqrt{s_{n,0}^2\log n}}{n}\right).$$

2. Concerning (ii), it performs along the same steps and by invoking the same arguments, just changing the variable $\Psi_i(\theta,x)$ into the following ones:

$$\Xi_i(\theta,t_\theta(\gamma),x)=H_i(t_\theta(\gamma))\Delta_i(\theta,x)-\mathbb{E}H_i(t_\theta(\gamma))\,\Delta_i(\theta,x).$$

Because H is a cumulative kernel, we have $H_i(t_\theta(\gamma)\leq 1$. By using systematically this fact to bound the variables H_i, all the calculus made previously with the variables $\Psi_i(\theta,x)$ remain valid with the variables $\Xi_i(\theta,t_\theta(\gamma),x)$.

Thus $\Psi_i(\theta,x) = K_i(\theta,x) - \mathbb{E}K_i(\theta,x)$ has zero mean and satisfies

$$|\Psi_i(\theta,x)| \leq C_{x,\theta}/\phi_{\theta,x}(h_K),$$

the Fuk-Nagaev's inequality [25] allows one to get

$$\widehat{F}_N(\theta,t_\theta(\gamma),x) - \mathbb{E}\widehat{F}_N(\theta,t_\theta(\gamma),x) = \mathcal{O}_{a.co.}\left(\frac{\sqrt{s_{n,1}^2 \log n}}{n}\right).$$

Consequently, the proof of Lemma 3.2 is achieved.

Proof of Lemma 3.3. We use again the same kind of decomposition as (6):

$$\begin{aligned}\widehat{F}^{(j)}(\theta,t_\theta(\gamma),x) - F^{(j)}(\theta,t_\theta(\gamma),x) = \frac{1}{\widehat{F}_D(\theta,x)}\Bigg\{ & \left(\widehat{F}_N^{(j)}(\theta,t_\theta(\gamma),x) - \mathbb{E}\widehat{F}_N^{(j)}(\theta,t_\theta(\gamma),x)\right) \\ & - \left(F^{(j)}(\theta,t_\theta(\gamma),x) - \mathbb{E}\widehat{F}_N^{(j)}(\theta,t_\theta(\gamma),x)\right)\Bigg\} \\ & + \frac{F^{(j)}(\theta,t_\theta(\gamma),x)}{\widehat{F}_D(\theta,x)}\left\{\mathbb{E}\widehat{F}_D(\theta,x) - \widehat{F}_D(\theta,x)\right\}. \end{aligned} \tag{14}$$

This proof is very similar to the one of Theorem 3.1.

First of all, lets consider the bias term $F^{(j)}(\theta,t_\theta(\gamma),x) - \mathbb{E}\widehat{F}_N^{(j)}(\theta,t_\theta(\gamma),x)$. Using the same arguments in the proof of Lemma 3.1, replacing $F(\theta,t_\theta(\gamma),x)$ (resp. $\widehat{F}(\theta,t_\theta(\gamma),x)$) with $F^{(j)}(\theta,t_\theta(\gamma),x)$ (resp. $\widehat{F}_N^j(\theta,t_\theta(\gamma),x)$) and considering hypotheses (H5), (H7) and (H12) we get:

$$F^{(j)}(\theta,t_\theta(\gamma),x) - \mathbb{E}\widehat{F}_N^j(\theta,t_\theta(\gamma),x) = \mathcal{O}\left(h_K^{b_1} + h_K^{b_2}\right). \tag{15}$$

Now, we focus on the term $\widehat{F}_N^{(j)}(\theta,t_\theta(\gamma),x) - \mathbb{E}\widehat{F}_N^{(j)}(\theta,t_\theta(\gamma),x)$. To get the asymptotic behaviour of this quantity, we comeback to the proof of Lemma 3.2, and we replace $F(\theta,t_\theta(\gamma),x)$ (resp. $\widehat{F}(\theta,t_\theta(\gamma),x)$) with $F^{(j)}(\theta,t_\theta(\gamma),x)$ (resp. $\widehat{F}_N^j(\theta,t_\theta(\gamma),x)$).

Note that (H11) and (H13) permit to show that

$$\mathbb{E}\Big(H^{(j)}\big(h_H^{-1}(t_\theta(\gamma) - Y_i)\big)H^{(j)}\big(h_H^{-1}(t_\gamma - Y_{i'})\big)\,\big|(X_i,X_i')\Big) = \mathcal{O}(h_H^2),$$

while (H1) and (H5) imply that

$$\mathbb{E}\Big(H^{(j)}\big(h_H^{-1}(t_{\theta(\gamma)} - Y_i)\big)\,\big|X_i\Big) = \mathcal{O}(h_H).$$

Consequently, we have by using successively (H8), (H0), (H2) and (H10)-(i)

$$Cov(\Xi_i^*(\theta,t_\theta(\gamma),x), \Xi_{i'}^*(\theta,t_\theta(\gamma),x)) = \mathcal{O}\left(h_H^2\left(\frac{\phi_{\theta,x}(h_K)}{n}\right)^{1/\alpha}\phi_{\theta,x}(h_K)\right),$$

where

$$\Xi_i^*(\theta,t_\theta(\gamma),x) = H^{(j)}\big(h_H^{-1}(t_\theta(\gamma) - Y_i)\big)K_i(\theta,x) - \mathbb{E}\Big(H^{(j)}(h_H^{-1}(t_\theta(\gamma) - Y_i))\,K_i(x)\Big).$$

has zero mean and satisfies

$$|\Xi_i^*(\theta, t_\theta(\gamma), x)| \leq Ch_H^{-j},$$

because $H^{(j)}$ is bounded. Indeed, it can be found that

$$\widehat{F}_N^{(j)}(\theta, t_\theta(\gamma), x) - \mathbb{E}\widehat{F}_N^{(j)}(\theta, t_\theta(\gamma), x) = \frac{h_H^{-j}}{n\mathbb{E}K_1(\theta, x)} \sum_{i=1}^n \Xi_i^*(\theta, t_\theta(\gamma), x),$$

then it allows us to use directly similar arguments of Lemma 3.2, we obtain

$$\widehat{F}_N^{(j)}(\theta, t_\theta(\gamma), x) - \mathbb{E}\widehat{F}_N^{(j)}(\theta, t_\theta(\gamma), x) = \mathscr{O}_{a.co.}\left(\frac{\sqrt{s_n^{*2}\log n}}{n}\right),$$

which leads directly to the result of Lemma 3.3.

Proof of Lemma 4.1. It is omitted as it very similar to that of Lemma 4.6 in Bouchentouf *et al.* [4].

Proof of Lemma 4.2. The proof can be completed following the same steps as of Lemmas 4.4 4.7 in Bouchentouf *et al.*

[4].

i. From (1), for $\forall x \in S_{\mathscr{H}}$ and $\forall \theta \in \Theta_{\mathscr{H}}$, we have the decomposition as follows.For all $x \in S_{\mathscr{H}}$ and $\theta \in \Theta_{\mathscr{H}}$, we set

Let us consider the following decomposition

$$\begin{aligned}
\sup_{x\in S_{\mathscr{H}}}\sup_{\theta\in\Theta_{\mathscr{H}}}\left|\widehat{F}_D(\theta,x)-1\right| &= \sup_{x\in S_{\mathscr{H}}}\sup_{\theta\in\Theta_{\mathscr{H}}}\left|\widehat{F}_D(\theta,x)-\mathbb{E}\left(\widehat{F}_D(\theta,x)\right)\right| \\
&\leq \sup_{x\in S_{\mathscr{H}}}\sup_{\theta\in\Theta_{\mathscr{H}}}\left|\widehat{F}_D(\theta,x)-\widehat{F}_D(\theta,x_{k(x)})\right| \\
&+ \sup_{x\in S_{\mathscr{H}}}\sup_{\theta\in\Theta_{\mathscr{H}}}\left|\widehat{F}_D(\theta,x_{k(x)})-\widehat{F}_D(t_{j(\theta)},x_{k(x)})\right| \\
&+ \sup_{x\in S_{\mathscr{H}}}\sup_{\theta\in\Theta_{\mathscr{H}}}\left|\widehat{F}_D(t_{j(\theta)},x_{k(x)})-\mathbb{E}\left(\widehat{F}_D(t_{j(\theta)},x_{k(x)})\right)\right| \\
&+ \sup_{x\in S_{\mathscr{H}}}\sup_{\theta\in\Theta_{\mathscr{H}}}\left|\mathbb{E}\left(\widehat{F}_D(t_{j(\theta)},x_{k(x)})\right)-\mathbb{E}\left(\widehat{F}_D(\theta,x_{k(x)})\right)\right| \\
&+ \sup_{x\in S_{\mathscr{H}}}\sup_{\theta\in\Theta_{\mathscr{H}}}\left|\mathbb{E}\left(\widehat{F}_D(\theta,x_{k(x)})\right)-\mathbb{E}\left(\widehat{F}_D(\theta,x)\right)\right| \\
&= F_1+F_2+F_3+F_4+F_5+ \qquad (16)
\end{aligned}$$

where $k(x) = \arg, \min_{k\in\{1\ldots r_n\}} \|x - x_k\|$ and $j(\theta) = \arg\min_{j\in\{1\ldots l_n\}} \|\theta - t_j\|$.

In order to complete the proof of Lemma 4.1, we only need to give the convergence rate of five terms in (16) respectively.

Firstly, we treat F_1. Let $\lambda = \lambda_0\sqrt{\frac{\log d_n^{S_{\mathscr{F}}} d_n^{\Theta_{\mathscr{F}}} S_{n,5}^2}{n_2}}$ for all $\lambda_0 > 0$, we have that

$$\begin{aligned}\mathbb{P}(F_3 > \lambda) &= \mathbb{P}\left(\max_{k\in\{1\ldots d_n^{S_{\mathscr{H}}}\}} \max_{j\in\{1\ldots d_n^{\Theta_{\mathscr{H}}}\}} F_3' > \lambda\right)\\ &\leq d_n^{S_{\mathscr{H}}} d_n^{\Theta_{\mathscr{H}}} \max_{k\in\{1\ldots d_n^{S_{\mathscr{H}}}\}} \max_{j\in\{1\ldots d_n^{\Theta_{\mathscr{H}}}\}} \mathbb{P}(F_3' > \lambda). \end{aligned} \tag{17}$$

where $F_3' = \left|\widehat{F}_D(t_{j(\theta)}, x_{k(x)}) - \mathbb{E}\left(\widehat{F}_D(t_{j(\theta)}, x_{k(x)})\right)\right|$.

By using the Fuk-Nagaev's inequality (Proposition A.11(ii), see Ferraty and Vieu [18]) with taking $r = (\log n)^2$ and $q = a+1$, one will obtain that

$$\mathbb{P}(F_3' > \lambda) \leq C_1 A_1 + C_2 A_2 \tag{18}$$

where

$$A_1 = \left(1 + \frac{\lambda_0^2(\log d_n^{S_{\mathscr{F}}} d_n^{\Theta_{\mathscr{F}}})}{(\log n)^2}\right)^{-\frac{(\log n)^2}{2}}$$

$$A_2 = \frac{n(\log n)^{2a}\lambda_0^{-(a+1)}}{(\log d_n^{S_{\mathscr{F}}} d_n^{\Theta_{\mathscr{F}}})^{(a+1)/2}} \frac{1}{s_{n,5}^{a+1}}.$$

By hypotheses (A4)-(i) and (iii), we get $\frac{\log d_n^{S_{\mathscr{H}}} + \log d_n^{\Theta_{\mathscr{H}}}}{(\log n)^2} \to 0$ as $n \to 0$, which leads to

$$A_1 \leq d_n^{S_{\mathscr{H}}} d_n^{\Theta_{\mathscr{H}}}, \tag{19}$$

for some $\beta > 1$ and $\lambda_0 > 0$ such that $\lambda_0^2 = 2\beta$. On the other hand,

$$A_2 \leq Cn(\log n)^{2a}(\log d_n^{S_{\mathscr{H}}} d_n^{\Theta_H})^{-(a+1)/2_n - \eta} \leq C' \frac{1}{n^{\eta-\tau-1}}, \tag{20}$$

where $\tau > 0$ such that $\eta > \eta - \tau > 2$. Meanwhile, by the selection of β and η, we can find that

$$(d_n^{S_{\mathscr{H}}} d_n^{\Theta_{\mathscr{H}}})^{\beta} = \mathscr{O}(n^{\eta-\tau-1}). \tag{21}$$

Combining the equations (5.5)-(5.9) with hypothesis (A4)-(iii), we have

$$F_3 = \mathscr{O}_{a.co.}\left(\sqrt{\frac{s_{n,5}^2 \log d_n^{S_{\mathscr{F}}} d_n^{\Theta_{\mathscr{F}}}}{n^2}}\right) \tag{22}$$

Next, let us treat F_1 and F_2, respectively. By Assumption (A1), it follows

$$\begin{aligned}\sup_{x\in S_{\mathscr{H}}} \sup_{\theta\in\Theta_{\mathscr{H}}} \frac{1}{n}\left|\sum_{i=1}^{n}\left(\Delta_i(x,\theta) - \Delta_i(x_{k(x)},\theta)\right)\right| &\leq \frac{C}{\phi(h_K)} \sup_{x\in S_{\mathscr{H}}} \sup_{\theta\in\Theta_{\mathscr{H}}} \frac{1}{n}\sum_{i=1}^{n} 1_{B_\theta(x,h)\cup B_\theta(x_{k(x)},h)}(X_i)\\ &= C \sup_{x\in S_{\mathscr{H}}} \sup_{\theta\in\Theta_{\mathscr{H}}} \frac{1}{n}\sum_{i=1}^{n}\Lambda_i(x,\theta)\end{aligned}$$

and

$$\sup_{x\in S_{\mathscr{H}}}\sup_{\theta\in\Theta_{\mathscr{H}}}\frac{1}{n}\left|\sum_{i=1}^{n}\left(\Delta_i(x_{k(x)},\theta)-\Delta_i(x_{k(x)},\theta_{m(\theta)})\right)\right|\leq\frac{C}{\phi(h_K)}\sup_{x\in S_{\mathscr{H}}}\sup_{\theta\in\Theta_{\mathscr{H}}}\frac{1}{n}\sum_{i=1}^{n}1_{B_\theta(x_{k(x)},h)\cup B_\theta(x_{k(x)},h)}(X_i)$$
$$=C\sup_{x\in S_{\mathscr{H}}}\sup_{\theta\in\Theta_{\mathscr{H}}}\frac{1}{n}\sum_{i=1}^{n}\Omega_i(x,\theta).$$

Therefore, similar to the argument for (22), we can get

$$F_1=\mathscr{O}_{a.co.}\left(\sqrt{\frac{s_{n,3}^2\log d_n^{S_{\mathscr{F}}}d_n^{\Theta_{\mathscr{F}}}}{n^2}}\right)\tag{23}$$

and

$$F_2=\mathscr{O}_{a.co.}\left(\sqrt{\frac{s_{n,4}^2\log d_n^{S_{\mathscr{F}}}d_n^{\Theta_{\mathscr{F}}}}{n^2}}\right)\tag{24}$$

Thus, by using the same arguments as that in Bouchentouf *et al.* [4], it leads $F_5\leq F_1$ and $F_4\leq F_1$, respectively. then, as $n\to\infty$,

$$F_4=\mathscr{O}_{a.co.}\left(\sqrt{\frac{s_{n,4}^2\log d_n^{S_{\mathscr{F}}}d_n^{\Theta_{\mathscr{F}}}}{n^2}}\right)\quad\text{and}\quad F_5=\mathscr{O}_{a.co.}\left(\sqrt{\frac{s_{n,3}^2\log d_n^{S_{\mathscr{F}}}d_n^{\Theta_{\mathscr{F}}}}{n^2}}\right).\tag{25}$$

Finally, the first part of Lemma 4.2 can be easily deduced from (22)-(24).

ii. Concerning (2), the proof follows the same steps as that in Ferraty *et al.* [13]. It is also adopted by Bouchentouf *et al.* [4]. In fact, by the compact property of $S_{\mathbb{R}}\subset\mathbb{R}$, we have $S_{\mathbb{R}}\subset\bigcup_{m=1}^{z_n}(y_m-l_n,y_m+l_n)$ and l_n, z_n can be selected such as $z_n=\mathscr{O}(l_n^{-}1)=\mathscr{O}(n^{\frac{1}{2b_2}})$. By taking $m(y)=\arg\min_{\{1,2,\ldots,z_n\}}|y-t_m|$, then similar to the decomposition given in Bouchentouf *et al.* [4], it leads

$$\sup_{\theta\in\Theta_{\mathscr{H}}}\sup_{x\in S_{\mathscr{H}}}\sup_{y\in S_{\mathbb{R}}}\left|\widehat{F}_N(\theta,y,x)-\mathbb{E}\left(\widehat{F}_N(\theta,y,x)\right)\right|=\Psi_1+\Psi_2+\Psi_3+\Psi_4+\Psi_5+\Psi_6+\Psi_7$$

where

$$\Psi_1=\sup_{\theta\in\Theta_{\mathscr{H}}}\sup_{x\in S_{\mathscr{H}}}\sup_{y\in S_{\mathbb{R}}}\left|\widehat{F}_N(\theta,y,x)-\widehat{F}_N\left(\theta,y,x_{k(x)}\right)\right|$$
$$\Psi_2=\sup_{\theta\in\Theta_{\mathscr{H}}}\sup_{x\in S_{\mathscr{H}}}\sup_{y\in S_{\mathbb{R}}}\left|\widehat{F}_N(\theta,y,x)-\widehat{F}_N\left(\theta,y,x_{k(x)}\right)\right|$$
$$\Psi_3=\sup_{\theta\in\Theta_{\mathscr{H}}}\sup_{x\in S_{\mathscr{H}}}\sup_{y\in S_{\mathbb{R}}}\left|\widehat{F}_N(t_{j(\theta)},y,x_{k(x)})-\widehat{F}_N(t_{j(\theta)},y_{m(y)},x_{k(x)})\right|,$$
$$\Psi_4=\sup_{\theta\in\Theta_H}\sup_{x\in S_{\mathscr{H}}}\sup_{y\in S_{\mathbb{R}}}\left|\widehat{F}_N(t_{j(\theta)},y_{m(y)},x_{k(x)})-\mathbb{E}\left(\widehat{F}_N(t_{j(\theta)},y_{m(y)},x_{k(x)})\right)\right|,$$
$$\Psi_5=\sup_{\theta\in\Theta_{\mathscr{H}}}\sup_{x\in S_{\mathscr{H}}}\sup_{y\in S_{\mathbb{R}}}\left|\mathbb{E}\left(\widehat{F}_N(t_{j(\theta)},y_{m(y)},x_{k(x)})\right)-\mathbb{E}\left(\widehat{F}_N(t_{j(\theta)},y,x_{k(x)})\right)\right|,$$
$$\Psi_6=\sup_{\theta\in\Theta_{\mathscr{H}}}\sup_{x\in S_{\mathscr{H}}}\sup_{y\in S_{\mathbb{R}}}\left|\mathbb{E}\left(\widehat{F}_N(t_{j(\theta)},y,x_{k(x)})\right)-\mathbb{E}\left(\widehat{F}_N(\theta,y,x_{k(x)})\right)\right|,$$
$$\Psi_7=\sup_{\theta\in\Theta_{\mathscr{H}}}\sup_{x\in S_{\mathscr{H}}}\sup_{y\in S_{\mathbb{R}}}\left|\mathbb{E}\left(\widehat{F}_N(\theta,y,x_{k(x)})\right)-\mathbb{E}\left(\widehat{F}_N(\theta,y,x)\right)\right|$$

Since

$$\Psi_1 \leq \sup_{\theta\in\Theta_{\mathscr{H}}} \sup_{x\in S_{\mathscr{H}}} \frac{1}{n}\left|\sum_{i=1}^{n}(\Delta_i(x,\theta)-\Delta_i(\theta,x_{k(x)}))\right| = F_1$$

and

$$\Psi_2 \leq \sup_{\theta\in\Theta_{\mathscr{H}}} \sup_{x\in S_{\mathscr{H}}} \frac{1}{n}\left|\sum_{i=1}^{n}\left(\Delta_i(\theta,x_{k(x)})-\Delta_i(t_{j(\theta)},x_{k(x)})\right)\right| = F_2$$

then using the fact that $\Gamma_1 \leq F_1$ and $\Psi_2 \leq F_2$ and using equations (23) and (24), we get

$$\Psi_1 = \mathscr{O}_{a.co.}\left(\sqrt{\frac{s_{n,3}^2 \log d_n^{S_{\mathscr{F}}} d_n^{\Theta_{\mathscr{F}}}}{n^2}}\right)$$

and

$$\Psi_2 = \mathscr{O}_{a.co.}\left(\sqrt{\frac{s_{n,4}^2 \log d_n^{S_{\mathscr{F}}} d_n^{\Theta_{\mathscr{F}}}}{n^2}}\right)$$

On the other hand, since $\Psi_7 \leq \Psi_1$ and $\Psi_6 \leq \Psi_2$, we get

$$\Psi_7 = \mathscr{O}_{a.co.}\left(\sqrt{\frac{s_{n,3}^2 \log d_n^{S_{\mathscr{F}}} d_n^{\Theta_{\mathscr{F}}}}{n^2}}\right)$$

and

$$\Psi_6 = \mathscr{O}_{a.co.}\left(\sqrt{\frac{s_{n,4}^2 \log d_n^{S_{\mathscr{F}}} d_n^{\Theta_{\mathscr{F}}}}{n^2}}\right)$$

respectively.

iii. Concerning Ψ_3 and Ψ_5; by conditions (H4) and (H5), boundness of K and selection of l_n, using the same arguments of Lemma 4.7 in Bouchentouf *et al.* [4], we get

$$\begin{aligned}
\left|\widehat{F}_N(t_{j(\theta)},y,x_{k(x)})-\widehat{F}_N(t_{j(\theta)},y_{m(y)},x_{k(x)})\right| &\leq \frac{C}{n}\left|\sum_{i=1}^{n}\left(\frac{K(h_K^{-1}<x_{k(x)}-X_i,\theta_{m(\theta)}>)}{\mathbb{E}K(h_K^{-1}<x_{k(x)}-X_i,\theta_{m(\theta)}>)}\right)\right|\left|\frac{y-y_{m(y)}}{h_H}\right| \\
&\leq \frac{C}{n}\left|\sum_{i=1}^{n}\Delta_i(x_{k(x)},\theta_{m(\theta)})\right|\left|\frac{y-y_{m(y)}}{h_H}\right| \\
&\leq \mathscr{O}\left(\frac{l_n}{h_H}\right) = \mathscr{O}\left(\sqrt{\frac{\log d_n^{S_{\mathscr{H}}} d_n^{\Theta_{\mathscr{H}}}}{n\phi(h_K)}}\right)
\end{aligned}$$

as $n\to\infty$, therefore, it follows

$$\Psi_5 \leq \Psi_3 = \mathscr{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{S_{\mathscr{H}}} d_n^{\Theta_{\mathscr{H}}}}{n\phi(h_K)}}\right)$$

iv. Concerning Ψ_4, let us consider $\varepsilon = \varepsilon_0\left(\sqrt{\frac{s_{n,6}^2 \log d_n^{S_{\mathscr{F}}} d_n^{\Theta_{\mathscr{F}}}}{n^2}}\right)$. Since

$$\mathbb{P}\left(\Psi_4 > \varepsilon_0 \sqrt{\frac{s_{n,6}^2 \log d_n^{S_{\mathscr{F}}} d_n^{\Theta_{\mathscr{F}}}}{n^2}}\right) = \mathbb{P}(\Psi_4 > \varepsilon)$$
$$= \mathbb{P}\left(\max_{j\in\{1\ldots d_n^{\Theta_{\mathscr{H}}}\}} \max_{k\in\{1\ldots d_n^{S_{\mathscr{H}}}\}} \max_{m(y)\in\{1,2,\ldots,z_n\}} |\Upsilon_n - \mathbb{E}\Upsilon_n| > \varepsilon\right)$$
$$\leq z_n d_n^{S_{\mathscr{H}}} d_n^{\Theta_{\mathscr{H}}} \mathbb{P}(|\Upsilon_n - \mathbb{E}\Upsilon_n| > \varepsilon)$$

where $\Upsilon_n = \widehat{F}_N(t_{j(\theta)}, y_{m(y)}, x_{k(x)})$, the application of Fuk-Nagaev's inequality (Proposition A.11-(ii), see Ferraty and Vieu [18]) with $r = (\log n)^2 > 1$ and $q = a+1$, we get that

$$\mathbb{P}\left(\left|\sum_{i=1}^{n}(\Gamma_i - \mathbb{E}\Gamma_i)\right| > \varepsilon\right) \leq C\left(1 + \frac{\varepsilon_0^2(\log d_n^{S_{\mathscr{F}}} d_n^{\Theta_{\mathscr{F}}})}{(\log n)^2}\right)^{-(\log n)^2/2} + \frac{n(\log n)^{2a}\varepsilon_0^{-(a+1)}}{(\log d_n^{S_{\mathscr{F}}} d_n^{\Theta_{\mathscr{F}}})^{(a+1)/2}} \frac{1}{s_{n,6}^{a+1}}$$
$$= C_1 B_1 + C_2 B_2$$

Similarly to (5.10), it yields

$$\Psi_4 = \mathscr{O}_{a.co.}\left(\sqrt{\frac{s_{n,6}^2 \log d_n^{S_{\mathscr{F}}}}{n^2}}\right).$$

Finally, the proof of Lemma 4.2 is achieved.

Proof of Lemma 4.3. The proof is an immediate consequence of the second part of Lemma 4.2, it suffices to replace the conditional cumulative distribution function by its successive derivatives.

6 Concluding remarks

In this article, we examine conditional quantile estimation in the single functional index model for $\alpha-$mixing functional data. The asymptotic properties such as pointwise almost complete consistency and the uniform almost complete convergence of the kernel estimator with rate are presented under some mild conditions. Although $\alpha-$mixing is reasonably weak among various weak dependence process and has many practical applications such as in time series prediction, we also address other dependence settings such as long memory dependence functional data (see Benhenni *et al.* [2]). In this case, the asymptotic properties of the estimation of successive derivatives of the conditional density function, conditional hazard function, conditional distribution function and conditional quantile in the single functional index model have been investigated in our other works.

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