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A Numerical approach based on exponential polinomials for solving of Fredholm Integro-Differential-Difference equations

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Abstract: In this study, a matrix method based on exponential polynomials by means of collocation points is proposed to solve the higher-order linear Fredholm integro-differential-difference equations under the initial-boundary conditions. In addition, an error analysis technique based on residual function is developed for our method. Illustrative examples are included to demostrate the validity and applicability of the presented technique.

Keywords: Exponential Polynomials, Fredholm Integro-Differential-Difference Equations, Matrix Method, Residual Error Analysis.

1 Introduction

In this study, we consider the high-order linear integro-differential-difference equations with constant arguments (advanced or delayed) and variable coefficients

$$\sum_{k=0}^{m_1}\sum_{j=0}^{n_1} P_{kj}(x)y^{(k)}(x+\tau_{kj}) = \sum_{r=0}^{m_2}\sum_{s=0}^{n_2}\int_0^b K_{rs}(x,t)y^{(r)}(t+\lambda_{rs})dt + g(x) \tag{1}$$

with the mixed conditions

$$\sum_{k=0}^{m_1-1}(a_{ik}y^{(k)}(0)+b_{ik}y^{(k)}(b)) = \mu_i, \quad i=0,1,...,m_1-1 \tag{2}$$

where $P_{kj}(x)$, $K_{rs}(x,t)$ and $g(x)$ are known functions defined on the interval $0 \le x,t \le b < \infty$; τ_{kj}, λ_{rs}, a_{ik} and b_{ik} are appropriate constants; $y(x)$ is an unknown function to be determined.

The equatioin defined by [1] is a combination of differential, difference and Fredholm integral equations. This is an important branch of modern mathematics and arises frequently in many applied areas which include engineering, mechanic, physics, chemistry, astronomy, biology, economics, elasticity, plasticity and oscillation theory, etc. [1-9]. In recent years, to solve the mentioned equations, several numerical methods were used such as the Successive Approximations, Adomian Decomposition, Haar Wavelet, Block-Pulse, Monte-Carlo, Tau and Walsh series methods,

[10-13].

Additionally, since the beginning of 1994, Taylor, Chebysher, Laguere, Hermite, Brstein and Bessel methods based on

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Sezer's method have been used by Sezer et all. [10-23] to solve linear differential, difference, integral and Fredholm integro-differential-difference equations. Our purpose in this study is to develop a new matrix method, which is based on the exponential basis set $\{1, e^{-x}, e^{-2x}, ..., e^{-nx}, ..\}$ [24] and collocation points, to obtain the approximate solution of the problem (1)-(2) in the exponential polynomial form

$$y(x) \cong y_N(x) = \sum_{n=0}^{N} a_n e^{-nx}, \quad 0 \le x \le b < \infty \tag{3}$$

where the exponential basis set is defined by $\{1, e^{-x}, e^{-2x}, ..., e^{-Nx}\}$; a_n $(n = 0, 1, 2, ..., N)$ are unknown coefficients to be determined.

The exponential polynomials based on the exponentials basis set are used to analyze successively many optices and quantum electronics, automatic control, electrical, circuits theory, hydro meteorology, turbulance and boundary layer, etc.[24-32].

The rest of this paper is organized as follows. The fundamential matrix relations with related to the exponential polynomials and their derivatives are presented in section 2.

The new exponential matrix method besed on collocation points is described in Section 3. In section 4, the error analysis technique related to residual function is developed for the present method. To support our findings, in section 5, we present the results of numerical experiments. Section 5 concludes this study with a brief summary.

2 Matrix Relations for Exponential Polynomials

Firstly, we can write the desired solution $y(x)$ defined by (3) of Eq. (1) in the matrix form as, for $n = 0, 1, ..., N$,

$$y(x) \cong y_N(x) = \mathbf{E}(x)\mathbf{A} \tag{4}$$

where $\mathbf{E}(x) = [1 \; e^{-x} \; e^{-2x} \; ... \; e^{-Nx}]$ and $\mathbf{A} = [a_0 \; a_1 \; a_2 \; ... \; a_N]^T$. Also, it is clearly seen that the relation between the matrix $\mathbf{E}(x)$ and its derivative $\mathbf{E}'(x)$ is

$$\mathbf{E}'(x) = \mathbf{E}(x)\mathbf{M}$$

and that, repeating the process

$$\begin{gathered} \mathbf{E}''(x) = \mathbf{E}'\mathbf{M} = \mathbf{E}(x)\mathbf{M}^2 \\ \mathbf{E}'''(x) = \mathbf{E}''\mathbf{M} = \mathbf{E}(x)\mathbf{M}^3 \\ \cdots \end{gathered}$$

$$\mathbf{E}^{(k)}(x) = \mathbf{E}^{(k-1)}(x)\mathbf{M} = \mathbf{E}(x)\mathbf{M}^k, \;\; k = 0, 1, 2, ... \tag{5}$$

where

$$\mathbf{M} = diag[0 \; (-1) \; (-2) \; ... \; (-N)]$$

$\mathbf{M}^0$ is unit matrix and

$$\mathbf{M}^k = diag[0 \; (-1)^k \; (-2)^k \; ... \; (-N)^k]$$

From the matrix relations (4) and (5), it follows that

$$y_N^{(k)}(x) \cong \mathbf{E}(x)\mathbf{M}^k\mathbf{A}, \quad k = 0, 1, 2, ..., m_1 \tag{6}$$

By putting $x \to x + \tau_{kj}$ in the relation (6) , we have, for $k = 0, 1, ..., m_1$,

$$y^{(k)}(x+\tau_{kj}) = \mathbf{E}(x+\tau_{kj})\mathbf{M}^k\mathbf{A} = \mathbf{E}(x)\mathbf{B}(\tau_{kj})\mathbf{M}^k\mathbf{A} \tag{7}$$

so that,

$$\mathbf{E}(x+\tau_{kj}) = \mathbf{E}(x)\mathbf{B}(\tau_{kj}) \text{ and } \mathbf{B}(\tau_{kj}) \ = diag[0 \ e^{-\tau_{kj}} \ e^{-2\tau_{kj}} \ ...e^{-N\tau_{kj}}]$$

Similarly , we get the matrix relation for $y^r(t+\lambda_{rs})$,

$$y^{(r)}(t+\lambda_{rs}) = \mathbf{E}(t+\lambda_{rs})\mathbf{M}^r\mathbf{A} = \mathbf{E}(t)\mathbf{B}(\lambda_{rs})\mathbf{M}^r\mathbf{A}, r = 0, 1, ..., m_2 \tag{8}$$

On the other hand, by using Taylor expansion, we get the matrix relation between the standard basis matrix

$$\mathbf{X}(x) = [1 \ x \ x^2 ... x^N]$$

and the exponention basis matrix

$$\mathbf{E}(x) = [1 \ e^{-x} \ e^{-2x} ... e^{-Nx}]$$

as follows:

$$\begin{pmatrix} 1 \\ e^{-x} \\ e^{-2x} \\ \vdots \\ e^{-Nx} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & \frac{-1}{1!} & \frac{(-1)^2}{2!} & \cdots & \frac{(-1)^N}{N!} \\ 1 & \frac{-2}{1!} & \frac{(-2)^2}{2!} & \cdots & \frac{(-2)^N}{N!} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \frac{-N}{1!} & \frac{(-N)^2}{2!} & \cdots & \frac{(-N)^N}{N!} \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^N \end{pmatrix}$$

or briefly

$$\mathbf{E}(x) = \mathbf{X}(x)\mathbf{T}^T \tag{9}$$

where

$$\mathbf{E}(x)^T = \begin{pmatrix} 1 \\ e^{-x} \\ e^{-2x} \\ \vdots \\ e^{-Nx} \end{pmatrix}, \mathbf{T} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & \frac{-1}{1!} & \frac{(-1)^2}{2!} & \cdots & \frac{(-1)^N}{N!} \\ 1 & \frac{-2}{1!} & \frac{(-2)^2}{2!} & \cdots & \frac{(-2)^N}{N!} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \frac{-N}{1!} & \frac{(-N)^2}{2!} & \cdots & \frac{(-N)^N}{N!} \end{pmatrix}$$

and

$$\mathbf{X}(x)^T = \begin{pmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^N \end{pmatrix}.$$

Now we can convert the kernel function $K_{rs}(x,t)$ to the matrix form, by means of the following procedure. The function

$K_{rs}(x,t)$ can be approximated by the truncated Taylor Series and the truncated exponontial series, respectively,

$$\begin{aligned} K_{rs}(x,t) &= \sum_{m=0}^{N}\sum_{n=0}^{N} k_{rs}x^m t^n \qquad (TaylorSeries) \\ &= \mathbf{X}(x)\mathbf{K}_{rs}^t\mathbf{X}^T(t), m,n=0,1,..,N \end{aligned} \tag{10}$$

$$\mathbf{K}_{rs}^t = \left[k_{rs}^{t_{mn}}\right], k_{rs}^{t_{mn}} = \frac{1}{m!n!}\frac{\partial^{m+n}K_{rs}(0,0)}{\partial x^m \partial t^n}$$

and

$$\begin{aligned} \mathbf{K}_{rs}(x,t) &= \sum_{m=0}^{N}\sum_{n=0}^{N} k_{rs}^{e,mn}\mathbf{E}_m(x)\mathbf{E}_n(t) \qquad (ExponontialSeries) \\ &= \mathbf{E}(x)\mathbf{K}_{rs}^e\mathbf{E}^T(t), \mathbf{K}_{rs}^e = [k_{rs}^{e,mn}] \end{aligned} \tag{11}$$

By using the relations (9),(10) and (11), we obtain the matrix equation

$$\mathbf{X}(x)\mathbf{K}_{rs}^t\mathbf{X}^T(t) = \mathbf{E}(x)\mathbf{K}_{rs}^e\mathbf{E}^T(t) = \mathbf{X}(x)\mathbf{T}^T\mathbf{K}_{rs}^e\mathbf{T}\mathbf{X}^T(t)$$

or briefly,

$$\mathbf{K}_{rs}^e = (\mathbf{T}^T)^{-1}\mathbf{K}_{rs}^t\mathbf{T}^{-1}, \mathbf{K}_{rs}^t = \left[k_{rs}^{t_{mn}}\right] \tag{12}$$

3 Method of Solution

For constructing the fundamental matrix equation, we first substitute the matrix relations (7),(8) and (11) into Eq.(1). Then, by simplifying the result equation, we obtain the fundamental matrix equation for Eq.(1) as follows:

$$\{\sum_{k=0}^{m_1}\sum_{j=0}^{n_1}\mathbf{P}_{kj}(x)\mathbf{E}(x)\mathbf{B}(\tau_{kj})\mathbf{M}^k - \sum_{r=0}^{m_2}\sum_{s=0}^{n_2}\mathbf{E}(t)\mathbf{K}_{rs}^e\mathbf{Q}\mathbf{B}(\lambda_{rs})\mathbf{M}^r\}\mathbf{A} = g(x) \tag{13}$$

where

$$\mathbf{Q} = [q_{mn}] = \int_0^b \mathbf{E}^T(t)\mathbf{E}(t)dt, q_{mn} = \left\{\begin{array}{c} b, \quad m=n=0 \\ \frac{1-e^{-(m+n)b}}{m+n}, \text{ otherwise} \end{array}\right\}$$

By using in Eq(13) the collacation points defined by

$$x_i = \frac{b}{N}i, i = 0,1,...,N,$$

we obtain the system of the matrix equations

$$\{\sum_{k=0}^{m_1}\sum_{j=0}^{n_1}\mathbf{P}_{kj}(x_i)\mathbf{E}(x_i)\mathbf{B}(\tau_{kj})\mathbf{M}^k - \sum_{r=0}^{m_2}\sum_{s=0}^{n_2}\mathbf{E}(x_i)\mathbf{K}_{rs}\mathbf{Q}\mathbf{B}(\lambda_{rs})\mathbf{M}^r\}\mathbf{A} = g(x_i)$$

and therefore the fundamental matrix equation

$$\{\sum_{k=0}^{m_1}\sum_{j=0}^{n_1}\mathbf{P}_{kj}\mathbf{E}\mathbf{B}(\tau_{kj})\mathbf{M}^k - \sum_{r=0}^{m_2}\sum_{s=0}^{n_2}\mathbf{E}\mathbf{K}_{rs}^e\mathbf{Q}\mathbf{B}(\lambda_{rs})\mathbf{M}^r\}\mathbf{A} = \mathbf{G} \tag{14}$$

where

$$\mathbf{P}_{kj} = diag[\mathbf{P}_{kj}(x_0)\ \mathbf{P}_{kj}(x_1)\ ...\ \mathbf{P}_{kj}(x_N)]$$

$$\mathbf{E} = \begin{pmatrix} \mathbf{E}(x_0) \\ \mathbf{E}(x_1) \\ \mathbf{E}(x_2) \\ \vdots \\ \mathbf{E}(x_N) \end{pmatrix} = \begin{pmatrix} 1 & e^{-x_0} & e^{-2x_0} & \cdots & e^{-Nx_0} \\ 1 & e^{-x_1} & e^{-2x_1} & \cdots & e^{-Nx_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & e^{-x_N} & e^{-2x_N} & \cdots & e^{-Nx_N} \end{pmatrix}, \mathbf{G} = \begin{pmatrix} g(x_0) \\ g(x_1) \\ g(x_2) \\ \vdots \\ g(x_N) \end{pmatrix}$$

The Fundemental Matrix equation (14) for Eq.(1) corresponds to a system of (N+1) algebraic equations for the (N+1) unknown coefficients $a_0, a_1, ..., a_N$. Briefly, we write Eq.(14) in the form

$$\mathbf{WA} = \mathbf{G} \text{ or } [\mathbf{W} : \mathbf{G}] \tag{15}$$

where

$$\mathbf{W} = [w_{pq}], p, q = 0, 1, ...N$$

On the other hand, we can obtain the matrix forms for the conditions (2), by means of the relation (6) as follows;

$$\sum_{k=0}^{m_1-1} (a_{ik}\mathbf{E}(0) + b_{ik}\mathbf{E}(b))\mathbf{M}^k\mathbf{A} = [\ \eta_i], i = 0, 1, ..., m_1 - 1$$

or briefly

$$\mathbf{U}_i\mathbf{A} = [\eta_i]$$

or

$$[\mathbf{U}_i : \eta_i], i = 0, 1, ..., m_1 - 1, \text{where, } \mathbf{U}_i = [u_{i0}, u_{i1}, ..., u_{iN}] \tag{16}$$

Consequently, to obtain the solution of Eq. (1) under the conditions (2), by replacing the row matrices (16) by the last m rows of the augmented matrix (15), we have the required augmented matrix [7,15,21]

$$[\widetilde{\mathbf{W}}; \widetilde{\mathbf{G}}]. \tag{17}$$

If rank$\widetilde{\mathbf{W}} = $ rank$[\widetilde{\mathbf{W}}; \widetilde{\mathbf{G}}] = N+1$, then we can write $A = \widetilde{W}^{-1}\widetilde{G}$. Thus the matrix A (there by the coefficients $a_0, a_1, ..., a_n$) is uniquely determined and the Eq. (1) under the conditions (2) has a unique solution. This solution is given by the truncated exponential series (3)

$$y_N(x) = \sum_{n=0}^{N} a_n e^{-nx}$$

On the other hand, when $|\widetilde{W}| = 0$, if rank $\widetilde{\mathbf{W}} = rank[\widetilde{\mathbf{W}}; \widetilde{\mathbf{G}}] < N+1$, then we may find a particular solution, otherwise if rank $\widetilde{\mathbf{W}} \neq$ rank$[\widetilde{\mathbf{W}}; \widetilde{G}] < N+1$, then, it is not a solution.

4 Error Analysis Technique Based on Residual Function:Accuracy of Solutions

We can easily check the accuracy of the obtained solutions as follows. As the truncated exponontial series in (3) is an approximate solution of Eq.(1), when the function $y_N(x)$ and its derivatives are substituted in Eq.(1), the resulting equation must be approximately satisfied; that is, for

$$x = x_i \ \varepsilon \ [0,b], i = 0,1,...,N$$

$$R_N(x_i\ ) = \sum_{k=0}^{m_1}\sum_{j=0}^{n_1}\mathbf{P}_{kj}(x_i)y_N^{^{K}}(x_i) - \sum_{r=0}^{m_2}\sum_{s=0}^{n_2}\int_0^b \mathbf{K}(x_i,t)y_N^{^{r}}(xt)dt - g(x_i) \approx 0 \tag{18}$$

or

$$R_N(x_i\ ) \leq 10^{-k_i}$$

(k_i is any positive integer). If max $10^{-k_i} = 10^{-k}$(k is any positive integer) is prescribed, then the truncation limit N is increased until the difference $R_N(x_i\ )$ at each of the points becomes smaller than the prescribed 10^{-k}.

On the other hand, by means of the residual function defined by $R_N(x)$ and the mean value of the function $|R_N(x)|$ on the interval $[0,b]$, the accuracy of the solution can be controlled and the error can be estimated. If $R_N(x) \to 0$ when N is sufficiently large enough, then the error decreases. Also, by using the Mean-Value Theorem, we can estimate the upper bound mean error $\overline{R}$ as follows:

$$\left|\int_0^b R_N(x)dx\right| \leq \int_0^b |R_N(x)|\,dx$$

and

$$\int_0^b R_N(x)dx = bR_N(c) \Rightarrow \left|\int_0^b R_N(x)dx\right| = b\,|R_N(c)|$$

$$\Rightarrow b\,|R_N(c)| \leq \int_0^b |R_N(x)|\,dx \Rightarrow$$

$$|R_N(c)| \leq \frac{\int_0^b |R_N(x)|\,dx}{b} = \overline{R}_N \ (0 \leq c \leq b)$$

5 Numerical Examples

In this section, we apply the method for some examples. All numerical computations have been done by using a computer program written in Maple.

Example 1: Let us consider the differential-difference equation given by

$$y'(x) + y(x) + e^{(x-1)}y(x-1) = 1$$

with initial condition $y(0) = 1$.
From equation (14) the colacation points for $N = 2$ are computed as

$$\left\{x_0 = 0, x_1 = \frac{1}{2}, x_2 = 1\right\}$$

and the fundamental matrix equation of the problem is

$$\left\{\mathbf{P}_{00}\mathbf{E}\mathbf{B}(\tau_{00})\mathbf{M}^0+\mathbf{P}_{01}\mathbf{E}\mathbf{B}(\tau_{01})\mathbf{M}^1+\mathbf{P}_{10}\mathbf{E}\mathbf{B}(\tau_{10})\mathbf{M}^1+\mathbf{P}_{11}\mathbf{E}\mathbf{B}(\tau_{11})\mathbf{M}^1\right\}\mathbf{A}=\mathbf{G}$$

where

$$\tau_{00}=0,\tau_{01}=-1,\tau_{10}=0,\tau_{11}=0$$

$$\mathbf{B}(\tau_{00})=\mathbf{B}(\tau_{10})=\mathbf{B}(\tau_{11})=\mathbf{B}(0)=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\mathbf{B}(\tau_{01})=\mathbf{B}(-1)=\begin{bmatrix}1&0&0\\0&e&0\\0&0&e^2\end{bmatrix},$$

$$\mathbf{P}_{00}=\mathbf{P}_{10}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\mathbf{P}_{01}=\begin{bmatrix}e^{-1}&0&0\\0&e^{\frac{-1}{2}}&0\\0&0&1\end{bmatrix},\mathbf{P}_{11}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

$$\mathbf{M}^0=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\mathbf{E}=\begin{bmatrix}1&1&1\\1&e^{\frac{-1}{2}}&e^{-1}\\1&e^{-1}&e^{-2}\end{bmatrix}.$$

The augmented matrix for this matrix equation is calculated as

$$[\mathbf{W}:\mathbf{G}]=\begin{bmatrix}1+e^{-1}&1&-1+e&:1\\1+e^{-1/2}&1&-e^{-1}+e^{1/2}&:1\\2&1&1-e^{-2}&:1\end{bmatrix}$$

The matrix form of initial condition is

$$\begin{bmatrix}1&1&1&:1\end{bmatrix}.$$

The new augmented matrix based on condition can be written as

$$\left[\widetilde{\mathbf{W}}:\widetilde{\mathbf{G}}\right]=\begin{bmatrix}1+e^{-1}&1&-1+e&:1\\1+e^{-1/2}&1&-e^{-1}+e^{1/2}&:1\\1&1&1&:1\end{bmatrix}.$$

Solving this system, the unknown coefficients are obtained as

$$\mathbf{A}=\begin{bmatrix}0&1&0\end{bmatrix}^T$$

By substituting the coefficients into (4), we have approximate solution

$$y_2(x)=e^{-x}$$

which is also exact solution.

Example 2: Now we consider the Firedholm integro-diferantial-difference equation with variable coefficients

$$y''(x) - e^{x+2}y'(x+1) = e^{-x} + \int_0^1 y(t-1)dt$$

with initial conditions $y(0) = 2, y'(0) = -1$.

The exact solution of this problem is $y(x) = 1 + e^{-x}$. Using the procedure in Section 3, for $n = 2$ the fundamental matrix equation of the problem is,

$$\left\{\mathbf{P}_{20}\mathbf{E}\mathbf{B}(\tau_{20})\mathbf{M}^2 + \mathbf{P}_{11}\mathbf{E}\mathbf{B}(\tau_{11})\mathbf{M}^1 - \mathbf{E}\mathbf{K}_{00}\mathbf{Q}\mathbf{B}(\lambda_{00})\mathbf{M}^0\right\}\mathbf{A} = \mathbf{G}$$
$$\left\{\mathbf{P}_{20}\mathbf{E}\mathbf{B}(0)\mathbf{M}^2 + \mathbf{P}_{11}\mathbf{E}\mathbf{B}(1)\mathbf{M} - \mathbf{E}\mathbf{K}_{00}\mathbf{Q}\mathbf{B}(-1)\mathbf{M}^0\right\}\mathbf{A} = \mathbf{G}$$

where

$$\mathbf{P}_{00} = \mathbf{P}_{01} = \mathbf{P}_{10} = \mathbf{P}_{21} = 0$$
$$\mathbf{P}_{20} = 1, \mathbf{P}_{11} = -e^{x+2}$$

$$\tau_{20} = 0, \tau_{11} = 1, \lambda_{00} = -1$$

$$\mathbf{P}_{20} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{P}_{11} = \begin{bmatrix} -e^2 & 0 & 0 \\ 0 & -e^{5/2} & 0 \\ 0 & 0 & -e^3 \end{bmatrix}$$

$$\mathbf{B}(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{B}(1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-1} & 0 \\ 0 & 0 & e^{-2} \end{bmatrix}, \mathbf{B}(-1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^1 & 0 \\ 0 & 0 & e^2 \end{bmatrix}$$

$$\mathbf{M}^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{M} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \mathbf{M}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\mathbf{E} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{\frac{-1}{2}} & e^{-1} \\ 1 & e^{-1} & e^{-2} \end{bmatrix}, \mathbf{K}_{00} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{G} = \begin{bmatrix} 1 \\ e^{-1/2} \\ e^{-1} \end{bmatrix}$$

$$\mathbf{Q} = \begin{bmatrix} 1 & 1 - e^{-1} & \frac{1}{2} - \frac{e^{-2}}{2} \\ 1 - e^{-1} & \frac{1}{2} - \frac{e^{-2}}{2} & \frac{1}{3} - \frac{e^{-2}}{3} \\ \frac{1}{2} - \frac{e^{-2}}{2} & \frac{1}{3} - \frac{e^{-2}}{3} & \frac{1}{4} - \frac{e^{-2}}{4} \end{bmatrix}$$

The augmented matrix for this matrix equation is

$$[\mathbf{W} : \mathbf{G}] = \begin{bmatrix} -1 & 2 & \frac{13-e^2}{2} & : & 1 \\ -1 & 1 + e^{-1/2} & 4e^{-1} + 2e^{-1/2} - \frac{e^2}{2} + \frac{1}{2} & : & e^{-1/2} \\ -1 & 1 + e^{-1} & 4e^{-2} + 2e^{-1} - \frac{e^2}{2} + \frac{1}{2} & : & e^{-1} \end{bmatrix}$$

The new augmented matrix based on conditions can be written as

$$\left[\widetilde{\mathbf{W}}:\widetilde{\mathbf{G}}\right]=\begin{bmatrix}-1 & 2 & \frac{13-e^2}{2} & : & 1\\ 1 & 1 & 1 & : & 2\\ 0 & -1 & -2 & : & -1\end{bmatrix}.$$

Solving this system with given conditions, the unknown coefficients are obtained as

$$\mathbf{A}=\left[1.0100\ 1.0041\ -0.00075\right]^T$$

By substituting the coefficients into (4), we have approximate solution

$$y_2(x)=1.0100+1.0041e^{-x}-0.00075e^{-2x}$$

For $n=4$ the approximate solution is found as

$$y_4(x)=1.0092383+0.9984129e^{-x}+0.00003046e^{-2x}-0.00003918e^{-3x}-0.000014341e^{-4x}$$

The graphics of exact and numerical solution for n=2 and n=4 is given in figure 1 and figure 2.

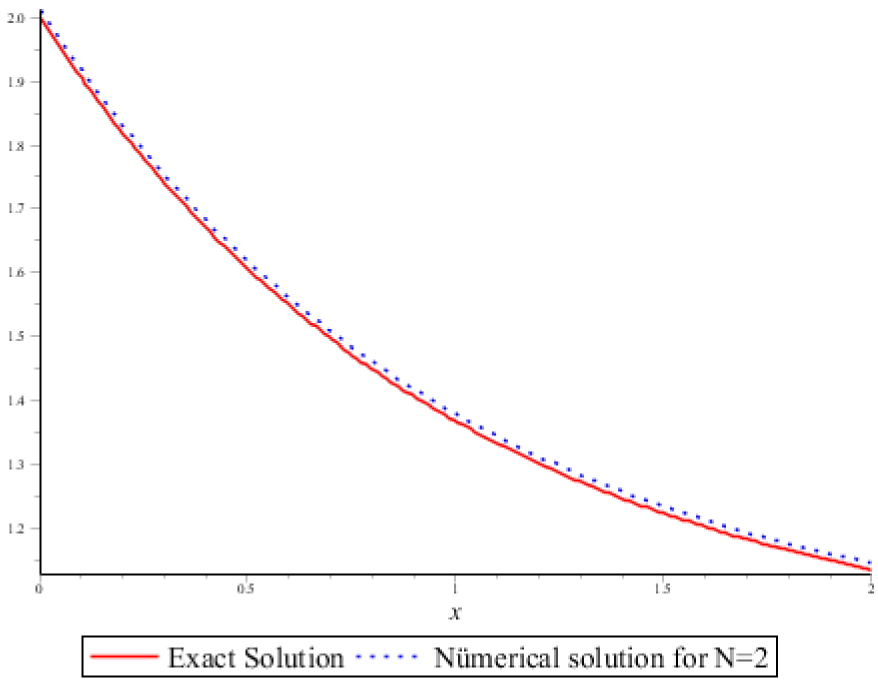

Fig. 1: Numerical and Exact Solution of Example 2 for N=2

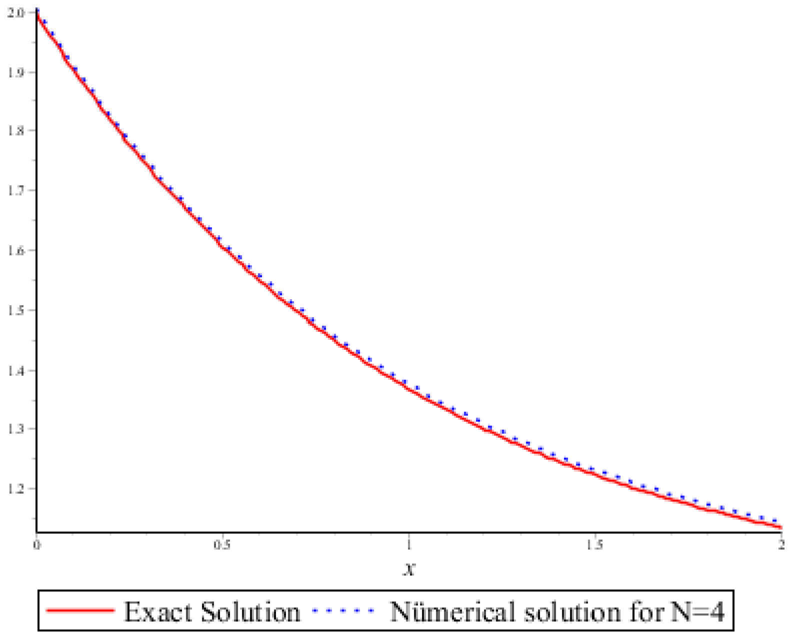

Fig. 2: Numerical and Exact Solution of Example 2 for N=4

6 Conclusions

In this paper, we have presented an exponential colacation method based on Sezer's Matrix Method for the solutions of the Fredholm Integro-differential-difference equations. Also, by using the tecnique given in section 4, the control of the solutions is performed. If the exact solution of the problem is exist and an exponential polynomial, such as in example 1, then the exact solution can be found by this tecnique. It is observed that the presented tecnique gives good result which is too close to exact solution base on values $\overline{R}_2 = 0.0115824$ and $\overline{R}_4 = 0.003157$ calculated for example 2 and figure1 and figure 2. This illustrative example also involves the approximate solutions for $N = 5$, $N = 8$, and $N = 10$, which yield us the exact solution. Therefore, the accuracy of the presented method is validated. This tecnique can be used to test reliability of the solutions of the other problems.

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