New Trends in Mathematical Sciences
http://www.ntmsci.com

Residual correction of the Hermite polynomial solutions of the generalized pantograph equations

Suayip Yuzbasi[1], *Emrah Gok*[2] *and Mehmet Sezer*[2]

[1]Department of Mathematics, Faculty of Science, Akdeniz University, 07058, Antalya, Turkey
[2]Department of Mathematics, Faculty of Science, Celal Bayar University, 45000, Manisa, Turkey

Received: 15 January 2015, Revised: 17 January 2015, Accepted: 10 February 2015
Published online: 15 February 2015

Abstract: In this paper, we consider the residual correction of the Hermite polynomial solutions of the generalized pantograph equations. The Hermite polynomial solutions are obtained by a collocation method. By means of this collocation method, the problem is into a system of algebraic equations and thus unknown coefficients are determined. An error problem is constructed by using the orginal problem and the residual function. Error problem is solved by the Hermite collocation method and thus the imroved approximate solutions are gained. The technique is illustrated by studying the problem for two examples. The obtained results show that the residual corrcetion method is very effective.

Keywords: Pantograph equations, Hermite polynomials, collocation method, residual function.

1 Introduction

In this study, we imrove the approximate solutions based on the Hermite polynomials of the generalized pantograph equation [1-9]

$$y^{(m)}(x) = \sum_{j=0}^{J}\sum_{k=0}^{m-1} P_{jk}(x)y^{(k)}(\alpha_j x+\beta_j)+g(x),\ -\infty < a \le x \le b < \infty \tag{1}$$

under the boundary conditions

$$\sum_{k=0}^{m-1} a_{ik}y^{(k)}(a)+b_{ik}y^{(k)}(b) = \lambda_i,\ i = 0,1,...,m-1 \tag{2}$$

where $y^{(0)}(x) = y(x)$ is a unknown function, $P_{jk}(x)$ and $g(x)$ are the functions defined on interval $a \le x \le b$ and α_j, β_j, a_{ik}, b_{ik}, b_{jk} and λ_i are real constants.

In this improvement, we obtain the approximate solutions of Eq.(1) under conditions (2) in the form

$$y(x) \cong y_{N,M}(x) = y_N(x)+e_{N,M}(x) \tag{3}$$

where

$$y_N(x) = \sum_{n=0}^{N} a_n H_n(x) \tag{4}$$

* Corresponding author e-mail: syuzbasi@akdeniz.edu.tr

is the Hermite polynomial solution and

$$e_{N,M}(x) = \sum_{n=0}^{M} a_n^* H_n(x)$$

is a approximation to error function $e_N(x)$. $e_{N,M}(x)$ is the Hermite polynomial solution of the error problem obtained by using the residual error function. Here, a_n and a_n^*, $(n = 0,1,2,...,N)$ are the unknown Hermite coefficients; N and M are any positive integers and $H_n(x)$, $(n = 0,1,2,...)$ denote the Hermite polynomials defined by

$$H_n(x) = \sum_{k=0}^{[\![\frac{n}{2}]\!]} \frac{(-1)^k (n)!}{k!(n-2k)!} (2x)^{n-2k}.$$

2 Fundamental matrix relations

Let us consider the pantograph equation (3) and find the matrix forms of each term in the equation. First we can convert the approximate solution (4) to matrix forms as

$$y(x) = H(x)A \tag{5}$$

where

$$H(x) = [H_0(x)\ H_1(x)\ \dots\ H_N(x)] \text{ and } A = [a_0\ a_1\ \dots\ a_N]^T.$$

In here, the matrix form $H(x)$ can be written as

$$H(x) = X(x)F^T \tag{6}$$

so that $X(x) = \left[1\ x\ \dots\ x^N\right]$ and for odd values of N:

$$\mathbf{F} = \begin{bmatrix} 2^0 & 0 & \cdots & 0 & 0 \\ 0 & 2^1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^{(\frac{N-5}{2})} \frac{2^0}{0!} \frac{(N-1)!}{(\frac{N-1}{2})!} & 0 & \cdots & 0 & 2^{N-1} \\ 0 & (-1)^{(\frac{N-1}{2})} \frac{2^1}{1!} \frac{N!}{(\frac{N-1}{2})!} & \cdots & 0 & 2^N \end{bmatrix},$$

By putting Eq.(6) into Eq.(5), we have the matrix form

$$y(x) = X(x)F^T A \tag{7}$$

The k-th order derivative of Eq.(7) is given by

$$y^{(k)}(x) = X(x)B^k F^T A \tag{8}$$

where

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

By placing $x \to \alpha_j x + \beta_j$ in Eq. (8), we obtain the matrix form

$$y^{(k)}(\alpha_j x + \beta_j) = X(x)B(\alpha_j, \beta_j)B^k F^T A \tag{9}$$

where

$X(\alpha_j x + \beta_j) = X(x)B(\alpha_j, \beta_j)$ and for $\alpha_j \neq 0$ and $\beta_j \neq 0$:

$$B(\alpha_j, \beta_j) = \begin{bmatrix} \binom{0}{0}(\alpha_j)^0(\beta_j)^0 & \binom{1}{0}(\alpha_j)^0(\beta_j)^1 & \binom{2}{0}(\alpha_j)^0(\beta_j)^2 & \cdots & \binom{N}{0}(\alpha_j)^0(\beta_j)^N \\ 0 & \binom{1}{1}(\alpha_j)^1(\beta_j)^0 & \binom{2}{1}(\alpha_j)^1(\beta_j)^1 & \cdots & \binom{N}{1}(\alpha_j)^1(\beta_j)^{N-1} \\ 0 & 0 & \binom{2}{2}(\alpha_j)^2(\beta_j)^0 & \cdots & \binom{N}{2}(\alpha_j)^2(\beta_j)^{N-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{N}{N}(\alpha_j)^N(\beta_j)^0 \end{bmatrix}$$

and for $\alpha_j \neq 0$ and$\beta_j = 0$:

$$B(\alpha_j, 0) = \begin{bmatrix} (\alpha_j)^0 & 0 & \cdots & 0 \\ 0 & (\alpha_j)^1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (\alpha_j)^N \end{bmatrix}.$$

3 Method of solution

Firstly, we substitute the matrix relations (8) and (9) into Eq.(1) and thus we obtain the matrix equation

$$X(x)B^m F^T A = \sum_{j=0}^{J} \sum_{k=0}^{m-1} P_{jk}(x)X(x)B(\alpha_j, \beta_j)B^k F^T A + g(x) \tag{10}$$

The collocation points defined by

$$x_i = a + \frac{b-a}{N} i, \; i = 0, 1, \ldots, N$$

are substituted into Eq. (10) and thus we obtain system of matrix equations as

$$X(x_i)B^m F^T A = \sum_{j=0}^{J} \sum_{k=0}^{m-1} P_{jk}(x_i)X(x_i)B(\alpha_j, \beta_j)B^k F^T A + g(x_i), \;\; i = 0, 1, \ldots, N$$

or briefly the fundamental matrix equation is

$$\left\{ X(x)B^m F^T - \sum_{j=0}^{J} \sum_{k=0}^{m-1} P_{jk}(x)X(x)B(\alpha_j, \beta_j)B^k F^T \right\} A = G \tag{11}$$

where

$$P_{jk} = \begin{bmatrix} P_{jk}(x_0) & 0 & \cdots & 0 \\ 0 & P_{jk}(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & P_{jk}(x_N) \end{bmatrix}, G = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix}, X = \begin{bmatrix} X(x_0) \\ X(x_1) \\ \vdots \\ X(x_N) \end{bmatrix} = \begin{bmatrix} 1 & x_0 & \cdots & x_0^N \\ 1 & x_1 & \cdots & x_1^N \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & \cdots & x_N^N \end{bmatrix}.$$

Briefly, Eq.(11) can be written in the form

$$W = X(x)B^m F^T - \sum_{j=0}^{J}\sum_{k=0}^{m-1} P_{jk}(x)X(x)B(\alpha_j,\beta_j)B^k F^T \tag{12}$$

$WA = G$Here$[W;G]$, Eq.(12) corresponds to a system of $(N+1)$ linear algebraic equations with the unknown Hermite coefficients $a_0, a_1, ..., a_N$.

By using the relation (8), the matrix form of the conditions (2) becomes

$$U_i A = [\lambda_i] \tag{13}$$

where

$$U_i = [u_{i0}\ u_{i1}\ ...\ u_{iN}] = \left[\sum_{k=0}^{m-1} a_{ik}X(a) + b_{ik}X(b)\right] B^k F^T = \lambda_i,\ i = 0,1,...,m-1.$$

To obtain the solution of Eq. (1) under the conditions (2), by replacing the m rows of matrix (12) by the m row matrices (13) we have the new augmented matrix

$$\tilde{W}A = \tilde{G} \text{ or } \left[\tilde{W};\tilde{G}\right].$$

If rank$\tilde{W}$ = rank$[\tilde{W};\tilde{G}] = N+1$, the unknown coefficients matrix **A** becomes

$$A = (\tilde{W})^{-1}\tilde{G}.$$

Thus, the Hermite coefficients matrix A is uniquely determined. Finally, by substituting the determined coefficients $a_0, a_1, \ldots, a_N$ into Eq.(4), we get the Hermite polynomial solution

$$y_N(x) = \sum_{n=0}^{N} a_n H_n(x) \tag{14}$$

4 Error estimation and improved approximate solutions

In this section, we develope an error estimaton for the Hermite approximate solution of Eq.(1) by means of the residual correction method [10,11] and we improve the approximate solution (14) by using this error estimation. The residual error estimation was presented for the Bessel approximate solutions of the system of the linear multi-pantograph equations [12]. For the problem (1)-(2), we modify the error estimation considered in [10-12].

Let us call $e_N(x) = y(x) - y_N(x)$ as the error function of the Hermite approximation $y_N(x)$ to $y(x)$, where $y(x)$ is the exact solution of problem (1)-(2). Hence, $y_N(x)$ satisfies the following problem:

$$y_N^{(m)}(x) - \sum_{j=0}^{J}\sum_{k=0}^{m-1} P_{jk}(x)y_N^{(k)}(\alpha_j x + \beta_j) = g(x) + R_N(x), \tag{15}$$

$$\sum_{k=0}^{m-1}\left(a_{jk}y_N^{(k)}(a) + b_{jk}y_N^{(k)}(b)\right) = \lambda_j, j = 0,1,...,m-1 \tag{16}$$

can be obtained by substituting $y_N(x)$ into the Eq. (1) and in here $R_N(x)$ is the residual function associated with $y_N(x)$.

By using the method defined in Section 3, we purpose to find an approximation $e_{N,M}(x)$ to the $e_N(x)$.

Subtracting (15) and (16) from (1) and (2), respectively, the error function $e_N(x)$ satisfy the equation

$$e_N^{(m)}(x) - \sum_{j=0}^{J}\sum_{k=0}^{m-1} P_{jk}(x) e_N^{(k)}(\alpha_j x + \beta_j) = -R_N(x), \tag{17}$$

with the homogeneous conditions

$$\sum_{k=0}^{m-1}\left(a_{jk} e_N^{(k)}(a) + b_{jk} e_N^{(k)}(b)\right) = 0, j = 0,1,\ldots,m-1 \tag{18}$$

Solving the error problem (17)-(18) by the method given in Section 3, we obtain the approximation $e_{N,M}(x)$ to $e_N(x)$.

Consequently, we have the improved approximate solution

$$y_{N,M}(x) = y_N(x) + e_{N,M}(x)$$

Note that if the exact solution of the problem is not known, then we can estimate the error function by $e_{N,M}(x)$.

5 Numerical Examples

In this section, we present two numerical examples to demonstrate the efficiency of the method.

Example 1 [2] Let us first consider the problem

$$\begin{cases} y^{(3)}(x) = xy''(2x) - y'(x) - y\left(\frac{x}{2}\right) + x\cos(2x) + \cos\left(\frac{x}{2}\right) \\ y(0) = 1, y'(0) = 0, y''(0) = -1. \end{cases} \tag{19}$$

The exact solution of the problem is given by$y(x) = \cos(x)$. In Table 1, we compare the exact solution and the approximate solutions for $N = 8$ and $M = 10,15$. The actual absolute errors are compared with the estimated absolute errors in Table 2. Also, we give the absolute errors of the improved approximate solutions for $N = 8$ and $M = 10,15$ in Table 2. The absoluter errors for $N = 8$ and $M = 15$ are compared with the Taylor Method (TM) [2] and the Bessel collocation method (BCM) [4] in Table 3. Figure 1-(a) shows a comparison of the actual and estimated absolute error functions. Figure 1-(b) is a plot of the improved absolute error functions for $N = 8$ and $M = 10,15$.

Table 1 Numerical results of the exact and the approximate solutions for $N = 8$ and $M = 10,15$ of the problem (19)

Exact solution		Hermite Polynomial solution	Improved Hermite polynomial solution	
x_i	$y(x_i) = \cos(x_i)$	$y_8(x_i)$	$y_{8,10}(x_i)$	$y_{8,15}(x_i)$
0	1	1	1	1
0.2	0.98006657784124163	0.98006657779186457	0.98006657783667039	0.98006657784124174
0.4	0.92106099400288510	0.92106099410744413	0.92106099402827402	0.92106099400288544
0.6	0.82533561490967822	0.82533562654399351	0.82533561646485876	0.82533561490968421
0.8	0.69670670934716539	0.69670681756116215	0.69670672357995023	0.69670670934722034
1	0.54030230586813976	0.54030280268159703	0.54030237104271506	0.54030230586838979

Table 2 Numerical results of the error functions for $N = 8$ and $M = 10, 15$ of the problem (19)

	Absolute errors	Estimated absolute errors		Improved absolute errors	
x_i	$\lvert e_8(x_i)\rvert = \lvert y(x_i) - y_8(x_i)\rvert$	$\lvert e_{8,10}(x_i)\rvert$	$\lvert e_{8,15}(x_i)\rvert$	$\lvert E_{8,10}(x_i)\rvert$	$\lvert E_{8,15}(x_i)\rvert$
0	0	1.4505e-020	3.8021e-020	0	0
0.2	4.9377e-011	4.4806e-011	4.9377e-011	4.5712e-012	1.1102e-016
0.4	1.0456e-010	7.9170e-011	1.0456e-010	2.5389e-011	3.3307e-016
0.6	1.1634e-008	1.0079e-008	1.1634e-008	1.5552e-009	5.9952e-015
0.8	1.0821e-007	9.3981e-008	1.0821e-007	1.4233e-008	5.4956e-014
1	4.9681e-007	4.3164e-007	4.9681e-007	6.5175e-008	2.5002e-013

Table 3 Comparison of the absolute errors for $N = 8$ and $M = 15$ of problem (19)

x_i	Present method for $N = 8$ and $M = 15$	TM [2] for$N = 8$	BCM [4] for$N = 8$
0	0	0	0
0.2	1.1102e-016	8.0000e-008	4.9377e-11
0.4	3.3307e-016	5.1200e-006	1.0456e-10
0.6	5.9952e-015	5.8322e-005	1.1634e-08
0.8	5.4956e-014	3.2771e-004	1.0821e-07
1	2.5002e-013	1.2503e-003	4.9681e-07

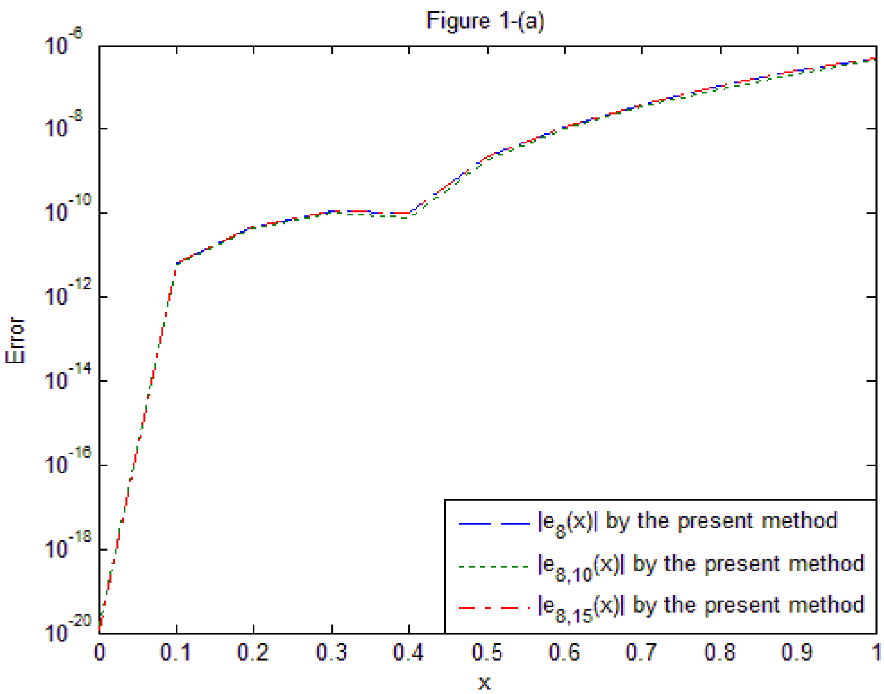

Figure 1. (a) Comparison of the absolute error functions $|e_N(x)| = |y(x) - y_N(x)|$ and the estimated absolute error functions $|e_{N,M}(x)|$ for $N = 8$ and $M = 10, 15$ of problem (19).

Example 2. [13]. We consider the pantograph equation

$$y''(x) = \frac{3}{4}y(x) + y\left(\frac{x}{2}\right) - x^2 + 2, \ \ 0 \le x \le 1$$

with the initial conditions $y(0) = 0$ and $y'(0) = 0$. By applying the method defined in Section 2, we obtain the approximate solution $y(x) = x^2$ which is the exact solution of the problem.

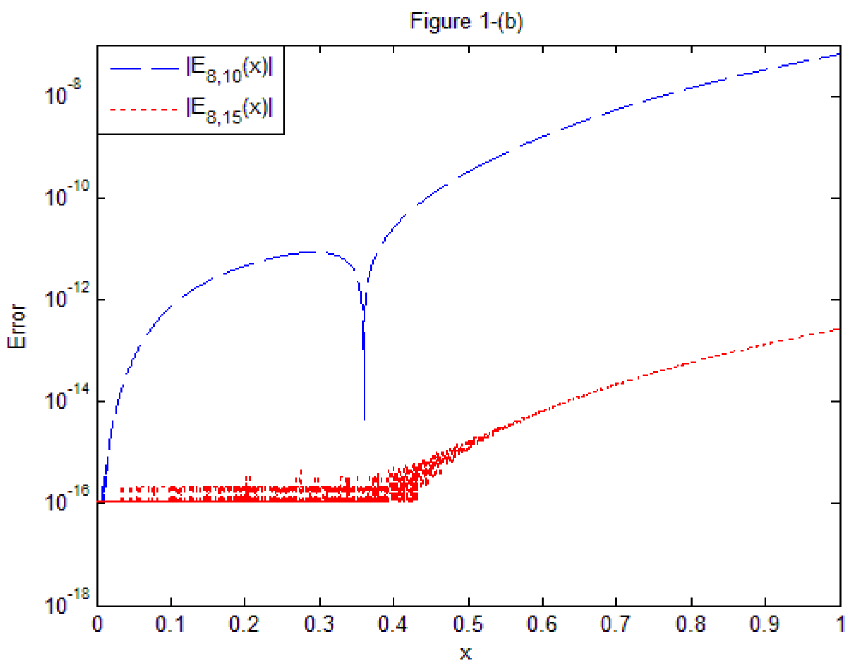

Figure 1. (b) Comparison of the corrected absolute error functions $|E_{N,M}(x)| = |y(x) - y_{N,M}(x)|$for $N = 8$ and $M = 12, 15$ of problem (19).

6 Conclusions

In this paper, we presented the Hermite collocation method for solving the linear generalized pantograph equations and we improved the Hermite polynomial solutions by means of the residual correction. By using the residual correction, an efficiently error estimation can be made for the Hermite collocation method. It is seen from table 2 and figure 1-(a) that the estimated absolute errors $|e_{N,M}(x_i)|$ and the actual absolute errors$|e_N(x_i)| = |y(x_i) - y_N(x_i)|$ are almost the same. Also, we see from table 2 and figure 1-(b) that the residual improvement for approximate solutions is very effective. Moreover, if the exact solution of the problem is not known, then the absolute errors $|e_N(x_i)| = |y(x_i) - y_N(x_i)|$ can be approximately computed with aid of the estimated absolute error function$|e_{N,M}(x)|$. If the problem has an exact solution which is the polynomial function, then it can be obtained by this method. The comparisons of the present method by the other methods show that our method is very effective.

Acknowledgement: The first author is supported by *the Scientific Research Project Administration* of Akdeniz University.

References

[1] Yusufoğlu E., An efficient algorithm for solving generalized pantograph equations with linear functional argument, *Appl. Math. Comput.*, **217** (2010) 3591-3595.

[2] Sezer M., .Akyüz-Daşcıoğlu A, A Taylor method for numerical solution of generalized pantograph equations with lineer functional argument, *J. Comput. Appl. Math.*, **200** (2007) 217-225.

[3] Yu Z.-H., Variational iteration method for solving the multi-pantograph delay equation, *Physics Letters* A **372** (2008) 6475-6479.

[4] Yüzbaşı Ş., Şahin N., Sezer M., A Bessel collocation method for numerical solution of generalized pantograph equations, *Numer Methods Partial Differential Equations* **28** (2012) 1105-1123.

[5] Liu M.Z., Li D., Properties of analytic solution and numerical solution of multi-pantograph equation, *Appl. Math. Comput*. **155** (2004) 853-871.

[6] Yalçınbaş S., Aynigül M., Sezer M., A collocation method using Hermite polynomials for approximate solution of pantograph equations, *J. Frank. Inst*. **348** (2011) 1128–1139.

[7] Saadatmandi A., Dehghan M., Variational iteration method for solving a generalized pantograph equation, *Comput. Math. Appl*. **58** (2009) 2190-2196.

[8] Keskin Y., Kurnaz A., Kiriş M.E., Oturanç G., Approximate solution of generalized pantograph equation by the differential transform method, *Int.J.Nonlinear Sci.* **8** (2007) 159-168.

[9] Işık O.R., Güney Z., Sezer M., Bernstein series solutions of pantograph equations using polynomial interpolation, *J. Dif. Equ. Appl,*. in press, (2010). DOI: 10.1080/10236198.2010.496456.

[10] Oliveira F.A., Collacation and residual correction, *Numer.Math*. **36** (1980) 27-31.

[11] Çelik İ., Collacation method and residual correction using Chebyshev series, *Appl. Math. Comput.* **174** (2006) 910-920.

[12] Yüzbaşı Ş., An efficient algorithm for solving multi-pantograph equation systems, *Computers and Mathematics with Applications*, in press (2012), doi:10.1016/j.camwa.2011.12.062.

[13] Evans D. J., Raslan K. R., The Adomain decomposition method for solving delay differential equation, *Int. J. Comput. Math.* **82** (2005), 49–54.