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Fractional integral inequalities for different functions

Cetin Yildiz, M.Emin Ozdemir and Havva Kavurmacı Onelan

Ataturk University, Faculty of Education of Kazim Karabekir, Department of Mathematics, 25240-Erzurum-Turkey

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Abstract: In this paper, we establish several inequalities for different convex mappings that are connected with the Riemann-Liouville fractional integrals. Our results have some relationships with certain integral inequalities in the literature.

Keywords: Hadamard Inequality, Riemann-Liouville Fractional Integration, Minkowski Inequality.

1 Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and let $a, b \in I$, with $a < b$. The following inequality;

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1}$$

is known in the literature as Hadamard's inequality. Both inequalities hold in reversed direction if f is concave.

In [1], Godunova and Levin introduced the following class of functions.

Definition 1. *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to belong to the class of $Q(I)$ if it is nonnegative and for all $x, y \in I$ and $\lambda \in (0,1)$ satisfies the inequality;*

$$f(\lambda x + (1-\lambda)y) \le \frac{f(x)}{\lambda} + \frac{f(y)}{1-\lambda}. \tag{2}$$

They also noted that all nonnegative monotonic and nonnegative convex functions belong to this class and also proved the following motivating result:

If $f \in Q(I)$ and $x, y, z \in I$, then

$$f(x)(x-y)(x-z) + f(y)(y-x)(y-z) + f(z)(z-x)(z-y) \ge 0. \tag{3}$$

In fact (3) is even equivalent to (2). So it can alternatively be used in the definition of the class $Q(I)$.

In [9], Dragomir *et.al.*, defined the following new class of functions.

* Corresponding author e-mail: cetin@atauni.edu.tr

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Definition 2. *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is P function or that f belongs to the class of $P(I)$, if it is nonnegative and for all $x, y \in I$ and $\lambda \in [0,1]$, satisfies the following inequality;*

$$f(\lambda x + (1-\lambda)y) \leq f(x) + f(y). \tag{4}$$

The power mean $M_r(x,y;\lambda)$ of order r of positive numbers x,y is defined by

$$M_r(x,y;\lambda) = \begin{cases} (\lambda x^r + (1-\lambda)y^r)^{\frac{1}{r}}, & r \neq 0 \\ x^{\lambda} y^{1-\lambda}, & r = 0. \end{cases}$$

In [14], Pearce *et al*. generalized this inequality to $r-$convex positive function f which is defined on an interval $[a,b]$, for all $x, y \in [a,b]$ and $\lambda \in [0,1]$;

$$f(\lambda x + (1-\lambda)y) \leq M_r(f(x), f(y); \lambda) = \begin{cases} (\lambda [f(x)]^r + (1-\lambda)[f(y)]^r)^{\frac{1}{r}}, & if\ r \neq 0 \\ [f(x)]^{\lambda} [f(y)]^{1-\lambda} & if\ r = 0 \end{cases}.$$

We have that $0-$convex functions are simply log-convex functions and $1-$convex functions are ordinary convex functions.

In [19], Varošanec introduced the following class of functions.

Definition 3. *Let $h : J \subset \mathbb{R} \to \mathbb{R}$ be a positive function. We say that $f : I \subset \mathbb{R} \to \mathbb{R}$ is $h-$convex function or that f belongs to the class $SX(h,I)$, if f is nonnegative and for all $x, y \in I$ and $\lambda \in (0,1)$, we have*

$$f(\lambda x + (1-\lambda)y) \leq h(\lambda) f(x) + h(1-\lambda) f(y). \tag{5}$$

If the inequality in (5) is reversed, then f is said to be $h-$concave, i.e., $f \in SV(h,I)$.

Obviously, if $h(\lambda) = \lambda$, then all nonnegative convex functions belong to $SX(h,I)$ and all nonnegative concave functions belong to $SV(h,I)$; if $h(\lambda) = \frac{1}{\lambda}$, then $SX(h,I) = Q(I)$; if $h(\lambda) = 1$, then $SX(h,I) \supseteq P(I)$ and if $h(\lambda) = \lambda^s$, where $s \in (0,1)$, then $SX(h,I) \supseteq K_s^2$. For some recent results for $h-$convex functions we refer to the interested reader to the papers [3], [4] and [15].

In [9], Dragomir *et.al*. proved two inequalities of Hadamard type for class of Godunova-Levin functions and $P-$functions.

Theorem 1. *Let $f \in Q(I)$, $a, b \in I$ with $a < b$ and $f \in L_1[a,b]$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{4}{b-a} \int_a^b f(x)dx. \tag{6}$$

Theorem 2. *Let $f \in P(I)$, $a, b \in I$ with $a < b$ and $f \in L_1[a,b]$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a} \int_a^b f(x)dx \leq 2[f(a) + f(b)]. \tag{7}$$

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In [11], Ngoc *et al*., established following theorem for $r-$convex functions:

Theorem 3. *Let $f:[a,b]\to(0,\infty)$ be $r-$convex function on $[a,b]$ with $a<b$.Then the following inequality holds for $0<r\leq 1$:*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \left(\frac{r}{r+1}\right)^{\frac{1}{r}}\left(f^r(a)+f^r(b)\right)^{\frac{1}{r}}. \tag{8}$$

For related results on $r-$convexity see the papers [10] and [20].

In [16], Sarıkaya *et al*. proved the following Hadamard type inequalities for $h-$convex functions.

Theorem 4. *Let $f\in SX(h,I)$, $a,b\in I$ with $a<b$ and $f\in L_1[a,b]$. Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq[f(a)+f(b)]\int_0^1 h(\alpha)d\alpha. \tag{9}$$

In [17], Sarıkaya *et al*. proved the following Hadamard type inequalities for fractional integrals as follows.

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$ be positive function with $0\leq a<b$ and $f\in L_1[a,b]$. If f is convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha(b)+J_{b^-}^\alpha(a)\right]\leq\frac{f(a)+f(b)}{2} \tag{10}$$

with $\alpha>0$.

Now we give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 4. *Let $f\in L_1[a,b]$. The Riemann-Liouville integrals $J_{a^+}^\alpha f$ and $J_{b^-}^\alpha f$ of order $\alpha>0$ with $a\geq 0$ are defined by*

$$J_{a^+}^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt,\ \ x>a$$

and

$$J_{b^-}^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt,\ \ x<b$$

respectively where $\Gamma(\alpha)=\int_0^\infty e^{-u}u^{\alpha-1}du$. Here is $J_{a^+}^0 f(x)=J_{b^-}^0 f(x)=f(x)$.

In the case of $\alpha=1$, the fractional integral reduces to the classical integral.

For some recent results connected with fractional integral inequalities see [2], [5]-[8], [12], [13], [17] and [18].

The main purpose of this paper is to present new Hadamard's inequalities for fractional integrals via functions that belongs to the classes of $Q(I)$, $P(I)$, $SX(h,I)$ and $r-$convex.

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2 MAIN RESULTS

Theorem 6. *Let $f \in Q(I)$, $a, b \in I$ with $0 \leq a < b$ and $f \in L_1[a,b]$. Then the following inequality for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}(b)+J_{b^-}^{\alpha}(a)\right] \tag{11}$$

with $\alpha > 0$.

Proof. Since $f \in Q(I)$, we have

$$2\left(f(x)+f(y)\right) \geq f\left(\frac{x+y}{2}\right)$$

for all $x, y \in I$ (with $\lambda = \frac{1}{2}$ in (1.2)).

If we choose $x = ta+(1-t)b$ and $y = (1-t)a+tb$ in above inequality, we get

$$2\left[f(ta+(1-t)b)+f((1-t)a+tb)\right] \geq f\left(\frac{a+b}{2}\right). \tag{12}$$

Then multiplying both sides of (12) by $t^{\alpha-1}$ and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{gathered}
2\int_0^1 t^{\alpha-1}\left[f(ta+(1-t)b)+f((1-t)a+tb)\right]dt \geq f\left(\frac{a+b}{2}\right)\int_0^1 t^{\alpha-1}dt \\
2\int_a^b\left(\frac{b-u}{b-a}\right)^{\alpha-1} f(u)\frac{du}{b-a}+2\int_a^b\left(\frac{v-a}{b-a}\right)^{\alpha-1} f(v)\frac{dv}{b-a} \geq \frac{1}{\alpha}f\left(\frac{a+b}{2}\right) \\
\frac{2\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}(b)+J_{b^-}^{\alpha}(a)\right] \geq f\left(\frac{a+b}{2}\right).
\end{gathered}$$

The proof is complete.

Remark. If we choose $\alpha = 1$ in Theorem 6, then the inequalities (11) become the inequalities (6).

Theorem 7. *Let $f \in P(I)$, $a, b \in I$ with $a < b$ and $f \in L_1[a,b]$. Then one has inequality for fractional integrals:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}(b)+J_{b^-}^{\alpha}(a)\right] \leq 2\left(f(a)+f(b)\right) \tag{13}$$

with $\alpha > 0$.

Proof. According to (4) with $x = ta+(1-t)b$, $y = (1-t)a+tb$ and $\lambda = \frac{1}{2}$, we find that

$$f\left(\frac{a+b}{2}\right) \leq f(ta+(1-t)b)+f((1-t)a+tb) \tag{14}$$

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for all $t \in [0,1]$. Thus multiplying both sides of (14) by $t^{\alpha-1}$ and integrating the resulting inequality with respect to t over $[0,1]$, we have

$$f\left(\frac{a+b}{2}\right)\int_0^1 t^{\alpha-1}dt \leq \int_0^1 t^{\alpha-1}\left[f(ta+(1-t)b)+f((1-t)a+tb)\right]dt$$
$$\frac{1}{\alpha}f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}(b)+J_{b^-}^{\alpha}(a)\right]$$
$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}(b)+J_{b^-}^{\alpha}(a)\right]$$

and the first inequality is proved.

Since $f \in P(I)$, we have

$$f(ta+(1-t)b) \leq f(a)+f(b)$$

and

$$f((1-t)a+tb) \leq f(a)+f(b).$$

By adding these inequalities, we get

$$f(ta+(1-t)b)+f((1-t)a+tb) \leq 2\left[f(a)+f(b)\right]. \tag{15}$$

Then multiplying both sides of (15) by $t^{\alpha-1}$ and integrating the resulting inequality with respect to t over $[0,1]$, we have

$$\int_0^1 t^{\alpha-1}\left[f(ta+(1-t)b)+f((1-t)a+tb)\right]dt \leq 2\left[f(a)+f(b)\right]\int_0^1 t^{\alpha-1}dt$$
$$\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}(b)+J_{b^-}^{\alpha}(a)\right] \leq 2\left(f(a)+f(b)\right)$$

and thus the second inequality is proved.

Remark. If we choose $\alpha = 1$ in Theorem 7, then the inequalities (13) become the inequalities (7).

Theorem 8. *Let $f : [a,b] \to (0,\infty)$ be $r-$convex function on $[a,b]$ with $a < b$ and $0 < r \leq 1$.Then the following inequality for fractional integral inequlities holds*:

$$\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}(b)+J_{b^-}^{\alpha}(a)\right] \leq \left[\left(\frac{1}{\alpha+\frac{1}{r}}\right)^{r}[f(a)]^{r}+\left(\beta(\alpha,\frac{r+1}{r})\right)^{r}[f(b)]^{r}\right]^{\frac{1}{r}}$$
$$+\left[\left(\beta(\alpha,\frac{r+1}{r})\right)^{r}[f(a)]^{r}+\left(\frac{1}{\alpha+\frac{1}{r}}\right)^{r}[f(b)]^{r}\right]^{\frac{1}{r}}.$$

Proof. Since f is r-convex function and $r > 0$, we have

$$f(ta+(1-t)b) \leq \left(t\,[f(a)]^{r}+(1-t)\,[f(b)]^{r}\right)^{\frac{1}{r}}$$

and

$$f((1-t)a+tb) \leq \left((1-t)\,[f(a)]^{r}+t\,[f(b)]^{r}\right)^{\frac{1}{r}}$$

for all $t \in [0,1]$.

By adding these inequalities we have

$$f(ta+(1-t)b)+f((1-t)a+tb) \leq (t[f(a)]^r+(1-t)[f(b)]^r)^{\frac{1}{r}}+((1-t)[f(a)]^r+t[f(b)]^r)^{\frac{1}{r}}.$$

Then multiplying both sides of above inequality by $t^{\alpha-1}$ and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}&\int_0^1 t^{\alpha-1}[f(ta+(1-t)b)+f((1-t)a+tb)]\,dt\\ &\leq \int_0^1 t^{\alpha-1}(t[f(a)]^r+(1-t)[f(b)]^r)^{\frac{1}{r}}\,dt+\int_0^1 t^{\alpha-1}((1-t)[f(a)]^r+t[f(b)]^r)^{\frac{1}{r}}\,dt.\end{aligned}$$

It is easy to observe that

$$\int_0^1 t^{\alpha-1}[f(ta+(1-t)b)+f((1-t)a+tb)]\,dt=\frac{\Gamma(\alpha)}{(b-a)^\alpha}\left[J_{a^+}^\alpha(b)+J_{b^-}^\alpha(a)\right]$$

Using Minkowski inequality, we have

$$\begin{aligned}\int_0^1 t^{\alpha-1}(t[f(a)]^r+(1-t)[f(b)]^r)^{\frac{1}{r}}\,dt &\leq \left[\left(\int_0^1 t^{\alpha+\frac{1}{r}-1}f(a)dt\right)^r+\left(\int_0^1 t^{\alpha-1}(1-t)^{\frac{1}{r}}f(b)dt\right)^r\right]^{\frac{1}{r}}\\ &=\left[\left(\frac{1}{\alpha+\frac{1}{r}}\right)^r[f(a)]^r+\left(\beta(\alpha,\frac{r+1}{r})\right)^r[f(b)]^r\right]^{\frac{1}{r}}\end{aligned}$$

and similarly

$$\begin{aligned}\int_0^1 t^{\alpha-1}((1-t)[f(a)]^r+t[f(b)]^r)^{\frac{1}{r}} &\leq \left[\left(\int_0^1 t^{\alpha-1}(1-t)^{\frac{1}{r}}f(a)dt\right)^r+\left(\int_0^1 t^{\alpha+\frac{1}{r}-1}f(b)dt\right)^r\right]^{\frac{1}{r}}\\ &=\left[\left(\beta(\alpha,\frac{r+1}{r})\right)^r[f(a)]^r+\left(\frac{1}{\alpha+\frac{1}{r}}\right)^r[f(b)]^r\right]^{\frac{1}{r}}.\end{aligned}$$

Thus

$$\begin{aligned}\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J_{a^+}^\alpha(b)+J_{b^-}^\alpha(a)\right] &\leq \left[\left(\frac{1}{\alpha+\frac{1}{r}}\right)^r[f(a)]^r+\left(\beta(\alpha,\frac{r+1}{r})\right)^r[f(b)]^r\right]^{\frac{1}{r}}\\ &\quad+\left[\left(\beta(\alpha,\frac{r+1}{r})\right)^r[f(a)]^r+\left(\frac{1}{\alpha+\frac{1}{r}}\right)^r[f(b)]^r\right]^{\frac{1}{r}}.\end{aligned}$$

This proof is complete.

Remark. In Theorem 8, if we choose $\alpha=1$, then we obtain the inequalities (8).

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Theorem 9. *Let $f \in SX(h,I)$, $a,b \in I$ with $a < b$ and $f \in L_1[a,b]$. Then one has inequality for $h-$convex functions via fractional integrals*

$$\frac{1}{\alpha h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}(b) + J_{b^-}^{\alpha}(a)\right] \tag{16}$$
$$\leq [f(a)+f(b)]\int_0^1 t^{\alpha-1}[h(t)+h(1-t)]\,dt.$$

Proof. According to (5) with $x = ta+(1-t)b$, $y = (1-t)a+tb$ and $\alpha = \frac{1}{2}$ we find that

$$f\left(\frac{a+b}{2}\right) \leq h\left(\frac{1}{2}\right) f(ta+(1-t)b) + h\left(\frac{1}{2}\right) f((1-t)a+tb) \tag{17}$$
$$\leq h\left(\frac{1}{2}\right)[f(ta+(1-t)b)+f((1-t)a+tb)].$$

Then multiplying the firts inequalitiy in (17) by $t^{\alpha-1}$ and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$f\left(\frac{a+b}{2}\right)\int_0^1 t^{\alpha-1}dt \leq h\left(\frac{1}{2}\right)\int_0^1 t^{\alpha-1}[f(ta+(1-t)b)+f((1-t)a+tb)]\,dt$$
$$\frac{1}{\alpha h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}(b) + J_{b^-}^{\alpha}(a)\right] \tag{18}$$

and the first inequality in (16) is proved.

Since $f \in SX(h,I)$, we have

$$f(tx+(1-t)y) \leq h(t)f(x)+h(1-t)f(y)$$

and

$$f((1-t)x+ty) \leq h(1-t)f(x)+h(t)f(y).$$

By adding these inequalities we get

$$f(tx+(1-t)y)+f((1-t)x+ty) \leq [h(t)+h(1-t)][f(x)+f(y)]. \tag{19}$$

By using (19) with $x = a$ and $y = b$ we have

$$f(ta+(1-t)b)+f((1-t)a+tb) \leq [h(t)+h(1-t)][f(a)+f(b)]. \tag{20}$$

Then multiplying both sides of (20) by $t^{\alpha-1}$ and integrating the resulting inequality with respect to t over $[0,1]$, we get

$$\int_0^1 t^{\alpha-1}[f(ta+(1-t)b)+f((1-t)a+tb)]\,dt \leq \int_0^1 t^{\alpha-1}[h(t)+h(1-t)][f(a)+f(b)]\,dt,$$

$$\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}(b)+J_{b^-}^{\alpha}(a)\right] \leq [f(a)+f(b)]\int_0^1 t^{\alpha-1}[h(t)+h(1-t)]\,dt \tag{21}$$

and thus the second inequality is proved. We obtain inequalities (16) from (18) and (21).The proof is complete.

Remark. In Theorem 9;

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–if we choose $h(t)=t$, then the inequalities (16) become the inequalities (10) of Theorem 5.
–if we take $\alpha=1$, then we obtain the inequalities (9).
–Let $\alpha=1$. If we choose $h(t)=t$ and $h(t)=1$, then (16) reduce to (1) and (7), respectively.

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