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Generalized Mittag-Leffler Function and Its Properties

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Abstract: Recently, Srivastava, Çetinkaya and Kıymaz [18] defined the generalized Pochhammer symbol and obtained some relations. In this paper, we define the generalized Mittag-Leffler function via the generalized Pochammer symbol and present some recurrence relation, derivative properties, integral representation. Moreover, we obtain a relation between wright hypergeometric function and the generalized Mittag-Leffler function.

Keywords: Pochhammer symbol, generalized Mittag-Leffler function, Wright Hypergeometric function.

1 Introduction

Mittag-Leffler function plays an important role in the solution of fractional order differential equations [13]. Applications of Mittag-Leffler function are given follows: fluid flow, electric networks, probability, statistical distrubition theory. Moreover, different kinds and properties of Mittag-Leffler functions were introduced and obtained in [4].

The well known Mittag-Leffler function was defined by Mittag-Leffler in [7], [8], [9]:

$$E_{\alpha}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}. \tag{1}$$

Then, Wiman, Agarwal and Humbert [1], [5], [15], [16] generalized the Mittag-Leffler function by

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!}. \tag{2}$$

Afterward, it was Prabhakar [12] who defined the generalization of the Mittag-Leffler function by

$$E^{\delta}_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{(\delta)_k}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!} \tag{3}$$

where $\alpha,\ \beta,\ \delta\in\mathbb{C}$ with $Re(\alpha)>0$. Recently, Özarslan and Yılmaz Yaşar [17] defined the extended Mittag-Leffler function by

$$E^{(\gamma;c)}_{\alpha,\beta}(z;p):=\sum_{k=0}^{\infty}\frac{B_p(\gamma+k,c-\gamma)}{B(\gamma,c-\gamma)}\frac{(c)_k}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!},\quad (p\geq 0;\ \ Re(c)>Re(\gamma)>0) \tag{4}$$

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where $B_p(x,y)$

$$B_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{\frac{-p}{t(1-t)}} dt, \ \ (Re(p) > 0, \ \ Re(x) > 0, \ \ Re(y) > 0)$$

is the extended Beta function defined in [2], [3]. They obtained some properties of the extended Mittag-Leffler function. Moreover, Özarslan and Özergin [10], [11] defined the extended Riemann-Liouville fractional derivative operator and obtained some generating relations for the extended hypergeometric function. On the other hand, Kurulay and Bayram [6] obtained some properties of the generalized Mittag-Leffler function.

In this paper, we define the generalized Mittag-Leffler function by

$$E_{\beta,\gamma}^{(\lambda,\rho)}(z) := \sum_{k=0}^{\infty} \frac{(\lambda;\rho)_k}{\Gamma(\lambda)\Gamma(\beta k+\gamma)} \frac{z^k}{k!}, \quad Re(\beta) > 0, \ \ Re(\lambda) > 0 \tag{5}$$

where

$$(\lambda;\rho)_k = \frac{1}{\Gamma(\lambda)} \int_0^{\infty} t^{\lambda+k-1} e^{-t-\frac{\rho}{t}} dt (Re(\rho) > 0, \ \ Re(\lambda+k) > 0 \ \ \text{when} \ \ \rho = 0) \tag{6}$$

is the generalized Pochhammer symbol defined in [18].

We organize the paper as follows: In section 2, we give some properties of the generalized Mittag-Lefler function. Furthermore, we give the Mellin transform of the generalized Mittag-Leffler function via the Wright hypergeometric function [14]. In section 3, we obtain some recurrence formula and derivatives of the generalized Mittag-Leffler function.

2 Some Properties of the Generalized Mittag-Leffler Function

In this section, we give integral representation and Mellin transform of the generalized Mittag-Leffler function.

Theorem 1. For the generalized Mittag-Leffler function, we have the following integral representation formula

$$E_{\beta,\gamma}^{(\lambda,\rho)}(z) = \frac{1}{[\Gamma(\lambda)]^2} \int_0^{\infty} t^{\lambda-1} e^{-t-\frac{\rho}{t}} E_{\beta,\gamma}(tz) dt \tag{7}$$

where $Re(\lambda) > 0, \ \ Re(\beta) > 0, \ \ Re(\gamma) > 0$.

Proof. Using (6) in (5), we have

$$E_{\beta,\gamma}^{(\lambda,\rho)}(z) = \sum_{n=0}^{\infty} \left(\frac{1}{\Gamma(\lambda)} \int_0^{\infty} t^{\lambda+n-1} e^{-t-\frac{\rho}{t}} dt\right) \frac{1}{\Gamma(\lambda)\Gamma(\beta n+\gamma)} \frac{z^n}{n!}.$$

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Interchanging the order of summation and integral, which is satisfied under the conditions of the theorem and using (2), we have

$$E_{\beta,\gamma}^{(\lambda,\rho)}(z) = \frac{1}{[\Gamma(\lambda)]^2}\int_0^\infty t^{\lambda-1}e^{-t-\frac{\rho}{t}}\sum_{n=0}^{\infty}\frac{(tz)^n}{\Gamma(\beta n+\gamma)n!}dt$$
$$= \frac{1}{[\Gamma(\lambda)]^2}\int_0^\infty t^{\lambda-1}e^{-t-\frac{\rho}{t}}E_{\beta,\gamma}(tz)dt.$$

Corollary 1. Taking $t = \frac{u}{1-u}$ in Theorem 1, we have

$$E_{\beta,\gamma}^{(\lambda,\rho)}(z) = \frac{1}{[\Gamma(\lambda)]^2}\int_0^1 u^{\lambda-1}(1-u)^{-\lambda-1}e^{\frac{-u^2-\rho(1-u)^2}{u(1-u)}}E_{\beta,\gamma}(\frac{u}{1-u}z)du.$$

In the following theorem, we obtain the Mellin transform of the generalized Mittag-Leffler function by means of Wright generalized hypergeometric function. Here, we choose to consider the Wright generalized hypergeometric function:

$${}_p\Psi_q(z) = {}_p\Psi_q\left[\begin{matrix}(a_1,A_1),\ (a_2,A_2),...,(a_p,A_p)\\(b_1,B_1),\ (b_2,B_2),...,(b_p,B_p)\end{matrix}\ ,z\right] \tag{8}$$
$$= \sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(a_j+A_jk)z^k}{\prod_{j=1}^{q}\Gamma(b_j+B_jk)k!},$$

where the coefficients $A_i(i=1,...,p)$ and $B_j(j=1,...,q)$ are positive real numbers such that

$$1+\sum_{j=1}^{q}B_j-\sum_{i=1}^{p}A_i \geq 0.$$

Theorem 2. Mellin transform of the generalized Mittag-Leffler function is given by

$$\mu\{E_{\beta,\gamma}^{(\lambda,\rho)}(z)\} = \frac{\Gamma(s)}{[\Gamma(\lambda)]^2}\ {}_1\Psi_1\left[\begin{matrix}(\lambda+s,1)\\(\beta,\alpha)\end{matrix}\ ,z\right] \tag{9}$$
$$(Re(\lambda)>0,\ Re(\gamma)>0,\ Re(\beta)>0,\ Re(\alpha)>0,\ Re(s)>0\ )$$

where ${}_1\Psi_1$ is the Wright generalized hypergeometric function.

Proof. Mellin transform is given by

$$\mu\{E_{\beta,\gamma}^{(\lambda,\rho)}(z)\} = \int_0^\infty p^{s-1}E_{\beta,\gamma}^{(\lambda,\rho)}(z)dp. \tag{10}$$

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Putting (7) into (10), we have

$$\begin{aligned}
\mu\{E_{\beta,\gamma}^{(\lambda,\rho)}(z)\} &= \int_0^\infty p^{s-1} E_{\beta,\gamma}^{(\lambda,\rho)}(z)dp \\
&= \int_0^\infty p^{s-1} \frac{1}{[\Gamma(\lambda)]^2}(\int_0^\infty t^{\lambda-1} e^{-t-\frac{\rho}{t}} E_{\beta,\gamma}(tz)dt)dp \\
&= \frac{1}{[\Gamma(\lambda)]^2}\int_0^\infty p^{s-1}(\int_0^\infty t^{\lambda-1} e^{-t-\frac{\rho}{t}} E_{\beta,\gamma}(tz)dt)dp \\
&= \frac{1}{[\Gamma(\lambda)]^2}\int_0^\infty t^{\lambda-1} E_{\beta,\gamma}(tz)\int_0^\infty p^{s-1} e^{-t-\frac{\rho}{t}} dpdt.
\end{aligned}$$

Taking $u=\frac{p}{t}$, and using the gamma function $\Gamma(s)=\int_0^\infty u^{s-1}e^{-u}du$, we have

$$\begin{aligned}
\mu\{E_{\beta,\gamma}^{(\lambda,\rho)}(z)\} &= \frac{1}{[\Gamma(\lambda)]^2}\int_0^\infty t^{\lambda-1} E_{\beta,\gamma}(tz)\int_0^\infty (ut)^{s-1} e^{-t-u} tdudt \\
&= \frac{1}{[\Gamma(\lambda)]^2}\int_0^\infty t^{\lambda+s-1} E_{\beta,\gamma}(tz)e^{-t}\ (\int_0^\infty u^{s-1} e^{-u}du)dt \\
&= \frac{\Gamma(s)}{[\Gamma(\lambda)]^2}\int_0^\infty t^{\lambda+s-1} e^{-t} E_{\beta,\gamma}(tz)dt.
\end{aligned}$$

Now, using the series form of Mittag-Leffler function $E_{\beta,\gamma}(tz)$, we have

$$\mu\{E_{\beta,\gamma}^{(\lambda,\rho)}(z)\} = \frac{\Gamma(s)}{[\Gamma(\lambda)]^2}\int_0^\infty t^{\lambda+s-1} e^{-t}(\sum_{k=0}^\infty \frac{t^k z^k}{\Gamma(\alpha k+\beta)k!})dt.$$

Interchanging the order of summation and integral, under the conditions, $Re(s)>0$, $Re(\lambda)>0$, $Re(\alpha)>0$, $Re(\beta)>0$, we get

$$\mu\{E_{\beta,\gamma}^{(\lambda,\rho)}(z)\} = \frac{\Gamma(s)}{[\Gamma(\lambda)]^2}\sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+\beta)k!}\int_0^\infty t^{\lambda+s+k-1} e^{-t}dt.$$

Using Gamma function, we have

$$\mu\{E_{\beta,\gamma}^{(\lambda,\rho)}(z)\} = \frac{\Gamma(s)}{[\Gamma(\lambda)]^2}\sum_{k=0}^\infty \frac{z^k\,\Gamma(\lambda+s+k)}{\Gamma(\alpha k+\beta)k!}.$$

Taking into consideration of Wright generalized hypergeometric function (8), we have

$$\mu\{E_{\beta,\gamma}^{(\lambda,\rho)}(z)\} = \frac{\Gamma(s)}{[\Gamma(\lambda)]^2}\ {}_1\Psi_1[\begin{matrix}(\lambda+s,1) \\ (\beta,\alpha)\end{matrix}\ ,z].$$

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Corollary 2. Taking $s = 1$ in theorem , we get

$$\int_0^\infty E_{\beta,\gamma}^{(\lambda,\rho)}(z)d\rho = \frac{1}{[\Gamma(\lambda)]^2}\, {}_1\Psi_1\left[\begin{array}{c}(\lambda+1,1)\\(\beta,\alpha)\end{array},z\right].$$

3 Recurrence Relation and Derivative Properties of Generalized Mittag-Leffler Function

In this section, we obtain derivatives of generalized Mittag-Leffler function and give the recurrence formula.

Theorem 3. For the generalized Mittag-Leffler function, we have

$$\frac{d^n}{dz^n}\{E_{\beta,\gamma}^{(\lambda,\rho)}(z)\} = \frac{[\Gamma(\lambda+n)]^2}{[\Gamma(\lambda)]^2}E_{\beta,n\beta+\gamma}^{(\lambda+n,\rho)}(z). \tag{11}$$

Proof. Considering integral representation of Mittag-Leffler function, we have

$$E_{\beta,\gamma}^{(\lambda,\rho)}(z) = \frac{1}{[\Gamma(\lambda)]^2}\int_0^\infty t^{\lambda-1}e^{-t-\frac{\rho}{t}}E_{\beta,\gamma}(tz)dt.$$

Writing $E_{\beta,\gamma}(tz)$ in above integral, we have

$$E_{\beta,\gamma}^{(\lambda,\rho)}(z) = \frac{1}{[\Gamma(\lambda)]^2}\int_0^\infty t^{\lambda-1}e^{-t-\frac{\rho}{t}}\left(\sum_{k=0}^\infty \frac{t^k z^k}{\Gamma(\beta k+\gamma)k!}\right)dt.$$

Taking derivative with respect to z, in the integral representation, we get

$$\begin{aligned}\frac{d}{dz}\{E_{\beta,\gamma}^{(\lambda,\rho)}(z)\} &= \frac{1}{[\Gamma(\lambda)]^2}\int_0^\infty t^{\lambda-1}e^{-t-\frac{\rho}{t}}\left(\sum_{k=1}^\infty \frac{t^k z^{k-1}}{\Gamma(\beta k+\gamma)(k-1)!}\right)dt\\ &= \frac{1}{[\Gamma(\lambda)]^2}\int_0^\infty t^{\lambda}e^{-t-\frac{\rho}{t}}\left(\sum_{k=0}^\infty \frac{t^k z^k}{\Gamma(\beta k+\beta+\gamma)k!}\right)dt\\ &= \frac{1}{[\Gamma(\lambda)]^2}\int_0^\infty t^{(\lambda+1)-1}e^{-t-\frac{\rho}{t}}E_{\beta,\beta+\gamma}(tz)dt\\ &= \frac{1}{[\Gamma(\lambda)]^2}[\Gamma(\lambda+1)]^2E_{\beta,\beta+\gamma}^{(\lambda+1,\rho)}(z)\\ &= \frac{[\Gamma(\lambda+1)]^2}{[\Gamma(\lambda)]^2}E_{\beta,\beta+\gamma}^{(\lambda+1,\rho)}(z). \end{aligned} \tag{12}$$

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Taking derivative with respect to z, in (12), we get

$$\begin{aligned}
\frac{d^2}{dz^2}\{E_{\beta,\gamma}^{(\lambda,\rho)}(z)\} &= \frac{[\Gamma(\lambda+1)]^2}{[\Gamma(\lambda)]^2}\frac{d}{dz}\{E_{\beta,\beta+\gamma}^{(\lambda+1,\rho)}(z)\} \\
&= \frac{[\Gamma(\lambda+1)]^2}{[\Gamma(\lambda)]^2}\frac{d}{dz}[\frac{1}{[\Gamma(\lambda+1)]^2}\int_0^\infty t^{(\lambda+1)-1}e^{-t-\frac{\rho}{t}}E_{\beta,\beta+\gamma}(tz)dt] \\
&= \frac{d}{dz}[\frac{1}{[\Gamma(\lambda)]^2}\int_0^\infty t^{(\lambda+1)-1}e^{-t-\frac{\rho}{t}}\sum_{k=0}^{\infty}\frac{t^k z^k}{\Gamma(\beta k+\beta+\gamma)k!}dt] \\
&= \frac{1}{[\Gamma(\lambda)]^2}[\int_0^\infty t^{(\lambda+1)-1}e^{-t-\frac{\rho}{t}}(\sum_{k=1}^{\infty}\frac{t^k z^{k-1}}{\Gamma(\beta k+\beta+\gamma)(k-1)!})dt] \\
&= \frac{1}{[\Gamma(\lambda)]^2}\int_0^\infty t^{(\lambda+1)-1}e^{-t-\frac{\rho}{t}}\,(\sum_{k=0}^{\infty}\frac{t^{k+1}z^k}{\Gamma(\beta k+2\beta+\gamma)k!})dt \\
&= \frac{1}{[\Gamma(\lambda)]^2}\int_0^\infty t^{\lambda+1}e^{-t-\frac{\rho}{t}}\,(\sum_{k=0}^{\infty}\frac{t^k z^k}{\Gamma(\beta k+2\beta+\gamma)k!})dt \\
&= \frac{1}{[\Gamma(\lambda)]^2}\int_0^\infty t^{\lambda+1}e^{-t-\frac{\rho}{t}}E_{\beta,2\beta+\gamma}(tz)dt \\
&= \frac{1}{[\Gamma(\lambda)]^2}[\Gamma(\lambda+2)]^2\,E_{\beta,2\beta+\gamma}^{(\lambda+2,\rho)}(z) \\
&= \frac{[\Gamma(\lambda+2)]^2}{[\Gamma(\lambda)]^2}E_{\beta,2\beta+\gamma}^{(\lambda+2,\rho)}(z).
\end{aligned}$$

Similar way, we can find

$$\frac{d^3}{dz^3}\{E_{\beta,\gamma}^{(\lambda,\rho)}(z)\} = \frac{[\Gamma(\lambda+3)]^2}{[\Gamma(\lambda)]^2}E_{\beta,3\beta+\gamma}^{(\lambda+3,\rho)}(z).$$

Continuing this procedure, we get

$$\frac{d^n}{dz^n}\{E_{\beta,\gamma}^{(\lambda,\rho)}(z)\} = \frac{[\Gamma(\lambda+n)]^2}{[\Gamma(\lambda)]^2}E_{\beta,n\beta+\gamma}^{(\lambda+n,\rho)}(z).$$

Theorem 4. For the generalized Mittag-Leffler function, the following differentiation formula hold

$$\frac{d^n}{dz^n}\{z^{\gamma-1}E_{\beta,\gamma}^{(\lambda,\rho)}(cz^\beta)\} = z^{\gamma-n-1}E_{\beta,\gamma-n}^{(\lambda,\rho)}(cz^\beta). \tag{13}$$

Proof. Taking cz^β in place of z, and multiplying with $z^{\gamma-1}$ in (11), we get the result.

Theorem 5. For $E_{\beta,\gamma}^{(\lambda,\rho)}(z)$, we have

$$\frac{d^n}{d\rho^n}\{E_{\beta,\gamma}^{(\lambda,\rho)}(z)\} = \frac{(-1)^n}{(\lambda-1)^2(\lambda-2)^2...(\lambda-n)^2}E_{\beta,\gamma}^{(\lambda-n,\rho)}(z). \tag{14}$$

Proof. Using the integral representation of the generalized Mittag-Leffler function and taking derivative with respect to ρ, n-times, we get the result.

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Theorem 6. (Recurrence Relation) The following recurrence formula holds for generalized Mittag-Leffler function

$$E_{\beta,\gamma-1}^{(\lambda,\rho)}(z^{\beta}) = \frac{z^{1-\gamma}}{2}\frac{d}{dz}\{z^{\gamma}E_{\beta,\gamma+1}^{(\lambda,\rho)}(z^{\beta})\} - \frac{\lambda^{2}}{2}\frac{d}{d\rho}\{E_{\beta,\gamma-1}^{(\lambda+1,\rho)}(z^{\beta})\}. \tag{15}$$

Proof. Taking into consideration of the definition of the generalized Mittag-Leffler function and the derivative properties, we get the result.

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