NEW TRENDS IN MATHEMATICAL SCIENCES
Vol.2, No. 3, p, 2014, p. 166-177
ISSN 2147-5520 - www.ntmsci.com

Joint Laplace-Fourier Transforms For Fractional PDEs

A. AGHILI* and H. ZEINALI

Department of Applied Mathematics, Faculty of Mathematical Sciences,University of Guilan.
P.O.Box, 1841, Rasht – Iran, arman.aghili@gmail.com, homa_zeinaly@yahoo.com

Abstract: In this paper, the authors implemented one dimensional Laplace transform to evaluate certain integrals, series and solve non homogeneous fractional PDEs. Illustrative examples are also provided. The results reveal that the integral transforms are very effective and convenient.

Keywords: Laplace transform; Bessel's functions; Parabolic cylindrical function; Wave equation; Caputo fractional derivative..

1. Introduction and Notations

In recent years, it has turned out that many phenomena in fluid mechanics, physics, biology, Engineering and other areas of sciences can be successfully modeled by the use of fractional derivatives. That is because of the fact that, a realistic modeling of a physical phenomenon having dependence not only at the time instant, but also the previous time history can be successfully achieved by using fractional calculus. Fractional differential equations arise in unification of diffusion and wave propagation phenomenon. The time fractional heat equation, which is a mathematical model of a wide range of important physical phenomena, is a partial differential equation obtained from the classical heat equation by replacing the first time derivative by a fractional derivative of order α, $0 < \alpha \le 1$. In the last part of this paper we consider the time fractional wave equation (time fractional in the -Caputo sense).

In this work, we consider methods and results for the partial fractional diffusion equations which arise in applications. Several methods have been introduced to solve fractional differential equations, the popular Laplace transform method , [1] , [2] , [3] , [5] the Fourier transform method [6], the iteration method and operational method [6].

Definition.1.1. Laplace transform of the function $f\ (t)$ is defined as follows

$$L\{f(t);t \to s\} = \int_0^\infty e^{-st} f(t)dt := F(s). \tag{1.1}$$

If $L\{f\ (t)\} = F(s)$, then $L^{-1}\{F(s)\}$ is given by

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}F(s)ds, \tag{1.2}$$

where $F(s)$ is analytic in the region $\mathrm{Re}(s) > c$.

Example.1.2. Evaluate Laplace transform of the parabolic cylindrical function

$$D_P(z)=\frac{\exp(-\frac{z^2}{4})}{\Gamma(-p)}\int_0^{+\infty} x^{-(p+1)}\exp(-xz-\frac{x^2}{4})dx\,.$$

Solution. By definition we have

$$L\{D_P(z);z\to s\}=\int_0^{\infty} e^{-sz}\left(\frac{\exp(-\frac{z^2}{4})}{\Gamma(-p)}\int_0^{+\infty} x^{-(p+1)}\exp(-xz-\frac{x^2}{4})dx\right)dz\,,$$

changing the order of integrals we get

$$L\{D_P(z);z\to s\}=\int_0^{\infty}\frac{x^{-(p+1)}e^{-\frac{x^2}{4}}}{\Gamma(-p)}\left\{\int_0^{\infty} e^{-z(s+x)-\frac{z^2}{4}}dz\right\}dx\,,$$

the inner integral is Laplace transform of the function $e^{-\frac{z^2}{4}}$, so we can write the final result as following

$$L\{D_P(z);z\to s\}=\frac{\sqrt{\pi}}{\Gamma(-p)}\int_0^{\infty} x^{-(p+1)}e^{(s+x)^2-\frac{x^2}{4}}erfc(s+x)dx\,.$$

Which can be written in the form

$$L\{D_P(z);z\to s\}=\frac{2}{\Gamma(-p)}\int_0^{\infty} e^{-t^2}\int_0^{t-s} x^{-(p+1)}e^{(s+x)^2-\frac{x^2}{4}}dxdt,$$

which can be evaluated by using partial method of integrating.

Definition.1.3. The left Caputo fractional derivative of order α ($n-1\le\alpha<n$) is defined as

$${}^C D_t^{\alpha} f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau.$$

Lemma.1.4. Let $F(s)$ be Laplace transform of the function $f(t)$ of exponential order with respect to t ,then we have

$$\begin{aligned}L^{-1}\{\frac{1}{s^{\alpha}}F(\frac{1}{s});s\to t\}=&\int_0^t\left(\int_0^{\infty}\sqrt{x}J_1(2\sqrt{\tau x})f(x)dx\right)\frac{-d\tau}{\Gamma(\alpha)\sqrt{\tau}(t-\tau)^{1-\alpha}}\\&+\left(L\{g(t);s=0\}\right)\delta(t).\end{aligned} \tag{1.3}$$

In which $0<\alpha<1$.

Proof. We can write

$$\frac{1}{s^{\alpha}}F(\frac{1}{s})=s^{1-\alpha}\left(\frac{1}{s}F(\frac{1}{s})\right), \tag{1.4}$$

where $0<1-\alpha=\beta<1$, on the other hand from Laplace transform table we know that

$$g(t)=L^{-1}\{\frac{1}{s}F(\frac{1}{s});s\to t\}=\int_0^{\infty}J_0(2\sqrt{tx})f(x)dx, \tag{1.5}$$

and also using the fact that (see [4])

$$L\{{}^{C}D^{\beta}f(t);t\rightarrow s\}=s^{\beta}F(s)-s^{\beta-1}f(0), \tag{1.6}$$

in which $0<\beta<1$. From relations (1.4), (1.5) and (1.6) we arrive at

$$L^{-1}\{\frac{1}{s^{\alpha}}F(\frac{1}{s});s\rightarrow t\}=L^{-1}\{s^{1-\alpha}L\{g(t)\}-g(0)+g(0)\}$$
$$={}^{C}D^{(1-\alpha)}g(t)+\left(L\{g(t);s=0\}\right)\delta(t),$$

in which $g(t)=\int_0^{\infty}J_0(2\sqrt{tx})f(x)dx$ and fractional derivation is considered in the Caputo sense. The final result will be obtained as below

$$L^{-1}\{\frac{1}{s^{\alpha}}F(\frac{1}{s});s\rightarrow t\}=\frac{1}{\Gamma(\alpha)}\int_0^{t}\frac{g'(\tau)}{(t-\tau)^{1-\alpha}}d\tau+\left(L\{g(t);s=0\}\right)\delta(t).$$

Which can be re written in the form

$$L^{-1}\{\frac{1}{s^{\alpha}}F(\frac{1}{s});s\rightarrow t\}=\int_0^{t}\left(\int_0^{\infty}\sqrt{x}J_1(2\sqrt{\tau x})f(x)dx\right)\frac{-d\tau}{\Gamma(\alpha)\sqrt{\tau}(t-\tau)^{1-\alpha}}$$
$$+\left(L\{g(t);s=0\}\right)\delta(t).$$

Definition.1.5. Laguerre differential equation is defined as

$$xy''+(1-x)y'+ny=0;\quad y(0)=n!,$$

which can be solved by using Laplace transform .Let us assume that

$$L\{y(x)\}=L\{L_n(x)\}=F(s),$$

taking Laplace transform of Laguerre differential equation we obtain

$$F(s)=\frac{1}{s}(1-\frac{1}{s})^{n}=L\{L_n(x)\}.$$

Lemma.1.6. (Schouten-Vanderpol) Consider a function $f(t)$ which has the Laplace transform $F(s)$ which is analytic in the half plane $\mathrm{Re}(s)>c$. If $q(s)$ is also analytic for $\mathrm{Re}(s)>c$, then the inverse of $F(q(s))$ is as follows

$$L^{-1}\{F(q(s));s\rightarrow t\}=\int_0^{\infty}f(\tau)\left[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{-q(s)\tau}e^{ts}ds\right]d\tau.$$

Special case: $q(s)=\sqrt{s}$;

$$L^{-1}\{F(\sqrt{s});s\rightarrow t\}=\frac{1}{2t\sqrt{\pi t}}\int_0^{\infty}\tau f(\tau)\exp(-\frac{\tau^{3}}{4t})d\tau.$$

Proof: See [6].

Lemma.1.7. Let X be an absolutely continuous random variable assuming non – negative values, $f(t)$ its density and $F(s)$ its Laplace transform (in such case $F(0)=1$ ($F(s)>0$ and $F^{'}(s)<0$ for real).The knowledge of $F(s)$ on the non – negative real line allows us to obtain some real moments of $f(t)$ through fractional integral and derivative of the α - th order of $F(s)$. Many other expected values may be found from $F(s)$ or $F^{'}(s)$.

The following relations hold true

$$1-E(X^{\alpha-1})=\int_0^\infty t^{\alpha-1}f(t)dt=\frac{1}{\Gamma(1-\alpha)}\int_0^\infty \frac{F(s)}{s^\alpha}ds,\quad 0<\alpha<1$$

$$2-E(X^{\alpha})=\int_0^{+\infty} t^{\alpha}f(t)dt=\frac{-1}{\Gamma(1-\alpha)}\int_0^{+\infty} \frac{F'(s)ds}{s^\alpha},\quad 0<\alpha<1$$

$$3-E(X^{\alpha})=\int_0^\infty t^{\alpha}f(t)dt=\frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \frac{1-F(s)}{s^{\alpha+1}}ds,\quad 0<\alpha<1$$

$$4-E(X^{n+\alpha-1})=\int_0^\infty t^{n+\alpha-1}f(t)dt=\frac{(-1)^n}{\Gamma(1-\alpha)}\int_0^\infty \frac{F^{(n)}(s)}{s^\alpha}ds,\quad 0<\alpha<1.$$

Proof : 1- By definition we have

$$\frac{1}{\Gamma(1-\alpha)}\int_0^\infty \frac{F(s)}{s^\alpha}ds=\frac{1}{\Gamma(1-\alpha)}\int_0^\infty s^{-\alpha}\{\int_0^\infty e^{-st}f(t)dt\}ds=\frac{1}{\Gamma(1-\alpha)}\int_0^\infty f(t)\{\int_0^\infty s^{-\alpha}e^{-st}ds\}dt,$$

which is equivalent to

$$\frac{1}{\Gamma(1-\alpha)}\int_0^\infty \frac{F(s)}{s^\alpha}ds=\int_0^\infty t^{\alpha-1}f(t)dt=E(X^{\alpha-1}).$$

2- We have

$$\frac{-1}{\Gamma(1-\alpha)}\int_0^{+\infty} \frac{F'(s)ds}{s^\alpha}=\frac{-1}{\Gamma(1-\alpha)}\int_0^\infty s^{-\alpha}\{\frac{d}{ds}\int_0^\infty e^{-st}f(t)dt\}ds=\frac{1}{\Gamma(1-\alpha)}\int_0^\infty s^{-\alpha}\{\int_0^\infty te^{-st}f(t)dt\}ds,$$

changing the order of integrals we get

$$\frac{-1}{\Gamma(1-\alpha)}\int_0^{+\infty} \frac{F'(s)ds}{s^\alpha}=\int_0^\infty t^\alpha f(t)dt=E(X^\alpha).$$

3- Regarding the definition of Laplace transform we know that

$$\frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \frac{1-F(s)}{s^{\alpha+1}}ds=\frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty s^{-(\alpha+1)}(\int_0^\infty \{\delta(t)-f(t)\}e^{-st}dt)ds,$$

changing the order of integrals and again using definition of Laplace transform the result will be obtained as following

$$\frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \frac{1-F(s)}{s^{\alpha+1}}ds=\frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty(\delta(t)-f(t))\left(\int_0^\infty e^{-st}s^{-\alpha-1}ds\right)dt$$

$$=-\frac{\alpha}{\Gamma(1-\alpha)}\Gamma(-\alpha)\int_0^\infty t^\alpha f(t)dt=\int_0^\infty t^\alpha f(t)dt=E(X^\alpha).$$

4-By definition we have

$$\frac{1}{\Gamma(1-\alpha)}\int_0^\infty \frac{F^{(n)}(s)}{s^\alpha}ds=\frac{1}{\Gamma(1-\alpha)}\int_0^\infty s^{-\alpha}\frac{\partial^n}{\partial s^n}\left(\int_0^\infty e^{-st}f(t)dt\right)ds$$

$$=\frac{1}{\Gamma(1-\alpha)}\int_0^\infty s^{-\alpha}\left((-1)^n\int_0^\infty t^n e^{-st}f(t)dt\right)ds,$$

changing the order of integrals we get

$$\frac{1}{\Gamma(1-\alpha)}\int_0^\infty \frac{F^{(n)}(s)}{s^\alpha}ds=\frac{(-1)^n}{\Gamma(1-\alpha)}\int_0^\infty t^n f(t)\left(\int_0^\infty s^{-\alpha}e^{-st}ds\right)dt=(-1)^n\int_0^\infty t^{n+\alpha-1}f(t)dt=(-1)^n E(X^\alpha).$$

Example.1.8. Evaluate the following integral

$$\int_0^\infty \frac{e^{-\sigma^2 s^2/2}}{\sqrt{s}}erfc(\frac{\sigma}{\sqrt{2}}s)ds.$$

Solution. Let $F(s)=e^{-\sigma^2 s^2/2}erfc(\frac{\sigma}{\sqrt{2}}s)$, then regarding table of Laplace transform (see [8]) we have

$$F(s)=e^{-\sigma^2 s^2/2}erfc(\frac{\sigma}{\sqrt{2}}s)=L\{f(t)=\frac{\sqrt{2}}{\sigma\sqrt{\pi}}e^{-t^2/2\sigma^2};t\to s\},$$

one can prove that the function $f(t)$ is a probability density function because

$$f(t)=\frac{\sqrt{2}}{\sigma\sqrt{\pi}}e^{-t^2/2\sigma^2}>0,\int_0^\infty f(t)dt=\int_0^\infty\frac{\sqrt{2}}{\sigma\sqrt{\pi}}e^{-t^2/2\sigma^2}dt=\frac{\sqrt{2}}{\sigma\sqrt{\pi}}(\sqrt{2}\sigma)\frac{\sqrt{\pi}}{2}=1,$$

now by using first part of the previous lemma for $\alpha=0.5$, we can write

$$\int_0^\infty \frac{e^{-\sigma^2 s^2/2}}{\sqrt{s}}erfc(\frac{\sigma}{\sqrt{2}}s)ds=\frac{\sqrt{2}}{\sigma}\int_0^\infty\frac{1}{\sqrt{t}}e^{-\frac{t^2}{2\sigma^2}}dt,$$

which can be evaluated by making a change of variable $t^2=u$ as below

$$\int_0^\infty \frac{e^{-\sigma^2 s^2/2}}{\sqrt{s}}erfc(\frac{\sigma}{\sqrt{2}}s)ds=\frac{\sqrt{2}}{\sigma}\int_0^\infty\frac{1}{\sqrt{t}}e^{-\frac{t^2}{2\sigma^2}}dt=\frac{2^{\frac{7}{4}}}{\sqrt{\sigma}}\Gamma(\frac{5}{4}).$$

Definition.1.9. The Fourier transform of the function $f(x)$ is defined as following

$$F(\alpha)=F\{f(x);x\to\alpha\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}f(x)e^{-i\alpha x}dx,$$

provided that the integral exists. The inverse of Fourier transform is

$$F^{-1}\{F(\alpha);\alpha\to x\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}F(\alpha)e^{i\alpha x}d\alpha.$$

Definition.1.10. The finite Fourier sine transform of $f\,(x\,)$ in $0<x\,<L$ is defined by

$$F_s(n)=\frac{2}{L}\int_0^L f\,(x\,)\sin\frac{n\pi x}{L}dx\,,$$

where n is an integer. The function $f\,(x\,)$ is then called the inverse finite Fourier sine transform of $F_s(n)$ and is given by

$$f\,(x\,)=\sum_{n=1}^{\infty}F_s(n)\sin\frac{n\pi x}{L}.$$

Similarly the finite Fourier cosine transform of $f\,(x\,)$ in $0<x\,<L$ is defined by

$$F_c(n)=\frac{2}{L}\int_0^L f\,(x\,)\cos\frac{n\pi x}{L}dx\,,$$

where n is an integer. The function $f\,(x\,)$ is then called the inverse finite Fourier cosine transform of $F_c(n)$ and is given by

$$f\,(x\,)=\frac{1}{2}F_c(0)+\sum_{n=1}^{\infty}F_c(n)\cos\frac{n\pi x}{L}.$$

Lemma 1.11.The following relation holds true

$$\sum_{n=0}^{\infty}L_n(t)P_n(x)=\frac{e^{\frac{t}{2}}}{\sqrt{2(1-x)}}J_0\left(\frac{t}{2}\sqrt{\frac{1+x}{1-x}}\right).$$

Proof. We know the generating function of Legendre polynomials as below

$$\sum_{n=0}^{\infty}t^nP_n(x)=\frac{1}{\sqrt{1-2tx+t^2}},$$

substituting $t=1-\frac{1}{p}$ in the above relationship we will have

$$\sum_{n=0}^{\infty}(1-\frac{1}{p})^nP_n(x)=\frac{1}{\sqrt{1-2x(1-\frac{1}{p})+(1-\frac{1}{p})^2}}=\frac{p}{\sqrt{p^2-2xp(p-1)+(p-1)^2}},$$

one can rewrite the above relationship as below

$$\sum_{n=0}^{\infty}\frac{1}{p}(1-\frac{1}{p})^nP_n(x)=\frac{1}{\sqrt{2(1-x)}}\cdot\frac{1}{\sqrt{(p-\frac{1}{2})^2+\frac{1+x}{4(1-x)}}},$$

on the other hand using definition 1.5 we know that $L\{L_n(t);p\}=\frac{1}{p}(1-\frac{1}{p})^n$ and

$$L\{J_0(\frac{1}{2}\sqrt{\frac{1+x}{1-x}}t);p\}=\frac{e^{-\frac{t}{2}}}{\sqrt{(p-\frac{1}{2})^2+\frac{1+x}{4(1-x)}}},$$

therefore if we take inverse Laplace transform of both sides of the above equation, we will have

$$\sum_{n=0}^{\infty}L_n(t)P_n(x)=\frac{e^{\frac{t}{2}}}{\sqrt{2(1-x)}}J_0(\frac{t}{2}\sqrt{\frac{1+x}{1-x}}).$$

2 One dimensional Laplace transform of certain special functions

The Bessel functions of the second kind, denoted by $Y_\alpha(x)$ or $N_\alpha(x)$ are solutions of the Bessel differential equation that have a singularity at the origin $x=0$. These are called Neumann or Weber functions as well. The Bessel functions are also valid for complex arguments x, and an important special case is that of a purely imaginary argument. In this case, the solutions to the Bessel equation are called the modified Bessel functions (or occasionally the hyperbolic Bessel functions) of the first and second kind, and are defined by any of these equivalent alternatives

$$I_\alpha(x)=i^{-\alpha}J_\alpha(ix),\ K_\alpha(x)=\frac{\pi}{2}i^{\alpha+1}H_\alpha^{(1)}(ix), \tag{2.1}$$

in which $H_\alpha^{(1)}(x)=J_\alpha(x)+iY_\alpha(x)$ is Hankel function, and $J_\alpha(x), Y_\alpha(x)$ are Bessel functions of the first and second kind.

Lemma.2.1. The following relationship holds true

$$L\{K_0(\beta x);x\to s\}=\frac{\cos^{-1}\left(\frac{s}{\beta}\right)}{\sqrt{\beta^2-s^2}}. \tag{2.2}$$

Proof. By the integral representation of modified Bessel function $K_0(\beta x)$, we have

$$K_0(\beta x)=\int_0^{+\infty}\cos(\beta x\sinh\theta)d\theta. \tag{2.3}$$

This leads to

$$L\{K_0(\beta x)\}=\int_0^{+\infty}\left(\int_0^{+\infty}\cos(\beta x\sinh\theta)d\theta\right)e^{-sx}dx. \tag{2.4}$$

By changing the order of integrals we have

$$L\{K_0(\beta x)\}=\int_0^{+\infty}\left(\int_0^{+\infty}\cos(\beta x\sinh\theta)e^{-sx}dx\right)d\theta=\int_0^{+\infty}\frac{s}{s^2+(\beta\sinh\theta)^2}d\theta. \tag{2.5}$$

It leads us to the following relationship

$$L\{K_0(\beta x)\}=\int_0^{+\infty}\frac{d\theta}{s\cosh^2\theta(1+\frac{\beta^2-s^2}{s^2}\tanh^2\theta)}, \tag{2.6}$$

At this point, let us introduce a change of variables $u=\frac{\sqrt{\beta^2-s^2}}{s}\tanh\theta$, we get

$$L\{K_0(\beta x)\} = \left. \frac{\tan^{-1}\left(\frac{\sqrt{\beta^2 - s^2}}{s} \tanh\theta \right)}{\sqrt{\beta^2 - s^2}} \right|_0^{+\infty}, \tag{2.7}$$

finally by using the fact that $\tanh^2\theta = 1 - \frac{1}{\cosh^2\theta}$ and some easy calculations one gets

$$L\{K_0(\beta x)\} = \left. \frac{\tan^{-1}\left(\frac{\sqrt{\beta^2 - s^2}}{s} \sqrt{1 - \frac{1}{\cosh^2\theta}} \right)}{\sqrt{\beta^2 - s^2}} \right|_0^{+\infty} = \frac{\tan^{-1}\left(\frac{\sqrt{\beta^2 - s^2}}{s} \right)}{\sqrt{\beta^2 - s^2}} = \frac{\cos^{-1}\left(\frac{s}{\beta} \right)}{\sqrt{\beta^2 - s^2}}.$$

Example.2.2. Show that

$$\int_0^\infty K_0(\beta x)dx = \frac{\pi}{2\beta}. \tag{2.8}$$

Solution. It suffices 173ol et $p = 0$ in lemma 2.1 to get the result.

Lemma.2.3. Assume $|\mathrm{Re}(v)| < 1, x > 0$, then we have the following integral representation

$$N_v(x) = -\frac{2}{\pi}\int_0^\infty \cos(x\cosh t - \frac{1}{2}v\pi)\cosh(vt)dt, \tag{2.9}$$

in special case $v = 0$, we have

$$N_0(x) = -\frac{2}{\pi}\int_0^\infty \cos(x\cosh t)dt. \tag{2.10}$$

Proof. See [5].

Lemma.2.4. The following relationship holds true

$$L\{N_0(x)\} = -\frac{2}{\pi}\frac{\ln\left(s \pm \sqrt{s^2+1}\right)}{\sqrt{s^2+1}}. \tag{2.11}$$

Proof. From lemma 2.3, we have

$$N_0(x) = -\frac{2}{\pi}\int_0^\infty \cos(x \cosh t)dt.$$

This leads to

$$L\{N_0(x); x \to s\} = -\frac{2}{\pi}\int_0^\infty \left(\int_0^\infty \cos(x \cosh t)dt \right) e^{-sx}dx.$$

Changing the order of integrals we get

$$L\{N_0(x); x \to s\} = -\frac{2}{\pi}\int_0^{+\infty} \left(\int_0^{+\infty} \cos(x\cosh t)e^{-sx}dx \right) dt = -\frac{2}{\pi}\int_0^{+\infty} \frac{s}{s^2 + \cosh^2 t}dt. \tag{2.12}$$

Consequently we get the following relationship

$$L\{N_0(x);x\to s\}=-\frac{2}{\pi}\int_0^{+\infty}\frac{dt}{s\sinh^2 t(\frac{s^2+1}{s^2}\coth^2 t-1)},\tag{2.13}$$

by a change of variables $u=\frac{\sqrt{s^2+1}}{s}\coth t$, we have

$$L\{N_0(x);x\to s\}=-\frac{2}{\pi}\left.\frac{\coth^{-1}\left(\frac{\sqrt{s^2+1}}{s}\coth t\right)}{\sqrt{s^2+1}}\right|_0^{+\infty},\tag{2.14}$$

finally by using the fact that $\coth^2 t=1+\frac{1}{\sinh^2 t}$ and some easy calculations we have

$$L\{N_0(x)\}=-\frac{2}{\pi}\left.\frac{\coth^{-1}\left(\frac{\sqrt{s^2+1}}{s}\sqrt{1+\frac{1}{\sinh^2 t}}\right)}{\sqrt{s^2+1}}\right|_0^{+\infty}=-\frac{2}{\pi}\frac{\coth^{-1}\left(\frac{\sqrt{s^2+1}}{s}\right)-\coth^{-1}\infty}{\sqrt{s^2+1}}=-\frac{2}{\pi}\frac{\coth^{-1}\left(\frac{\sqrt{s^2+1}}{s}\right)}{\sqrt{s^2+1}},$$

by some manipulations we get finally

$$L\{N_0(x)\}=-\frac{2}{\pi}\frac{\sinh^{-1}(s)}{\sqrt{s^2+1}}.\tag{2.15}$$

Now let $\sinh^{-1}s=z$ and $e^z=y$ to get the relationship $y^2-2xy-1=0$,it is provided that

$$y=x\pm\sqrt{x^2+1},$$

and consequently it means that

$$L\{N_0(x)\}=-\frac{2}{\pi}\frac{\ln\left(s\pm\sqrt{s^2+1}\right)}{\sqrt{s^2+1}}.$$

Lemma.2.5. (Bobylev-Cercignani) Let F(p) be an analytic function having no singularities in the cut plane $C\setminus R_-$.
Assume that $\overline{F(p)}=F(\overline{p})$ and the limiting value

$$F^{\pm}(t)=\lim_{\phi\to\pi^-}F(te^{\pm i\phi}),\quad F^+(t)=\overline{F^-(t)}$$

exist for almost all

(i) $F(p)=o(1)$ for $|p|\to\infty$ and $F(p)=o(|p|^{-1})$ for $|p|\to 0$,uniformly in any sector $|\arg p|<\pi-\eta,\pi>\eta>0;$

(ii) there exists $\varepsilon>0$ such that for every $\pi-\varepsilon<\phi\le\pi,$

$$\frac{F(re^{\pm i\phi})}{1+r}\in L^1(R_+),\quad \left|F(re^{\pm i\phi})\right|\le a(r),$$

where a(r) does not depend on ϕ and $a(r)e^{-\delta r}\in L^1(R_+)$ for any $\delta>0$. Then, in the notation of the problem,

$$f(t)=L^{-1}[F(s)]=\frac{1}{\pi}\int_0^{\infty}\mathrm{Im}[F^-(\eta)]e^{-t\eta}d\eta.$$

Proof. See[6].

Example.2.6. Using the previous lemma let $F(s)=\sqrt{s}e^{-\sqrt{s}}$, one can check that $F(s)$ satisfies the conditions of lemma. Hence ,we may easily find the inverse of $F(s)$ by using the formula

$$L^{-1}\{F(s);t\}=f(t)=\frac{1}{\pi}\int_0^\infty \operatorname{Im}[\lim_{\phi\to\pi}F(\eta e^{-i\phi})]e^{-t\eta}d\eta,$$

substituting in the above formula leads to

$$L^{-1}\{\sqrt{s}e^{-\sqrt{s}};t\}=\frac{1}{\pi}\int_0^\infty \sqrt{\eta}\cos\sqrt{\eta}e^{-t\eta}d\eta,$$

making a change of variable $\eta=4u$ and using table of integrals, we have

$$L^{-1}\{\sqrt{s}e^{-\sqrt{s}};t\}=\frac{8}{\pi}\int_0^\infty \sqrt{u}\cos 2\sqrt{u}e^{-4tu}du=\frac{\sqrt{\pi}}{(2\sqrt{t})^5}(2t-1)e^{-\frac{1}{4t}}.$$

3 Main results

The dynamic behavior of an overhead power wire which is connected to elecric locomotives by the panthograph can be simulated by a fractional wave partial differential equation which contains a term that shows the instant forces pushed towards the wire in certain moments.

Problem.3.1. Consider the following fractional PDE which describes the vibrations of an overhead wire under the power of an electric locomotive as a pantograph

$$\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}}=c^2\frac{\partial^2 u}{\partial x^2}+\frac{a}{\rho V}\delta(t-\frac{x}{V}),\quad 0<x<L, 0>t, 0.5<\alpha<1, \tag{6.1}$$

under the following initial and 175oundary conditions

$$\begin{cases}u(x,0)=u_t(x,0)=0, & 0<x<L\\ u(0,t)=u(L,t)=0, & t>0\end{cases}. \tag{6.2}$$

Solution. We solve this fractional PDE by using joint Laplace-Fourier finite sine transform. Taking Laplace transform of (6.1) with respect to t we have

$$p^{2\alpha}U(x,p)=c^2\frac{\partial^2 U(x,p)}{\partial x^2}+\frac{a}{\rho V}e^{-\frac{p}{V}x}, \tag{6.3}$$

now taking finite Fourier sine transform of the above relationship with respect to x we get

$$p^{2\alpha}\bar{U}(n,p)=c^2\left\{-\frac{n^2\pi^2}{L^2}\bar{U}(n,p)\right\}+\frac{a}{\rho V}\frac{2n\pi V^2}{(V^2n^2\pi^2+p^2L^2)}[1-(-1)^n e^{-\frac{p}{V}L}], \tag{6.4}$$

one can rewrite the above equation as below

$$\bar{U}(n,p)=\frac{2n\pi Va}{\rho L^2(p^2+\frac{V^2n^2\pi^2}{L^2})(p^{2\alpha}+\frac{c^2n^2\pi^2}{L^2})}[1-(-1)^n e^{-\frac{p}{V}L}]. \tag{6.5}$$

By using lemma 1.6 we have

$$L^{-1}\{\frac{1}{p^{2\alpha}+\frac{c^2n^2\pi^2}{L^2}};p\to t\}=\frac{1}{2\pi i}\int_0^\infty e^{-t\eta}\{\frac{1}{\eta^{2\alpha}e^{2i\alpha\pi}+\frac{c^2n^2\pi^2}{L^2}}-\frac{1}{\eta^{2\alpha}e^{-2i\alpha\pi}+\frac{c^2n^2\pi^2}{L^2}}\}d\eta, \tag{6.6}$$

from (6.5), (6.6) and convolution theorem for Laplace transform we have

$$L^{-1}\{\overline{U}(n,p);p\to t\}=\frac{a}{\rho L\pi i}\int_0^\infty \sin(\frac{Vn\pi}{L}(t-\xi))[1-(-1)^n H(t-\xi-\frac{L}{V})]\times$$

$$\int_0^\infty e^{-\xi\eta}\{\frac{1}{\eta^{2\alpha}e^{2i\alpha\pi}+\frac{c^2n^2\pi^2}{L^2}}-\frac{1}{\eta^{2\alpha}e^{-2i\alpha\pi}+\frac{c^2n^2\pi^2}{L^2}}\}d\eta d\xi,$$

taking inverse finite Fourier sine transform, the result will be

$$u(x,t)=-\frac{2ai}{L^2\rho\pi}\sum_{n=1}^\infty \sin\frac{n\pi x}{L}\int_0^\infty \sin(\frac{Vn\pi}{L}(t-\xi))[1-H(t-\xi-\frac{L}{V})]\times$$

$$\int_0^\infty e^{-\xi\eta}\{\frac{1}{\eta^{2\alpha}e^{2i\alpha\pi}+\frac{c^2n^2\pi^2}{L^2}}-\frac{1}{\eta^{2\alpha}e^{-2i\alpha\pi}+\frac{c^2n^2\pi^2}{L^2}}\}d\eta d\xi.$$

Problem.3.2. Let us consider the following non homogenous time fractional PDE

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\lambda\frac{\partial u}{\partial x}+\mu u+\int_0^t h(\xi)d\xi,\quad x\in R,0>t,0<\alpha\le 1,\ \ u(x,0)=\kappa \tag{6.7}$$

Solution. (Joint Laplace - Fourier transform) Taking Laplace transform with respect to t of (6.7). We have,

$$s^\alpha\overline{U}(x,s)-\kappa s^{\alpha-1}=\lambda\frac{\partial\overline{U}}{\partial x}(x,s)+\mu\overline{U}(x,s)+\frac{H(s)}{s},$$

in which $\overline{U}(x,s)=L\{u(x,t);t\to s\}$. Now taking Fourier transform with respect to x we have

$$s^\alpha\hat{\overline{U}}(w,s)-\kappa s^{\alpha-1}\sqrt{2\pi}\delta(w)=i\lambda w\hat{\overline{U}}(w,s)+\mu\hat{\overline{U}}(w,s)+\sqrt{2\pi}\delta(w)\frac{H(s)}{s},$$

or

$$\hat{\overline{U}}(w,s)=\sqrt{2\pi}\delta(w)\{\frac{H(s)}{s(s^\alpha-(\mu+iw\lambda))}\}+\kappa\sqrt{2\pi}\delta(w)\{\frac{s^{\alpha-1}}{(s^\alpha-(\mu+iw\lambda))}\}, \tag{6.8}$$

and consequently

$$\hat{\overline{U}}(w,s)=\sqrt{2\pi}\delta(w)\{\frac{H(s)}{s}.\frac{1}{(s^\alpha-(\mu+iw\lambda))}\}+\kappa\sqrt{2\pi}\delta(w)\{\frac{1}{s^{1-\alpha}}.\frac{1}{s^\alpha-(\mu+iw\lambda)}\}, \tag{6.9}$$

$$\hat{\overline{U}}(w,s)=\sqrt{2\pi}\delta(w)\{\frac{H(s)}{s}+\frac{\kappa}{s^{1-\alpha}})(\frac{1}{s^\alpha-(\mu+iw\lambda)})\}, \tag{6.10}$$

in which $\hat{\overline{U}}(w,s)=F\{\overline{U}(x,s);x\to w\}$. Now invert $\hat{\overline{U}}(w,s)$ with respect to s,w respectively. By using Schouten-Vander pol theorem we know that

$$L^{-1}\{\frac{1}{s^\alpha-(\mu+iw\lambda)};s\to t\}=\chi(t)=\frac{1}{\pi}\int_0^\infty e^{(\mu+iw\lambda)\beta}(\int_0^\infty e^{-t\eta}e^{-\tau\eta^\alpha\cos\alpha\pi}\sin((\sin\alpha\pi)\tau\eta^\alpha)d\eta)d\beta,$$

also we know that

$$L^{-1}\{\frac{\kappa}{s^{1-\alpha}}+\frac{H(s)}{s}\}=\frac{\kappa}{\Gamma(1-\alpha)t^\alpha}+\int_0^t h(\xi)d\xi=\phi(t)$$

which can be rewritten as below

$$L^{-1}\{(\frac{\kappa}{s^{1-\alpha}}+\frac{H(s)}{s}).(\frac{1}{s^{\alpha}-(\mu+iw\lambda)});s\to t\}$$

$$=\int_0^t \phi(\sigma)\chi(t-\sigma)d\sigma$$

$$=\int_0^t \phi(\sigma)\frac{1}{\pi}(\int_0^\infty e^{(\mu+iw\lambda)\beta}(\int_0^\infty e^{-(t-\sigma)\eta}e^{-\tau\eta^{\alpha}\cos\alpha\pi}\sin((\sin\alpha\pi)\tau\eta^{\alpha})d\eta)d\beta)d\sigma$$

Now invert the above relation (6.10) with respect to w . By using the definition of inverse Fourier transform we can write

$$u(x,t)=\int_{-\infty}^{+\infty} e^{-ixw}\delta(w)\{\int_0^t \phi(\sigma)\frac{1}{\pi}(\int_0^\infty e^{(\mu+iw\lambda)\beta}(\int_0^\infty e^{-(t-\sigma)\eta}e^{-\tau\eta^{\alpha}\cos\alpha\pi}\sin((\sin\alpha\pi)\tau\eta^{\alpha})d\eta)d\beta)d\sigma\}dw$$

which can be evaluated and simplified after change of integrals as following

$$u(x,t)=\int_0^t \phi(\sigma)\frac{1}{\pi}(\int_0^\infty e^{\mu\beta}(\int_{-\infty}^{+\infty} e^{iw\lambda\beta-ixw}\delta(w)(\int_0^\infty e^{-(t-\sigma)\eta}e^{-\tau\eta^{\alpha}\cos\alpha\pi}\sin((\sin\alpha\pi)\tau\eta^{\alpha})d\eta)dw)d\beta)d\sigma$$

or,

$$u(x,t)=\int_0^t \phi(\sigma)\frac{1}{\pi}(\int_0^\infty e^{\mu\beta}(\int_0^\infty e^{-(t-\sigma)\eta}e^{-\tau\eta^{\alpha}\cos\alpha\pi}\sin((\sin\alpha\pi)\tau\eta^{\alpha})(\int_{-\infty}^{+\infty} e^{i(\lambda\beta-x)w}\delta(w)dw)d\eta)d\beta)d\sigma$$

Finally, we get

$$u(x,t)=\frac{1}{\pi}\int_0^t \phi(\sigma)(\int_0^\infty e^{\mu\beta}(\int_0^\infty e^{-(t-\sigma)\eta}e^{-\tau\eta^{\alpha}\cos\alpha\pi}\sin((\sin\alpha\pi)\tau\eta^{\alpha})d\eta)d\beta)d\sigma$$

In case of $\alpha=0.5$ one has simply

$$u(x,t)=\frac{1}{\pi}\int_0^t \phi(\sigma)(\int_0^\infty e^{\mu\beta}(\int_0^\infty e^{-(t-\sigma)\eta}\sin(\tau\sqrt{\eta})d\eta)d\beta)d\sigma.$$

where $$\phi(\sigma)=\frac{\kappa}{\sqrt{\pi\sigma}}+\int_0^\sigma h(\xi)d\xi .$$

4 Conclusion

The paper is devoted to study applications of one dimensional Laplace transforms in details.
One dimensional Laplace transform provides a powerful method for analyzing linear systems. Certain time fractional wave equations with boundary conditions is solved. The method could lead to a promising approach for many applications in applied sciences.

5 Acknowledgments

The authors would like to thank the referees for their constructive comments and helps.

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