NEW TRENDS IN MATHEMATICAL SCIENCES
Vol. 2, No. 1, 2014, p.23-34
ISSN 2147-5520 - www.ntmsci.com

The best approximation of metric P −space of χ^2 −defined by Musielak

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Abstract: In this paper, we introduce the idea of constructing sequence space χ^2 of best approximation in p −metric defined by Musielak and also construct some general topological properties of approximation of χ^2.

Keywords: Analytic sequence, modulus function, double sequences χ^2 space, Musielak - modulus function, p −metric space p −best approximation, p −orthogonality.

1 Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.

We write w^2 for the set of all complex sequences(x_{mn}); where $m, n \in N$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [4]. Later on, they were investigated by Hardy [8], Moricz [16], Moricz and Rhoades [17], Basarir and Solankan [3], Tripathy [20], Turkmenoglu [21], and many others.

We procure the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{(x_{mn}) \in w^2 : sup_{m,n \in N} |x_{mn}|^{t_{mn}} < \infty\right\},$$
$$\mathcal{C}_p(t) := \left\{(x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn} - 1|^{t_{mn}} = 1 \, for \, some \, l \in \mathbb{C}\right\},$$
$$\mathcal{C}_{0p}(t) := \left\{(x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1\right\},$$
$$\mathcal{L}_u(t) := \left\{(x_{mn}) \in w^2 : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty\right\},$$
$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \cap \mathcal{M}_u(t) \, and \, \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \cap \mathcal{M}_u(t);$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in N$ and $p - lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in N$; $\mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$ respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [6,7] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [23] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [14] and Tripathy [20] have independently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesaro summable double sequences. Altay and Başar [1] have defied the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the $\alpha -$ duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and $\beta(\vartheta) -$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Başar and Sever [2] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [19] have studied the space $\chi_M^2(p,q,u)$ of double sequences and gave some inclusion relations.

The class of sequences which are strongly Cesaro summable with respect to a modulus was introduced by Maddox [15] as an extension of the definition of strongly Cesaro summable sequences. Cannor [5] further extended this definition to a definition of strong $A -$ summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong $A -$ summability, strong $A -$ summability with respect to a modulus, and $A -$ statistical convergence. In [24] the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [9]-[10], and [11] the four dimensional matrix transformation $(Ax)_{k,l} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{kl}^{mn} x_{mn}$ was studied extensively by Robison and Hamilton.

We need the following inequality in the sequel of the paper. For $a, b \geq 0$ and $0 < p < 1$, we have

$$(a+b)^p \leq a^p + b^p.$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij}$ $(m,n \in N)$.

A sequence $x = (x_{mn})$ is said to be double analytic if $sup_{mn}|x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $\left((m+n)!\,|x_{mn}|\right)^{1/m+n} \to 0$ $as\ m,n \to \infty$. The double gai sequences will be denoted by χ^2. Let $\phi = \{all\ finite\ sequences\}$.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\mathfrak{F}_{ij}$ for all $m,n \in \mathbb{N}$; where $\mathfrak{F}_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i,j \in \mathbb{N}$.

An FK-space (or a metric space) X is said to have AK property if $(\mathfrak{F}_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})(m,n \in \mathbb{N})$ are also continuous.

Let M and ϕ are mutually complementary modulus functions. Then, we have

(i) For all $u, y \geq 0$,

$$uy \leq M(u) + \Phi(y), \text{(Young's inequality) [See[12]]}$$

(ii) For all $u \geq 0$,

$$u\eta(u) = M(u) + \Phi(\eta(u)).$$

(iii) For all $u \geq 0$, and $0 < \lambda < 1$,

$$M(\lambda u) \leq \lambda M(u).$$

Lindenstrauss and Tzafriri [13] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{x \in w: \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, for\ some\ \rho > 0\right\}.$$

The space ℓ_M with the norm

$$\|x\| = \inf\left\{\rho > 0: \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p$ $(1 \leq p < 1)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.

A sequence $f = (f_{mn})$ of modulus function is called a Musielak-modulus function. A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = \sup\{|v|u - (f_{mn})(u): u \geq 0\}, m, n = 1,2, \ldots$$

is called the complementary function of a Musielak-modulus function f. For a given Musielak modulus function f, the Musielak-modulus sequence space t_f and its subspace h_f are defined as follows

$$t_f = \left\{x \in w^2 : I_f(|x_{mn}|)^{1/m+n} \to 0 \; as \; m, n \to \infty\right\},$$

$$h_f = \left\{x \in w^2 : I_f(|x_{mn}|)^{1/m+n} \to 0 \; as \; m, n \to \infty\right\},$$

where I_f is a convex modular defined by

$$I_f(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}(|x_{mn}|)^{1/m+n}, x = (x_{mn}) \in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d(x,y) = sup_{mn}\left\{\inf\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left(\frac{(|x_{mn}|)^{1/m+n}}{mn}\right)\right) \leq 1\right\}$$

If X is a sequence space, we give the following definitions:

(i) X'= the continuous dual of X;
(ii) $X^{\alpha} = \left\{a = (a_{mn}): \sum_{m,n=1}^{\infty}|a_{mn}x_{mn}| < \infty, for \; each \; x \in X\right\}$;
(iii) $X^{\beta} = \left\{a = (a_{mn}): \sum_{m,n=1}^{\infty} a_{mn}x_{mn} \; is \; convegent, for \; each \; x \in X\right\}$;
(iv) $X^{\gamma} = \left\{a = (a_{mn}): sup_{mn} \geq 1\left|\sum_{m,n=1}^{M,N} a_{mn}x_{mn}\right| < \infty, for \; each \; x \in X\right\}$;
(v) Let X be an FK $-$space $\supset \phi$; then $X^{f} = \{f(\mathfrak{F}_{mn}); f \in X'\}$;
(vi) $X^{\delta} = \left\{a = (a_{mn}): sup_{mn}|a_{mn}x_{mn}|^{1/m+n} < \infty, for \; each \; x \in X\right\}$;

$X^{\alpha}, X^{\beta}, X^{\gamma}$ are called $\alpha -$ (orKöthe $-$ Toeplitz) dual of $X, \beta -$ (or generalized $-$ Köthe $-$ Toeplitz) dual of $X, \gamma -$ dual of $X, \delta -$ dual of X respectively. X^{α} is defined by Gupta and Kamptan [13]. It is clear that $X^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\beta} \subset X^{\gamma}$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_{∞} where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

Here $Z = c, c_0$ and ℓ_{∞} denote the classes of convergent, null and bounded sclar valued single sequences respectively. The difference sequence space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \leq p \leq \infty$ by Başar and Altay and in the case $0 < p < 1$ by Altay and Başar in [1]. The spaces $c(\Delta), c_0(\Delta), \ell_{\infty}(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + \sup_{k\geq 1}|\Delta x_k| \text{ and } \|x\|_{bv_p} = \left(\sum_{k=1}^{\infty}|x_k|^p\right)^{1/p}, (1 \leq p \leq \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \{x = (x_{mn}) \in w^{w} : (\Delta x_{mn}) \in Z\},$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$.

2. Definition and Preliminaries

Let $n \in \mathbb{N}$ and X be a real vector space of dimension w, where $n \leq w$. A real valued function $d_p(x_1, \ldots, x_n) = \|d_1(x_1), \ldots, d_n(x_n)\|_p$ on X satisfying the following four conditions:

(i) $\|d_1(x_1),\ldots,d_n(x_n)\|_p = 0$ if and and only if $d_1(x_1),\ldots,d_n(x_n)$ are linearly dependent,
(ii) $\|d_1(x_1),\ldots,d_n(x_n)\|_p$ is invariant under permutation,
(iii) $\|\alpha d_1(x_1),\ldots,d_n(x_n)\|_p = |\alpha|\|d_1(x_1),\ldots,d_n(x_n)\|_p, \alpha \in \mathbb{R}$
(iv) $d_p\big((x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)\big) = \left(d_X(x_1,x_2,\ldots,x_n)^p + d_Y(y_1,y_2,\ldots,y_n)^p\right)^{1/p}$ for $1 \leq p \leq \infty; (or)$
(v) $d\big((x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)\big) := \sup\{d_X(x_1,x_2,\ldots,x_n), d_Y(y_1,y_2,\ldots,y_n)\}$,

for $x_1,x_2,\ldots,x_n \in X, y_1,y_2,\ldots,y_n \in Y$ is called the p product metric of the Cartesian product of n metric spaces is the p norm of the n-vector of the norms of the n subspaces.

A trivial example of p product metric of n metric space is the p norm space is $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space is the p norm:

$$\|d_1(x_1),\ldots,d_n(x_n)\|_E = \sup\big(\big|det\big(d_{mn}(x_{mn})\big)\big|\big) = \sup\left(\begin{vmatrix} d_{11}(x_{11}) & d_{12}(x_{12}) & \ldots & d_{1n}(x_{1n}) \\ d_{21}(x_{21}) & d_{22}(x_{22}) & \ldots & d_{2n}(x_{1n}) \\ \vdots & \vdots & \vdots & \vdots \\ d_{n1}(x_{n1}) & d_{n2}(x_{n2}) & \ldots & d_{nn}(x_{nn}) \end{vmatrix}\right)$$

where $x_i = (x_{i1},\ldots,x_{in}) \in \mathbb{R}^2$ for each $i = 1,2,\ldots,n$.

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the $p-$ metric. Any complete $p-$ metric space is said to be $p-$ Banach metric space.

Let $\big(X, \|d(x_1),d(x_2),\ldots,d(x_{n-1})\|_p\big)$ be an $p-$ metric space and $W_1,W_2,\ldots,W_p$ be $p-$ subspaces of X. A map $f: W_1 \times W_2 \times W_3 \times \ldots \times W_p \to \mathbb{R}$ is called $p-$ functional on $W_1 \times W_2 \times W_3 \times \ldots \times W_p$, whenever for all $x_{11},x_{12},x_{13},\ldots,x_{1n} \in W_1, x_1,x_{22},x_{23},\ldots,x_{2n} \in W_2,\ldots,x_{n1},x_{n2},x_{n3},\ldots,x_{nn} \in W_p$ and $\lambda_1,\lambda_2,\ldots,\lambda_n \in \mathbb{R}$.

(i) $$f\begin{pmatrix} x_{11}+ & x_{12}+ & \ldots & +x_{1n} \\ x_{21}+ & x_{22}+ & \ldots & +x_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1}+ & x_{n2}+ & \ldots & +x_{nn} \end{pmatrix}$$

(ii) $$f\begin{pmatrix} \lambda_1 x_{11} & \lambda_1 x_{12} & \ldots & \lambda_1 x_{1n} \\ \lambda_2 x_{21} & \lambda_2 x_{22} & \ldots & \lambda_2 x_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_n x_{n1} & \lambda_n x_{n2} & \ldots & \lambda_n x_{nn} \end{pmatrix} = (\lambda_1,\lambda_2,\ldots,\lambda_n) f\begin{pmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & \ldots & x_{nn} \end{pmatrix}$$

Let $\big(X, \|d(x_1),d(x_2),\ldots,d(x_{n-1})\|_p\big)$ be an $p-$ metric space and $0 \neq u_2,u_3,\ldots,u_n \in X$ we denote by X_B^* the Banach metric space of all bounded functionals on $X \times \langle u_2\rangle \times \langle u_3\rangle \times \ldots \times \langle u_n\rangle$ where $\langle Z\rangle$ be the subspace of X generated by Z and $B = \{u_2,u_3,\ldots,u_p\}$.

A sequence (x_{mn}) in an $p-$ metric space $\big(X, \|d(x_1),d(x_2),\ldots,d(x_{n-1})\|_p\big)$ is said to converge in the $p-$ metric if

$$lim_{m,n\to\infty}\left(\left\|x_{mn}, \big(d(u_2),d(u_3),\ldots,d(u_p)\big)\right\|_p\right) = 0,$$

for every $u_2,u_3,\ldots,u_p \in X$.

Any complete $p-$ metric space is said to be $p-$ Banach metric space.

A sequence (x_{mn}) in an $p-$ metric space $\big(X, \|d(x_1),d(x_2),\ldots,d(x_{n-1})\|_p\big)$ is said to be Cauchy with respect to the $p-$ metric if

$$lim_{m,n,u,v\to\infty}\left(\left\|x_{mn} - x_{uv}, \big(d(u_2),d(u_3),\ldots,d(u_p)\big)\right\|_p\right) = 0,$$

for every $u_2,u_3,\ldots,u_p \in X$.

2.1. Definition. Let $\big(X, \|d(x_1),d(x_2),\ldots,d(x_{n-1})\|_p\big)$ be an $p-$ metric space we say that x is $p-$ orthogonal to y if

$$\left\|x, \big(d(u_2),d(u_3),\ldots,d(u_p)\big)\right\|_p \leq \left\|x + \alpha y, \big(d(u_2),d(u_3),\ldots,d(u_p)\big)\right\|_p$$

for all $u_2, u_3, \ldots, u_p \in X, \alpha \in \mathbb{R}$ and we call x is $p-$ orthogonal to y.

2.2. Definition. Let $\left(X, \|d(x_1), d(x_2), \ldots, d(x_{n-1})\|_p\right)$ be an $p-$ metric space, M a non-empty subspace of X and $x \in X$ then $g_0 \in M$ is called $p-$ best approximation of $x \in X$ in M, if for every $g \in M$ and $u_2, u_3, \ldots, u_p \in X$.

$$\left\|x - g_0, \left(d(u_2), d(u_3), \ldots, d\left(u_p\right)\right)\right\|_p \leq \left\|x - g, \left(d(u_2), d(u_3), \ldots, d\left(u_p\right)\right)\right\|_p.$$

If for every $x \in X \backslash \bar{M}$ there exists at least one $p-$ best approximation in M, then M is called $p-$ proximinal subspace of X.

If for every $x \in X \backslash \bar{M}$ there exists a unique $p-$ best approximation in M, then M is called an $p-$Chebyshev subspace of X.

For $x \in X$ we write,

$$P_M^p(x) = \{g_0 \in M: g_0 \ is \ an \ p - best \ approximation \ of \ x\}$$

2.3. Definition. Let $\left(X, \|d(x_1), d(x_2), \ldots, d(x_{n-1})\|_p\right)$ be a real linear $p-$ metric space and $w^2(X)$ denotes $X-$ valued sequence space. Then for an Musielak modulus function $f = (f_{mn})$ we define the following sequence spaces for every every $u_2, u_3, \ldots, u_p \in X$:

$$\left[\chi_f^2\right] \left\|\left(d(u_2), d(u_3), \ldots, d\left(u_p\right)\right)\right\|_p =$$

$$\left\{x = (x_{mn}) \in w^2(X): lim_{m,n\to\infty} f\left(\left((m+n)!\,|x_{mn}|\right)^{1/m+n}, \left\|\left(d(u_2), d(u_3), \ldots, d\left(u_p\right)\right)\right\|_p\right) = 0\right\},$$

$$\left[\Lambda_f^2\right] \left\|\left(d(u_2), d(u_3), \ldots, d\left(u_p\right)\right)\right\|_p =$$

$$\left\{x = (x_{mn}) \in w^2(X): sup_{m,n} f\left(|x_{mn}|^{1/m+n}, \left\|\left(d(u_2), d(u_3), \ldots, d\left(u_p\right)\right)\right\|_p\right) < \infty\right\}.$$

Let X be a linear metric space. A function $w: X \to \mathbb{R}$ is called paranorm, if

(1) $w(x) \geq 0$, for all $x \in X$;
(2) $w(-x) = w(x)$, for all $x \in X$;
(3) $w(x+y) \leq w(x) + w(y)$, for all $x, y \in X$;
(4) If (σ_{mn}) is a sequence of scalars with $\sigma_{mn} \to \sigma$ as $m, n \to \infty$ and (x_{mn}) is a sequence of vectors with $w(x_{mn} - x) \to 0$ as $m, n \to \infty$, then $w(\sigma_{mn} x_{mn} - \sigma x) \to 0$ as $m, n \to \infty$.

A paranorm w for which $w(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, w) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [22], Theorem 10.4.2, p.183).

The following inequality will be used throughout the paper. If $0 \leq q_{mn} \leq \sup q_{mn} = H, K = \max(1, 2^{H-1})$ then

$$|a_{mn} + b_{mn}|^{q_{mn}} \leq K\{|a_{mn}|^{q_{mn}} + |b_{mn}|^{q_{mn}}\}$$

for all m, n and $a_{mn}, b_{mn} \in \mathbb{C}$. Also $|a|^{q_{mn}} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

The main aim of this paper is to study some sequence spaces defined by a Musielakmodulus function over $p-$metric spaces also study some topological properties and some inclusion relations between these spaces.

3. Main Results

3.1.Theorem. Let $f = (f_{mn})$ be a Musielak-modulus function. Then then spaces $\left[X_f^2, \left\|\left(d(u_2), d(u_3), \ldots, d\left(u_p\right)\right)\right\|_p\right]$ and $\left[\Lambda_f^2, \left\|\left(d(u_2), d(u_3), \ldots, d\left(u_p\right)\right)\right\|_p\right]$ are linear spaces.

Proof: The proof is a routine verification and so omitted.

3.2. Theorem. Let $f = (f_{mn})$ be a Musielak-modulus function, $q = (q_{mn})$ be double analytic sequence of positive real numbers. Then then spaces $\left[X_f^{2q}, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]$ is a paranormed space with respect to the paranorm defined by

$$g(x) = in\,f\left\{\left(\left[f_{mn}\left(\left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H}\right\} \leq 1,$$

where H = $max(1, sup_{mn} q_{mn} < \infty)$.

Proof: Clearly $g(x) \geq 0$ for $x = (x_{mn}) \in \left[X_f^{2q}, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]$.

Since $f_{mn}(0) = 0$, we get $g(0)$

Conversely, suppose that $g(x) = 0$, then

$$inf\left\{\left(\left[f_{mn}\left(\left\|X^2(x), \left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H}\right\} \leq 1 = 0.$$

Suppose that $X^2(x) \neq 0$ for each $m, n \in \mathbb{N}$. Then $\left\|X^2(x), \left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p \to \infty$. It follows that

$$\left(\left[f_{mn}\left(\left\|X^2(x), \left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H} \to \infty$$

which is a contradiction. Therefore $X^2(x) = 0$. Let

$$\left(\left[f_{mn}\left(\left\|X^2(x), \left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H} \leq 1$$

and

$$\left(\left[f_{mn}\left(\left\|X^2(y), \left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H} \leq 1.$$

Then by using Minkowski's inequality, we have

$$\left(\left[f_{mn}\left(\left\|X^2(x+y), \left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H} \leq$$
$$\left(\left[f_{mn}\left(\left\|X^2(x), \left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right)\right]^{q_{mn}}\right)^{\frac{1}{H}} +$$
$$\left(\left[f_{mn}\left(\left\|X^2(y), \left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H}.$$

So we have

$$g(x+y) = inf\left\{\left(\left[f_{mn}\left(\left\|X^2(x+y), \left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H} \leq 1\right\} \leq$$

$$inf\left\{\left(\left[f_{mn}\left(\left\|X^2(x), \left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H} \leq 1\right\} +$$

$$inf\left\{\left(\left[f_{mn}\left(\left\|X^2(y), \left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H} \leq 1\right\}$$

Therefore,

$$g(x+y) \leq g(x) + g(y).$$

Finally, to prove that the scalar multiplication is continuous. Let λ be any complex number. By definition,

$$g(\lambda x) = inf\left\{\left(\left[f_{mn}\left(\left\|X^2(\lambda x), \left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H} \leq 1\right\}$$

Then

$$g(\lambda x) = inf\left\{((|\lambda|t)^{q_{mn}/H} : \left(\left[f_{mn}\left(\left\|X^2(\lambda x), \left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H} \leq 1\right\}$$

where $t = \frac{1}{|\lambda|}$. Since $|\lambda|^{q_{mn}} \leq max(1, |\lambda|^{supp_{mn}})$, we have

$$g(\lambda x) \leq \max(1, |\lambda|^{supp_{mn}})$$

$$inf\left\{t^{q_{mn}/H} : \left(\left[f_{mn}\left(\left\|X^2(\lambda x), \left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H} \leq 1\right\}$$

This completes the proof.

3.3. Theorem. The $\beta - 1$ dual space of

$$\left[X_f^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]^\beta = \left[\Lambda_f^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]$$

Proof: First, we observe that

$$\left[X_f^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]^\beta \subset \left[\Gamma_f^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right].$$

Therefore

$$\left[\Gamma_f^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]^\beta \subset \left[X_f^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]^\beta.$$

But $\left[\Gamma_f^2\right]^\beta \underset{\neq}{\subset} \left[\Lambda_f^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]$. Hence

$$\left[\Lambda_f^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right] \subset \left[X_f^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]^\beta \qquad (3.1)$$

Next we show that

$$\left[X_f^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]^\beta \subset \left[\Lambda_f^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right].$$

Let $y = (y_{mn}) \in \left[X_f^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]^\beta$.

Consider $f(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{mn}y_{mn}$ with $x = (x_{mn}) \in \left[X_f^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]$

$$x = [(\lambda_{mn} - \lambda_{mn+1}) - (\lambda_{m+1n} - \lambda_{m+1n+1})]$$

$$= \begin{pmatrix} 0 & 0 & ...0 & 0 & ... & 0 \\ 0 & 0 & ...0 & 0 & ... & 0 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 0 & 0 & ...\frac{1}{(m+n)!} & \frac{-1}{(m+n)!} & ... & 0 \\ 0 & 0 & ...0 & 0 & ... & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & ...0 & 0 & ... & 0 \\ 0 & 0 & ...0 & 0 & ... & 0 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 0 & 0 & ...\frac{1}{(m+n)!} & \frac{-1}{(m+n)!} & ... & 0 \\ 0 & 0 & ...0 & 0 & ... & 0 \end{pmatrix}$$

$$\left[f_{mn}\left(\left\|\left(d(u_2), d(u_3), \ldots, d(u_p)\right)\right\|_p\right)\right] =$$

$$\begin{pmatrix} 0 & 0 & ...0 & 0 & ... & 0 \\ 0 & 0 & ...0 & 0, & ... & 0 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 0 & 0 & ...f_{mn}\left(\frac{1}{(m+n)!}\right) & f_{mn}\left(\frac{-1}{(m+n)!}\right) & ... & 0 \\ 0 & 0 & ...f_{mn}\left(\frac{-1}{(m+n)!}\right) & f_{mn}\left(\frac{1}{(m+n)!}\right) & ... & 0 \\ 0 & 0 & ...0 & 0, & ... & 0 \end{pmatrix}$$. Hence converges to zero.

Therefore $[(\lambda_{mn} - \lambda_{mn+1}) - (\lambda_{m+1n} - \lambda_{m+1n+1})] \in \left[X_f^2, \left\|\left(d(u_2), d(u_3), \ldots, d(u_p)\right)\right\|_p\right]$.

Hence $d\big((\lambda_{mn} - \lambda_{mn+1}) - (\lambda_{m+1n} - \lambda_{m+1n+1}), 0\big) = 1$.

But $|y_{mn}| \le \|f\| d\big((\lambda_{mn} - \lambda_{mn+1}) - (\lambda_{m+1n} - \lambda_{m+1n+1}), 0\big) \le \|f\|. 1 < \infty$ for each m, n. Thus (y_{mn}) is a best approximation of $p-$ metric double analytic sequence.

In other words $y \in \left[\Lambda_f^2, \left\|\left(d(u_2), d(u_3), \ldots, d(u_p)\right)\right\|_p\right]$. But $y = (y_{mn})$ is arbitrary in $\left[X_f^2, \left\|\left(d(u_2), d(u_3), \ldots, d(u_p)\right)\right\|_p\right]^\beta$.

Therefore

$$\left[X_f^2, \left\|\left(d(u_2), d(u_3), \ldots, d(u_p)\right)\right\|_p\right]^\beta \subset \left[\Lambda_f^2, \left\|\left(d(u_2), d(u_3), \ldots, d(u_p)\right)\right\|_p\right] \tag{3.2}$$

From (3.1) and (3.2) we get

$$\left[X_f^2, \left\|\left(d(u_2), d(u_3), \ldots, d(u_p)\right)\right\|_p\right]^\beta = \left[\Lambda_f^2, \left\|\left(d(u_2), d(u_3), \ldots, d(u_p)\right)\right\|_p\right]$$

3.4. Theorem. The dual space of $\left[X_f^2, \left\|\left(d(u_2), d(u_3), \ldots, d(u_p)\right)\right\|_p\right]$ is $\left[\Lambda_f^2, \left\|\left(d(u_2), d(u_3), \ldots, d(u_p)\right)\right\|_p\right]$. In other words $\left[X_f^2, \left\|\left(d(u_2), d(u_3), \ldots, d(u_p)\right)\right\|_p\right]^* = \left[\Lambda_f^2, \left\|\left(d(u_2), d(u_3), \ldots, d(u_p)\right)\right\|_p\right]$.

Proof: We recall that $x_{mn} = \begin{pmatrix} 0 & 0 & ...0 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \\ . & & & & \\ . & & & & \\ . & & & & \\ 0 & 0 & ...\frac{1}{(m+n)!} & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \end{pmatrix}$

With $\frac{1}{(m+n)!}$ in the $(m,n)th$ position and zero's else where, with

$$\left[X_f^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right] = \begin{pmatrix} 0. & . & . & 0 \\ . & & & \\ . & & & \\ . & & & \\ 0 & f\left(\frac{1}{(m+n)!}\right)^{1/m+n} & . & 0 \\ & (m,n)^{th} & & \\ 0 & . & . & 0 \end{pmatrix}$$

which is a $p-$ metric of double gai sequence. Hence,

$$x_{mn} \in \left[X_f^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]. f(x) = \sum_{m,n-1}^{\infty} x_{mn} y_{mn}$$

with $x \in \left[X_f^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]$ and $f \in \left[X_f^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]^*$, where

$\left[X_f^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]^*$ is the dual space of $\left[X_f^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]$.

Take $x = (x_{mn}) \in \left[X_f^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]$. Then,

$$|y_{mn}| \le \|f\| d(x_{mn}, 0) < \infty \; \forall m, n \tag{3.3}$$

Thus, (y_{mn}) is a $p-$ metric of double analytic sequence and hence an $p-$ metric of double analytic sequence.In other words, $y \in \left[\Lambda_f^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]$.

Therefore $\left[X_f^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]^* = \left[\Lambda_f^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]$. This completes the proof.

3.5. Proposition. $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} x_{mn} a_{mn}$ converges for all

$$x = \{x_{mn}\} \in \left[X_f^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right] \Leftrightarrow \{a_{mn}\} \in \left[X_g^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right].$$

Proof: $|x_{mn} a_{mn}| \le f_{mn}(|x_{mn}|) + g_{mn}(a_{mn})$

$\Leftrightarrow \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn} a_{mn}| \le \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn}(|x_{mn}|) + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} g_{mn}(|a_{mn}|)$.

Since $a = \{a_{mn}\} \in \left[X_g^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]$ we have

$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} g_{mn}(|a_{mn}|) < \infty$. Hence $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{mn}a_{mn}$ converges $\Leftrightarrow$ $a = \{a_{mn}\} \in \left[X_g^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]$. This completes the proof.

3.6. Theorem.

(i) If the sequence (f_{mn}) satisfies uniform $\Delta_2 -$ condition, then

$$\left[X_f^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]^{\propto} = \left[X_g^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right].$$

(ii) If the sequence (g_{mn}) satisfies uniform $\Delta_2 -$ condition, then

$$\left[X_f^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]^{\propto} = \left[X_g^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]$$

Proof: Let the sequence (f_{mn}) satisfies uniform $\Delta_2 -$ condition, we get

$$\left[X_g^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right] = \left[X_f^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]^{\propto} \quad (3.4)$$

To prove the inclusion

$$\left[X_f^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]^{\propto} \subset \left[X_g^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right],$$

let $a \in \left[X_f^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]^{\propto}$.

Then for all $\{x_{mn}\}$ with $(x_{mn}) \in \left[X_f^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]$ we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |x_{mn}a_{mn}| < \infty \quad (3.5)$$

Since the sequence (f_{mn}) satisfies uniform $\Delta_2 -$ condition, then

$(y_{mn}) \in \left[X_f^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]$, we get $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left|\frac{y_{mn}a_{mn}}{(m+n)!}\right| < \infty$. by (3.5). Thus $(a_{mn}) \in \left[X_f^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right] = \left[X_g^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]$

and hence

$(a_{mn}) \in \left[X_g^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]$. This gives that

$$\left[X_f^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]^{\propto} \subset \left[X_g^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right] \quad (3.6)$$

we are granted with (3.4) and (3.6)

$$\left[X_f^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]^{\propto} = \left[X_g^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]$$

(ii) Similarly ,one can prove that

$$\left[X_g^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]^{\propto} \subset \left[X_f^2, \left\|\left(d(u_2), d(u_3), \dots, d(u_p)\right)\right\|_p\right]$$

If the sequence (g_{mn}) satisfies uniform $\Delta_2 -$ condition.

3.7 Proposition. Let $\left[X_f^2, \left\|\left(d(u_2), d(u_3), \ldots, d(u_p)\right)\right\|_p\right]$ be an $p-$ metric linear space and $0 \neq x, \in X_f^2$. Then the following statements are equivalent:

(i) x is $p-$ orthognal to y

(ii) There exist $d(u_2), d(u_3), \ldots, d(u_p) \in X_f^2$ and $F \in \left(X_f^2\right)_B^*$ such that $d(F, 0) = 1$, $F\left[X_f^2(x), \left\|\left(d(u_2), d(u_3), \ldots, d(u_p)\right)\right\|_p\right] = F\left[X_f^2(y), \left\|\left(d(u_2), d(u_3), \ldots, d(u_p)\right)\right\|_p\right] = 0$

and

$$B = \left\{d(u_2), d(u_3), \ldots, d(u_p)\right\}.$$

3.8.Corollary. Let $\left[X_f^2(x), \left\|\left(d(u_2), d(u_3), \ldots, d(u_p)\right)\right\|_p\right]$ be an $p-$ metric linear space, f a non empty subspace of $X_f^2, 0 \neq x \in X_f^2$ and $g_0 \in f$. Then the following statements are equivalent:

(i) $g_0 \in P_f^P(x)$

(ii) There exist $d(u_2), d(u_3), \ldots, d(u_p) \in X_f^2$ and $F \in \left(X_f^2\right)_B^*$ such that $d(F, 0) = 1$,

$$F\left[X_f^2 - g_0, \left(d(u_2), d(u_3), \ldots, d(u_p)\right)_p\right] = \left\|X_f^2 - g_0, \left(d(u_2), d(u_3), \ldots, d(u_p)\right)\right\|_p$$

and

$$F\left[g, \left(d(u_2), d(u_3), \ldots, d(u_p)\right)\right] = 0, \forall f \in f \text{ and } B = \left\{d(u_2), d(u_3), \ldots, d(u_p)\right\}.$$

3.9.Lemma. We define the following function

$\left[X_f^2(x), \left\|\left(d(u_2), d(u_3), \ldots, d(u_p)\right)\right\|_p^Y\right]$ on $Y \times Y \times \ldots \times Y(p - factors)$ by $\left\|\left((m+n)!\,|x_{mn}|\right)_1^{1/m+n}, \left((m+n)!\,|x_{mn}|\right)_2^{1/m+n}, \ldots \left((m+n)!\,|x_{mn}|\right)_p^{\frac{1}{m}+n},\right\|_P$

are linearly dependent, and

$$\left\|\left((m+n)!\,|x_{mn}|\right)_1^{1/m+n}, \left((m+n)!\,|x_{mn}|\right)_2^{1/m+n}, \ldots \left((m+n)!\,|x_{mn}|\right)_p^{\frac{1}{m}+n},\right\|_P = inf\left\{m, n \geq 1, u_2 \ldots u_p \in X_f^2 f\left(\left\|\left((m+n)!\,|x_{mn}|\right)^{\frac{1}{m}+n}, \left(d(u_2), d(u_3), \ldots, d(u_p)\right)\right\|_p\right) < 1\right\}$$

if $\left((m+n)!\,|x_{mn}|\right)_1^{1/m+n}, \left((m+n)!\,|x_{mn}|\right)_2^{1/m+n}, \ldots \left((m+n)!\,|x_{mn}|\right)_p^{\frac{1}{m}+n}$ are linearly independent.

3.10.Example. Consider the space X^2 of real sequences with only finite number of non-zero terms. Let us define : $\left\|x_1, x_2, \ldots, x_p\right\|_p = 0$, if $x_1, x_2, \ldots, x_p$ are are linearly dependent,

$$= \lim_{m,n\to\infty}\left(\left((m+n)!\,|x_{mn}|\right)_1^{\frac{1}{m}+n}, \left((m+n)!\,|x_{mn}|\right)_2^{\frac{1}{m}+n}, \ldots \left((m+n)!\,|x_{mn}|\right)_p^{\frac{1}{m}+n}\right),$$

if $x_1, x_2, \ldots, x_p$ are are linearly independent. Then $\left\|X_f^2(x), \left(d(u_2), d(u_3), \ldots, d(u_p)\right)\right\|_p$ is an $-p$ metric on x^2 consisting of real sequences.

Acknowledgement: I wish to thank the referee's for their several remarks and valuable suggestions that improved the presentation of the paper.

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