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Estimation of survival and mean residual life functions from dependent random censored data

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Abstract: In this article we consider the problem of estimating the survival and mean residual life functions for dependent random censoring observations on the right.

Keywords: Random censorship, survival function, mean residual life function, Archimedean copulas.

1 Introduction

During a long time statisticians have been interested on the relationship between a multivariate distribution function and its lower dimensional margins. M. Fr´echet and G. Dall'Aglio (see [5]) did some interesting works about this matter in the fifties, studying the bivariate and trivariate distribution functions with given univariate margins. The answer to this problem for the univariate margins case was given by A. Sklar in 1959 (see [5]) creating a new class of functions which he called copulas. In probability theory and mathematical statistics, a copula is a multivariate probability distribution for which the marginal probability distribution of each variable is uniform. Copulas are used to describe the dependence between random variables. At the beginning, copulas were mainly used in the development of the theory of probabilistic metric spaces. Later, they were of interest to define nonparametric measures of dependence between random variables (r.v.-s) and since then, they began to play a important role in probability theory and mathematical statistics. In this article we consider the problem of estimating of survival function and mean residual life function in the case of random censoring from the right. In order to propose our estimators we need to introduce of definition of copulas.

Definition 1. [4,5]. A copula $C(u,v): [0,1]^2 \to [0,1]$ is a bivariate distribution function with uniform marginals.

A first example of copulas is the product copula $C(u,v) = uv$, which characterizes independent random variables when the distribution functions are continuous. The importance of copulas in Mathematical Statistics is described in Sklar's Theorem (see [5]).

Theorem 1. (Sklar, 1959). Let *H* be a joint distribution function with margins *F* and *G*. Then there exists a copula *C* such that for all *x*, *y* in *R*,

$$H(x,y) = C\big(F(x), G(y)\big). \tag{1}$$

If *F* and *G* are continuous, then *C* is unique; otherwise, *C* is uniquely determined on $Ran(F) \times Ran(G)$. Conversely, if *C* is a copula and *F* and *G* are distribution functions, then the function *H* defined by (1) is a joint distribution function with margins *F* and *G*. Thus copulas link joint distribution functions to their one-dimensional margins. A proof of this theorem can be found in [5].

Furthermore, the representation (1) suggests that if the copula C were known, then substituting continuous marginal estimators for F and G would yield a plug-in estimate of their associated joint distribution function H. Moreover, in light of Sklar's result with arrive at the following functional definition of a copula.

Definition 2. [5]. Given a bivariate distribution function H with marginals F and G , the function defined as

$$C(u,v) = H(F^{-1}(u), G^{-1}(v)),$$

For $(u,v) \in [0,1]^2$, where $F^{-1}(u)$ and $G^{-1}(v)$ are the inverse functions of F and G respectively, is the copula corresponding to H . In many applications, the r.v.-s of interest represent the lifetimes of individuals or objects in some population. The probability of an individual living or surviving beyond time x is given by the survival function (or survivor function, or reliability function) $S(x) = P(X > x) = 1 - F(x)$ where, as before, F denotes the distribution function of X . Let C be the copula function of the bivariate distribution of (X,Y). We have

$$\begin{aligned}\overline{H}(x,y) = P(X > x, Y > y) = 1 - F(x) - G(y) + H(x,y) = S(x) + S(y) - 1 + C(1 - S(x), 1 - S(y)) \\ = C^*\big(S(x), S(y)\big),\end{aligned}$$

where $C^*(u,v) = u + v - 1 + C(1-u, 1-v)$ -survival copula function.

Definition 3. [5]. Let φ be a continuous, strictly decreasing function from $[0,1]$ to $[0,\infty]$ such that $\varphi(1) = 0$. The pseudo-inverse of φ is the function $\varphi^{[-1]}$ with $Dom\varphi^{[-1]} = [0,\infty]$ and $Ran\ (f)$ given by

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{-1}(t), & 0 < t < \varphi(0), \\ 0 \quad , & \varphi(0) \le t \le \infty, \end{cases}$$

Note that $\varphi^{[-1]}$ is continuous and no incteasing on $[0,\infty]$, and strictly decteasing on $[0,\varphi(0)]$. Furthermore, $\varphi^{[-1]}(\varphi(u))$ on I, and

$$\varphi(\varphi^{[-1]}(t)) = \begin{cases} t \quad , & 0 < t < \varphi(0), \\ 0 \quad , & \varphi(0) \le t \le \infty, \end{cases} = \min(t, \varphi(0)).$$

If $\varphi(0) = \infty$, then $\varphi^{[-1]} = \varphi^{-1}$.

Definition 4.[5]. Copulas of the form $C(u,v) = \varphi^{[-1]}\big[\varphi(u) + \varphi(v)\big)$ are called Archimedean copulas, where the function φ is called a generator of the copula.

For a detailed study of these copulas, see [5].

2 The model and estimation of survival function

In survival analysis our interest focuses on a nonnegative r.v.-s denoting death times of biological organisms or failure times of mechanical systems. A difficulty in the analysis of survival data is the possibility that the survival times can be subjected to random censoring by other nonnegative r.v.-s and therefore we observe incomplete data. There are various types of censoring mechanisms. In this article we consider only right censoring model and problem of estimation of survival and mean residual life functions when the survival times and censoring times are dependent and propose new estimates of survival functions assuming that the dependence structure is described by a known copula function.

On the probability space (Ω, A, P) we consider $\{(X_k, Y_k), k \ge 1\}$ - a sequence of independent and identically distributed pairs of nonnegative r.v.-s with common joint distribution function (d.f.) $H(x,y) = P(X_1 \le x, Y \le y)$, $(x,y) \in \overline{R}^{+2} = [0,\infty]^2$. We suppose that the marginal d.f.-s $(x) = P(X_1 \le x,) = H(x,\infty)$ and $G(y) = P(Y_1 \le y) = H(+\infty, y), x, y \in \overline{R}^{+}$, are continuous and $F(0) = G(0) = 0$. Assume that the sequence $\{Y_k, k \ge 1\}$ and at n -th stage of the experiment the observation is available the sample $\mathbb{V}^{(n)} = \{(Z_k, \delta_k), 1 \le k \le n\}$, where $Z_k = \min(X_k, Y_k)$, $\delta_k = I(Z_k = X_k)$ and I(A) is the indicator of the event A . Should be noted that it does not require independence of sequences $\{X_k\}$ and $\{Y_k\}$. The problem is consist in estimating of the survival function $S^x(x) = \mathbb{P}(X_1 > x) = 1 - F(x), x \in \overline{R}^{+}$ from the sample

$\mathbb{V}^{(n)}$. Let $\overline{H}(x,y) = P(X_1 > x, Y_1 > y), (x,y) \in \overline{R}^{+2}$- a joint survival function of the pairs (X_k, Y_k). According to Theorem of Sclar H and $\overline{H}$ can be submitted through the appropriate copula functions (see [4, 5]):

$$H(x,y) = C\big(F(x); G(y)\big), \ \ (x,y) \in \overline{R}^{+2},$$

$$\overline{H}(x,y) = C^*(S^X(x); S^Y(y)), (x,y) \in \overline{R}^{+2}, \tag{2}$$

where copulas C and C^* are related as

$$C^*(u,v) = u + v - 1 + C(1-u; 1-v), (u,v) \in [0,1]^2. \tag{3}$$

In the sequel in order to construct estimates for the survival function S^X , assume that C^* is Archimedean copula, i.e. $C^*(u,v) = \varphi^{[-1]}[\varphi(u) + \varphi(v)], (u,v) \in [0,1]^2$ where $\varphi: [0,1] \to \overline{R}^+$ is some generator function with the pseude inverse $\varphi^{[-1]}$. Thus, by (2) and (3)

$$\overline{H}(x,y) = \varphi^{[-1]}[\varphi((S^X(x)) + \varphi(S^Y(y))], (x,y) \in \overline{R}^{+2},$$

$$S^Z(x) = \varphi^{[-1]}[\varphi((S^X(x)) + \varphi(S^Y(y))], x \in \overline{R}^{+}. \tag{4}$$

We introduce a usual $\lambda^X \lambda^Z$ and “crude” $\lambda -$ hazard functions

$$\lambda^X(x) = \lim_{\Delta \downarrow 0} \frac{1}{\Delta} P\left(x < X_1 \le x + \frac{\Delta}{X_1} > x\right),$$

$$\lambda^Z(x) = \lim_{\Delta \downarrow 0} \frac{1}{\Delta} P\left(x < Z_1 \le x + \frac{\Delta}{X_1} > x, Y_1 > x\right),$$

$$\lambda(x) = \lim_{\Delta \downarrow 0} \frac{1}{\Delta} P\left(x < X_1 \le x + \frac{\Delta}{X_1} > x, Y_1 > x\right).$$

In order to construct a copula estimates for S^X consider the following easily verifiable equality:

$$\lambda^X(x) S^X(x) \varphi'\big(S^X(x)\big) = \lambda(x) S^Z(x) \varphi'\big(S^Z(x)\big). \tag{5}$$

Integrating (5) over the interval [0, x] and denoting by $\Lambda(x) = \int_0^x \lambda(t)dt$ and $\Lambda^X(x) = \int_0^x \lambda^X(t)dt$ corresponding cumulative hazard functions we obtain the integral equation

$$\int_0^x S^X(t) \varphi'\big(S^X(t)\big)\, d\Lambda^X(t) = \int_0^x S^Z(t) \varphi'\big(S^Z(t)\big)\, d\Lambda(t), \qquad x \in \overline{R}^{+}. \tag{6}$$

Integral on the left side of (6) is equal to $-\varphi\big(S^X(t)\big)$ and then (6) takes the form

$$\varphi\big(S^X(t)\big) = -\int_0^x S^Z(t) \varphi'\big(S^Z(t)\big)\, d\Lambda(t), \qquad x \in \overline{R}^{+}. \tag{7}$$

Hence we find the expression for the survival function S^X :

Denklemi buraya yazın. (8)

$$S^X(x) = \varphi^{[-1]}\big[-\textstyle\int_0^x S^Z(t) \varphi'\big(S^Z(t)\big)\, d\Lambda(t)\big], \ \ x \in \overline{R}^{+}. \tag{8}$$

Note that the survival function S^Z permit usual empirical estimation by the values Z_k observed in the sample $\mathbb{V}^{(n)}$:

$$S_n^Z(x) = \frac{1}{n}\sum_{k=1}^{n} I(Z_k > x), \qquad x \in \overline{R}^{+}. \tag{9}$$

Substituting (9) to the right of representation (8), we obtain a preliminary estimate of S^X as

$$\tilde{S}_n^X(x) = \varphi^{[-1]}\left[-\int_0^x I(S_n^Z(t-) > 0)\, S_n^Z(t-)\varphi'\big(S_n^Z(t)\big)\, d\Lambda_{\mathrm{n}}(t)\right], \tag{10}$$

where

$$\Lambda_{\mathrm{n}}(t) = \frac{1}{n}\sum_{k=1}^{n} \frac{I(Z_k \le t, \delta_k = 1)}{S_n^Z(Z_k) - \frac{1}{n}}, \tag{11}$$

-the corresponding estimate for $\Lambda(\mathrm{t}) = \int_0^{\mathrm{t}} \frac{\mathrm{dP}(\mathrm{Z}_1 \le \mathrm{s}, \delta_1 = 1)}{\mathrm{P}(\mathrm{Z}_1 > s)}$ Estimate (10) plays a supporting role in the construction of the main estimates for S^X in the future. Let $N_k(t) == I(Z_k \le t, \delta_k = 1)$. Define the countin processes $\overline{\mathbb{N}}_n(t) = \sum_{k=1}^{n} N_k(t)$ and $\mathbb{J}_n(t) = nS_n^Z(t-) == \sum_{k=1}^{n} I(Z_k \ge t)$. Then the estimates (10) and (11) can be represented as

$$\tilde{S}_n^X(x) = \varphi^{[-1]}\left[-\frac{1}{n}\int_0^x I(\mathbb{J}_n(t) > 0)\, \varphi'\left(\frac{\mathbb{J}_n(t)}{n}\right) d\overline{\mathbb{N}}_n(t)\right], \tag{12}$$

$$\Lambda_{\mathrm{n}}(t) = \int_0^t \frac{I(\mathbb{J}_n(s) > 0)}{\mathbb{J}_n(s)}\, d\overline{\mathbb{N}}_n(s).$$

Given the analog left side of (6), i.e.

$$\varphi\big(S^Z(x)\big) = -\int_0^x S^Z(t)\varphi'\big(S^Z(t)\big)\, d\Lambda^Z(t), \tag{13}$$

where $\Lambda^Z(t) = \int_0^x \lambda^Z\, ds$, together with (9) also obtain other estimate for S^Z as

$$\tilde{S}_n^X(x) = \varphi^{[-1]}\left[-\frac{1}{n}\int_0^x I(\mathbb{J}_n(t) > 0)\, \varphi'\left(\frac{\mathbb{J}_n(t)}{n}\right) d\overline{\mathbb{N}}_n{}^Z(t)\right], \tag{14}$$

where $\Lambda_n^Z(t) = \int_0^t \frac{I(\mathbb{J}_n(s) > 0)}{\mathbb{J}_n(s)}\, d\overline{\mathbb{N}}_n{}^Z(s)$, is estimate for $\Lambda^Z(t)$ and $\overline{N}_n^Z(t) = n\big(1 - S_n^Z(t)\big) = \mathbb{J}_n(t+) == \sum_{k=1}^{n} N_k^Z(t) = \sum_{k=1}^{n} I(Z_k \le t)$ - the counting process. For S^X have the following obvious identity obtained from the representations (7) and (13):

$$S^X(x) = \varphi^{[-1]}\left[\varphi\big(S^Z(t)\big)\frac{\left(-\int_0^x S^Z(t)\varphi'\big(S^Z(t)\big)\, d\Lambda(t)\right)}{\left(-\int_0^x S^Z(t)\varphi'\big(S^Z(t)\big)\, d\Lambda^Z(t)\right)}\right]. \tag{15}$$

Now substituting the empirical estimate of (9) under the first factor on the right of representation (15) and the corresponding estimates (12) and (14) instead of integrals we obtain the final estimate of S^X in the form

$$S_n^X(x) = \varphi^{[-1]}\left[\varphi\big(S^Z(t)\big)\frac{\left(-\int_0^x I(\mathbb{J}_n(t) > 0)\, \varphi'\left(\frac{\mathbb{J}_n(t)}{n}\right) d\overline{\mathbb{N}}_n(t)\right)}{\left(-\int_0^x I(\mathbb{J}_n(t) > 0)\, \varphi'\left(\frac{\mathbb{J}_n(t)}{n}\right) d\overline{\mathbb{N}}_n{}^Z(t)\right)}\right], \tag{16}$$

where

$$\varphi\big(S_n^Z(x)\big) = -\int_0^x I(\mathbb{J}_n(s) > 0)\ \left[\varphi\left(\frac{\mathbb{J}_n(s)}{n}\right) - \varphi\left(\frac{\mathbb{J}_n(s)}{n} - \frac{1}{n}\right)\right] d\overline{\mathbb{N}}_n{}^Z(t),$$

is estimator of $\varphi\big(S^Z(x)\big)$.

3 Main Results

In fact, we suppose that in (15) the generator function φ is strong (that is φ (0)=∞) and hence $\varphi^{[-1]} = \varphi^{-1}$ is usual inverse function. Denote $Z^{(n)} = \sup\{x \geq 0: J_n(x) > 0\}$, $T^Z = \sup\{x \geq 0: S^Z(x) > 0\}, \Psi(x) = -x\varphi'(x)$. Introduce the regularity conditions with respect to S^X, S^Z and the copula generator φ. By Λ^* in conditions below denote both of Λ and Λ^Z :

$(C1)$The strong generator function $\varphi(.)$ is strictly decreasing on $(0,1]$ and is sufficiently smooth in the sense that the first two derivatives of the functions $\varphi(x)$ and $\Psi(x)$ are bounded for $x \in [\varepsilon, 1]$, where $\varepsilon > 0$ is arbitrary. Moreover, the first derivative φ' is bounded away from zero on [0,1] ;

$(C2)0 < \int_0^{T_z}\big[\Psi\big(S^Z(x)\big)\big]^m\ d\Lambda^*(x) < \infty$ for $m = 1,2$;

$(C3)\ \int_0^{T_z}\big|\Psi'\big(S^z(x)\big)\big| d\Lambda^*(x) < \infty$;

$(C4)\ \limsup_{x\to T_z} \int_x^{T_z} \frac{\Psi\big(S^Z(t)\big)}{S^Z(t)} d\Lambda^*(t) = 0$;

$(C5)\ S^X(.) -$ continuous on $[0, T_Z]$ if $T_z < \infty$. Otherwise, $S^Z(\infty) = \lim_{x\to\infty} S^X(x)$.

At first we state the strong consistency of estimator (16) on the interval $[0, T]$ where $T = T_Z$ if $T_Z < \infty$ and $T = Z^n$, if $T_Z = \infty$. In fact, these results are also valid throughout half $[0, \infty)$ and $T_Z = \infty$, because $S_n^X(x) = 0$ for $x > Z^{(n)}$, $Z^{(n)^P} \to T_z = \infty$ and $S^X\big(Z^{(n)}\big)^P \to S^X(T^Z) = S^X(\infty) = 0$ for $n \to \infty$.

Theorem 2. Let conditions $(C1) - (C3)$ are hold. Then for $n \to \infty$

$(A)\ \sup_{0\leq x\leq T} |\Lambda_n^Z(x) - \Lambda^Z(x)| \overset{P}{\to} 0$

$(B)\ \sup_{0\leq x\leq T} |\Lambda_n(x) - \Lambda(x)| \overset{P}{\to} 0$

$(C)\ \sup_{0\leq x\leq T} \left|\varphi\left(\tilde{S}_n^z(x)\right) - \varphi\big(S^z(x)\big)\right| \overset{P}{\to} 0$

$(D)\ \sup_{0\leq x\leq T} \left|\varphi\big(S_n^Z(x)\big) - \varphi\left(\tilde{S}_n^z(x)\right)\right|^{a.s.} = O\left(\frac{1}{n}\right)$

$(E)\ \sup_{0\leq x\leq T} \left|\varphi\left(\tilde{S}_n^z(x)\right) - \varphi\big(S^X(x)\big)\right| \overset{P}{\to} 0$

$(G)\ \sup_{0\leq x\leq \infty} |S_n^X(x) - S^x(x)| \overset{P}{\to} 0$

The proof of the theorem 2. (A) For all $x \in [0, T]$ by using the identity $1 \equiv I(\mathbb{J}_n(x) \geq 0 = I\ (\mathbb{J}_n(x) > 0) +$ $+I(\mathbb{J}_n(x) = 0)$ we have

$$\Lambda_n^Z(x) - \Lambda^Z(x) = \int_0^x \frac{I(\mathbb{J}_n(t) > 0)}{\mathbb{J}_n(t)} d\bar{N}_n^Z(t) -$$
$$- \int_0^x I(\mathbb{J}_n(t) \geq 0)\, d\Lambda^Z(t) == \int_0^x \frac{I(\mathbb{J}_n(t) > 0)}{\mathbb{J}_n(t)} d\mathbb{M}_n^Z(t) - \tag{17}$$
$$- \int_0^x I(\mathbb{J}_n(t) = 0)\, d\Lambda^Z(t) = \mathbb{A}_{ln}(x) + \mathbb{R}_{ln}(x)$$

Let $\tau \leq T$ so that $S^Z(\tau) > 0$. Then for $x \in [0, \tau]$ (using (C2) when m = 0) we have, $|\mathbb{R}_{ln}(x)| \leq I(\mathbb{J}_n(x) = 0). \int_0^x d\Lambda^Z(t) < I(\mathbb{J}_n(\tau) = 0)\Lambda^Z(\tau)$, where $\Lambda^Z(\tau) < \infty$ and in accordance with SLLN under $n \to \infty$ have $\frac{\mathbb{J}_{n(\tau)}}{n} = S_n^Z(\tau) > 0$. Consequently $\mathbb{J}_n(\tau) \xrightarrow{a.s.} \infty$ and from here $I(\mathbb{J}_n(\tau) = 0) \xrightarrow{a.s.} 0.$ Thus, when $n \to \infty$

$$\sup_{0 \leq x \leq T} |\mathbb{R}_{ln}(x)| \xrightarrow{a.s.} 0. \tag{18}$$

Integrand in $\mathbb{A}_{1n}(x)$ is bounded predictable random process (since it is adapted process on $[0, x]$ and continuous from the left) and, therefore, $\mathbb{A}_{1n}(x)$ is a locally square-integrable martingale $\left(\mathbb{A}_{1n}(x) \in M_{loc}^2\left(F_p^{(n)}\right)\right)$ with quadratic characteristics$< A_{ln}, \mathbb{A}_{ln} > (x) = \int_0^x \frac{I(\mathbb{J}_n(t)>0)}{\mathbb{J}_n(t)} d\Lambda^Z(t)$. Then $\mathbb{A}_{1n}^2(x) -< A_{1n}, \mathbb{A}_{1n} > (x)$ is also o a martingale with respect to filtration $F_p^{(n)}$ and by the Lenglart's inequality $\forall \varepsilon$ and $\eta > 0$[1]:

$$P\left(\sup_{0 \leq x \leq \tau} |< A_{1n}(x)| > \varepsilon\right) \leq P\left(\sup_{0 \leq x \leq \tau} \mathbb{A}_{1n}^2(x) > \varepsilon^2\right) \leq \frac{\eta}{\varepsilon^2} +$$

$$P\left(\int_0^\tau \frac{I(\mathbb{J}_n(t)>0)}{\mathbb{J}_n(t)} d\Lambda^Z(t) > \eta\right) < \frac{\eta}{\varepsilon^2} + P\left(\frac{\Lambda^Z(\tau)}{\mathbb{J}_n(\tau)}\right) = o(1), \ n \to \infty,$$

Because $\varepsilon, \eta -$ arbitrary and $\mathbb{J}_n(\tau) \xrightarrow{a.s.} \infty$. Thus, when $n \to \infty$

$$\sup_{0 \leq x \leq \tau} |\mathbb{A}_{1n}(x)| \xrightarrow{P} 0. \tag{19}$$

Now consider the interval $(\tau, T]$. If $T_Z = \infty$, then the proof is obvious. Let $T_Z < \infty$. Then we choose $\varepsilon > 0$ a sufficiently small and for $\tau = \mathrm{T} - \varepsilon$ have

$$\begin{aligned} \sup_{0 \leq x \leq T} |\Lambda_n^Z(x) - \Lambda^Z(x)| &\leq \sup_{0 \leq x \leq \tau} |\mathbb{A}_{1n}(x)| + \sup_{\tau \leq x \leq T} |\mathbb{A}_{1n}(x)| + \\ + \sup_{0 \leq x \leq T} |\mathbb{R}_{1n}(x)| &\leq \sup_{0 \leq x \leq \tau} |\mathbb{A}_{1n}(x)| + \sup_{0 \leq x \leq T} |\mathbb{R}_{1n}(x)| + \\ &+ |\mathbb{A}_{1n}(T_Z) - \mathbb{A}_{1n}(T_Z - \varepsilon)| + |\mathbb{A}_{1n}(T_{Z-\varepsilon})|. \end{aligned} \tag{20}$$

Now, using (25), (26) and tending $\varepsilon > 0$ to zero from (27) we obtain the assertion (A) of the theorem. (B) repeats the proof of (A), we need only replace $\bar{N}_n^Z$ and Λ^Z and respectively on $\bar{N}_n$ and Λ. Let us prove (C). It is easy to verify (see (14)), the following representations using to the indicator identity from proof of (A):

$$\begin{aligned} \varphi\left(\tilde{S}_n^Z(x)\right) - \varphi\left(S^Z(x)\right) &= -\frac{1}{n} \int_0^x I(\mathbb{J}_n(t) > 0) \varphi'\left(\frac{\mathbb{J}_n(t)}{n}\right). d\mathbb{M}_n^2(t) + \\ &+ \int_0^x I(\mathbb{J}_n(t) > 0) \left[\Psi\left(\frac{\mathbb{J}_n(t)}{n}\right) - \Psi\left(S^Z(t)\right)\right]. d\Lambda^Z(t) - \\ &- \int_0^x I(\mathbb{J}_n(t) = 0) \Psi\left(S^Z(t)\right) d\Lambda^Z(t) = \mathbb{A}_{2n}(x) + \mathbb{A}_{3n}(x) + \mathbb{A}_{4n}(x). \end{aligned} \tag{21}$$

For $x \in [0, \tau]$ at $n \to \infty$

$$|\mathbb{A}_{4n}(x)| \le I(\mathbb{J}_n(x)=0)\int_0^x \Psi\left(S^Z(t)\right)d\Lambda^Z(t) < I(\mathbb{J}_n(\tau)=0)\int_0^\tau \Psi\left(S^Z(t)\right)d\Lambda^Z(t) \xrightarrow{P} 0, \tag{22}$$

where we use condition (C2), under $m = 1$ and arguments of the proof of (25). Obviously, $\mathbb{A}_{2n}(x) \in M^2_{loc}\left(F_p^{(n)}\right)$ and quadratic characteristic of this martingale is

$$< \mathbb{A}_{2n}, \mathbb{A}_{2n} > (x) = \int_0^x I(\mathbb{J}_n(t)>0)\left[\varphi'\left(\frac{\mathbb{J}_n(t)}{n}\right)\right]^2 \frac{\mathbb{J}_n(t)}{n^2}\, d\mathbb{M}_n^Z(t) = \int_0^x \frac{\mathbb{J}_n(t)>0}{\mathbb{J}_n(t)}\left[\Psi\frac{\mathbb{J}_n(t)}{n}\right]^2 d\Lambda^Z(t).$$

Therefore, $\mathbb{A}^2_{2n}(x) - < \mathbb{A}_{2n}, \mathbb{A}_{2n} > (x)$ is a also martingale and by Lenglart's inequality, $\forall \varepsilon, \eta > 0$ we have

$$\mathbb{P}\left(\sup_{0\le x\le\tau}|\mathbb{A}_{2n}(x)| > \varepsilon\right) \le \frac{\eta}{\varepsilon^2} + \mathbb{P}\left(\int_0^\tau \frac{I(\mathbb{J}_n(t)>0)}{\mathbb{J}_n(t)}\left[\Psi\left(\frac{\mathbb{J}_n(t)}{n}\right)\right]^2 d\Lambda^Z(t) > \eta\right)$$
$$< \frac{\eta}{\varepsilon^2} + P\left(\frac{1}{\mathbb{J}_n(\tau)}\int_0^\tau \left[\Psi\left(\frac{\mathbb{J}_n(t)}{n}\right)\right]^2 d\Lambda^Z(t) > \eta\right).$$

According to Glivenko-Cantelli theorem for $n \to \infty$

$$\sup_{0\le x\le\infty}\left|\frac{\mathbb{J}_n(x)}{n} - S^Z(x)\right| \xrightarrow{a.s.} 0. \tag{23}$$

Moreover, due to the boundedness of Ψ and Ψ' on $[S^Z(\tau), 1]$ (condition (C1)), for $n \to \infty$ we have

$$\sup_{0\le x\le\tau}\left|\Psi^2\left(\frac{\mathbb{J}_n(x)}{n}\right) - \Psi^2\left(S^Z(x)\right)\right| \xrightarrow{a.s.} 0$$

Then by condition (C2) with $n \to \infty$

$$\int_0^\tau \left[\Psi\left(\frac{\mathbb{J}_{n(t)}}{n}\right)\right]^2 d\Lambda^Z(t) \xrightarrow{a.s.} \int_0^\tau \Psi^2\left(S^Z(t)\right)d\Lambda^Z(t) < \infty,$$

and consequently, taking into account $\mathbb{J}_{n(\tau)\xrightarrow{a.s.}\infty}$ we have convergence to zero of probability in the right side of (30), i.e.

$$\sup_{0\le x\le\tau}|\mathbb{A}_{2n}(x)| \xrightarrow{P} 0. \tag{24}$$

By mean value theorem

$$\sup_{0\le x\le\tau}|\mathbb{A}_{3n}(x)| \le \sup_{0\le x\le\tau}\int_0^x I(\mathbb{J}_n(t)>0)\,.\left|\Psi'\left(\theta_n(t)\right)\right|.\left|\frac{\mathbb{J}_n(t)}{n} - S^Z(t)\right| d\Lambda^Z(t)$$
$$\le \sup_{0\le x\le\infty}\left|\frac{\mathbb{J}_n(x)}{n} - S^Z(x)\right|.\int_0^\tau \left|\Psi'\left(\theta_n(t)\right)\right| d\Lambda^Z(t),$$

where $\theta_n(t) \in \left[min\left\{\frac{\mathbb{J}_n(t)}{n}, S^Z(t)\right\}; max\left\{\frac{\mathbb{J}_n(t)}{n}, S^Z(t)\right\}\right]$.

Now, by using (31) and condition (C3) at $n \to \infty$ have

$$\sup_{0\le x\le\tau}|\mathbb{A}_{3n}(x)| \xrightarrow{P} 0. \tag{25}$$

From (29), (32) and (33) at $n \to \infty$

$$\sup_{0\le x\le\tau}\left|\varphi\left(\tilde{S}_n^Z(x)\right) - \varphi\left(S^Z(x)\right)\right| \xrightarrow{P} 0, \tag{26}$$

where the number $\tau < T$ so that $S^Z(\tau) > 0$. In order to proof (C) as in the proof of (A) set $\tau = T - \varepsilon$. Given the monotonity of S^Z and φ we have

$$\sup_{0\le x\le T}\left|\varphi\left(\tilde{S}_n^Z(x)\right)-\varphi\big(S^Z(x)\big)\right| \le \sup_{0\le x\le T-\varepsilon}\left|\varphi\left(\tilde{S}_n^Z(x)\right)-\varphi\big(S^Z(x)\big)\right| +$$

$$+\sup_{T-\varepsilon\le x\le T}\left|\varphi\left(\tilde{S}_n^Z(x)\right)-\varphi\big(S^Z(x)\big)\right| \le \sup_{0\le x\le T-\varepsilon}\left|\varphi\left(\tilde{S}_n^Z(x)\right)-\varphi\big(S^Z(x)\big)\right| +$$

$$+\left|\varphi\left(\tilde{S}_n^Z(T)\right)-\varphi\left(\tilde{S}_n^Z(T-\varepsilon)\right)\right| + \left|\varphi\left(\tilde{S}_n^Z(T-\varepsilon)\right)-\varphi\big(S^Z(T-\varepsilon)\big)\right| +$$

$$+\left|\varphi\big(S^Z(T-\varepsilon)\big)-\varphi\big(S^Z(T)\big)\right|.$$

Now, using (34) and letting ε tend to zero, we obtain (C). Let us prove (D). According to Taylor's expansion, condition (C1) and (31) under $n \to \infty$ we get

$$\sup_{0\le x\le T}\left|\varphi\big(S_n^Z(x)\big)-\varphi\left(\tilde{S}_n^Z(x)\right)\right| \le \int_0^T \left|\varphi\left(\frac{\mathbb{J}_n(t)}{n}\right)-\varphi\left(\frac{\mathbb{J}_n(t)}{n}-\frac{1}{n}\right)-\right.$$

$$\left.-\frac{1}{n}\varphi'\left(\frac{\mathbb{J}_n(t)}{n}\right)\right| d\overline{\mathbb{N}}_n^Z(t) \le \frac{1}{2n^2}\int_0^T \left|\varphi''\big(\theta_n(t)\big)\right| d\overline{\mathbb{N}}_n^Z(t) \le$$

$$\le \frac{1}{2n}\sup_{0\le x\le T}\left|\varphi''\big(\theta_n(x)\big)\right| \overset{a.s.}{=} 0\left(\frac{1}{n}\right),$$

i.e. (D) is true. Clearly (E) is a consequence of (C) and (D). The proof of (F) is identical with that of (C). Now turn to the proof of the main statement (G) of uniform consistency of S_n^X. Consider the representation (23), where according to (D), (F) and (31) at $n \to \infty$

$$\sup_{0\le x\le T}|\mathbb{A}_{1n}^*(x)| \overset{P}{\to} 0,\ \sup_{0\le x\le T}|\mathbb{A}_{2n}^*(x)| = 0\left(\frac{1}{n}\right),\ \sup_{0\le x\le T}|\mathbb{A}_{3n}^*(x)| = 0_P\left(\frac{1}{n}\right), \tag{27}$$

Hence, by (23),

$$\sup_{0\le x\le T}\left|\varphi\big(S_n^X(x)\big)-\varphi\big(S^X(x)\big)\right| \overset{P}{\to} 0. \tag{28}$$

Now by the mean value theorem and condition (C1) from (28) we obtain (G). The theorem 2 is proved.

Now we demonstrate result on asymptotic normality of estimator (16). Introduce the stopped processes

$$q_n(x) = n^{1/2}\left(S_n^X\big(x\wedge Z^{(n)}\big) - S^X\big(x\wedge Z^{(n)}\big)\right),$$

where $a \wedge b = \min(a,b)$. Let $q(x) = e(x)\left[\varphi'\big(S^X(n)\big)\right]^{-1} + \xi\left[\varphi'\big(S^X(T_Z)\big)\right]^{-1}$, where $e(x)$ is mean zero Gaussian process with covariance function

$$A(x_1,x_2) = \int_0^{x_1\wedge x_2} S^Z(t)\left[\varphi'\big(S^Z(t)\big)\right]^2 d\Lambda(t) +$$

$$+2\int_0^{x_1\wedge x_2}\int_0^t S^Z(t)\big(1-S^Z(s)\big)\Psi'\left(S^Z(s)\right)d\Lambda(s)d\Lambda(t) +$$

$$+\int_{x_1\wedge x_2}^{x_1\vee x_2} S^Z(t)\,\Psi'\big(S^Z(t)\big)d\Lambda(t)\int_0^{x_1\wedge x_2}\left[\big(1-S^Z(s)\big)\Psi'\big(S^Z(s)\big)+\varphi'\big(S^Z(s)\big)\right]d\Lambda(s),$$

$x_1 \vee x_2 = max(x_1,x_2), \xi^D = \mathbb{N}(0,\sigma_0^2)$ and $\sigma_0^2(x) = = \lim_{x\to T_Z} A(x,x)$. Let $C(x) = \lim_{x\to T_Z} A(t,x)$.

Theorem 3. Let conditions (C1)-(C5) are hold, $\sigma_0^2 < \infty$ and for every $x \epsilon [0, T_Z)$: $C(x) < \infty$. Then for $n \to \infty$:

$$q_n(x) \overset{D}{\Rightarrow} q(x) \text{ in } D[0, T_Z]. \tag{29}$$

The proof of the theorem 3. First, examine the process

$$\mathbb{D}_n(x) = n^{1/2}\left(\varphi\left(S_n^X(x \Lambda Z^{(n)})\right) - \varphi\left(S_n^X(x \Lambda Z^{(n)})\right)\right)$$

and show that when $n \to \infty$

$$\mathbb{D}_n(x) \overset{D}{\Rightarrow} \mathbb{D}(x) \text{ in } D[0, T_z]. \tag{30}$$

According to the representation (21)

$$\mathbb{D}_n(x) = \sum_{m=1}^{3} n^{1/2}\, \mathbb{A}_{mn}^*\left(x \Lambda Z^{(n)}\right) \tag{31}$$

Since $Z^{(n)} \overset{a.s.}{\longrightarrow} T_z$ when $n \to \infty$, according to (27)

$$\sup_{0 \le x \le T_z} n^{1/2}\left|\mathbb{A}_{mn}^*\left(x \wedge z^{(n)}\right)\right| = 0_p\left(n^{-1/2}\right), m = 2,3 \tag{32}$$

Therefore, to establish (30), taking into account (31) and (32), it suffices to prove

$$n^{1/2}\mathbb{A}_{ln}^*\left(x \wedge Z^{(n)}\right) \overset{D}{\Rightarrow} \mathbb{D}(x) \text{ in } D[0, T_z]. \tag{33}$$

Using formulas (7), (10) and (11), we have

$$n^{1/2}\mathbb{A}_{ln}^*\left(x \wedge Z^{(n)}\right) = n^{1/2}\left(-\frac{1}{n}\int_0^{x \Lambda Z^{(n)}} I(S_n^Z(t) > 0) S_n^Z(t -).\right.$$

$$\left.\varphi'\left(S_n^Z(t)\right) d\Lambda_n(t) + \int_0^{x \Lambda Z^{(n)}} I(S_n^Z(t) > 0)\, S^Z(t) \varphi'\left(S^Z(t)\right) d\Lambda(t)\right) =$$

$$= n^{1/2}\left(-\frac{1}{n}\int_0^{x \Lambda Z^{(n)}} I(S_n^Z(t) > 0)\,.\,\varphi'\left(\left(S_n^Z(t)\right)\right) dM_n(t) + \int_0^{x \Lambda Z^{(n)}} I(S_n^Z(t) > 0).\right.$$

$$.\left[\Psi\left(S_n^Z(t)\right) - \Psi\left(S^Z(t)\right)\right] d\Lambda(t) + 0\left(\frac{1}{n}\right),$$

where have used (2.2.5) and the equation $S_n^Z(t -) = S_n^Z(t) + \frac{1}{n}$. By (1.2.4) subject to the conditions (C1),

$$\varphi'\left(S_n^Z(t)\right) = \varphi'\left(S^Z(t)\right) + 0_p\left((n^{-1}\ln n)^{1/2}\right),$$

$$\Psi\left(S_n^Z(t)\right) - \Psi\left(S^Z(t)\right) = \Psi'\left(S^Z(t)\right)\left(S_n^Z(t) - S^Z(t)\right) + 0_p(n^{-1}\ln n). \tag{34}$$

we obtain from (33), (34) we have

$$n^{1/2}\mathbb{A}_{ln}^*\left(x \wedge Z^{(n)}\right) = B_{1n}(x) + B_{2n}(x) + o_p(1), \tag{35}$$

where

$$B_{1n}(x) = -n^{1/2}\int_0^{x \Lambda Z^{(n)}} \varphi'\left(S^Z(t)\right) dM_n(t),$$

$$B_{2n}(x) = \int_0^{x\wedge Z^{(n)}} \Psi'\left(S^Z(t)\right)\varepsilon_n(t)d\Lambda(t).$$

$$\varepsilon_n(t) = n^{1/2}\left(S^Z(t) - S^Z(t)\right).$$

According to (35), the convergence (33) follows from the convergence

$$B_n(x) \overset{D}{\Rightarrow} \mathbb{D}(x) \text{ in } D[0, T_Z], \tag{36}$$

where $B_n(x) = B_{1n}(x) + B_{2n}(x)$. In the paper [6] (see equation (16)) states that for any x_0, such that $S^Z(x_0) > 0, B_n(x)$ converges weakly to a $\mathbb{D}(x)$ in $D[0, x_0]$. Therefore, to prove (36) according to the criterion of weak convergence the density is $B_n(x)$, for $k = 1{,}2$ and for any $\varepsilon > 0$:

$$\lim_{y\to T_Z} \limsup_{n\to\infty} P\left(\sup_{x\in(y,T_Z)} |B_{kn}(x) - B_{kn}(y)| > \varepsilon\right) = 0. \tag{37}$$

For $k = 1$

$$B_{1n}(x) - B_{1n}(y) = -n^{-\frac{1}{2}} \int_{y\wedge Z^{(n)}}^{x\wedge Z^{(n)}} \varphi'\left(S^Z(t)\right)dM_n(t). \tag{38}$$

Note that (38) is a martingale integral form with the stopping time, and then by the inequality Lenglart for $\forall\varepsilon, n > 0$ we have

$$\begin{gathered} P\left(\sup_{x\epsilon[y,T_Z]} |B_{1n}(x) - B_{1n}(y)| > \varepsilon\right) \le \frac{\eta}{\varepsilon^2} + \\ +P\left(\frac{1}{n}\int_{y\wedge Z^{(n)}}^{T_Z} \left(\varphi'\left(S^Z(t)\right)\right)^2 \mathbb{J}_n(t)d\Lambda(t) > \eta\right) \le \frac{\eta}{\varepsilon^2} + \\ +P\left(\int_y^{T_Z} \left(\varphi'\left(S^Z(t)\right)^2 S_n^Z(t)\right) d\Lambda(t) > \eta\right). \end{gathered} \tag{39}$$

According to theorem Glivenko-Cantelli for $n \to \infty$

$$\int_y^{T_Z} \left(\varphi'\left(S^Z(t)\right)\right)^2 S_n^Z(t)d\Lambda(t) \overset{P}{\to} \int_y^{T_Z} \left(\frac{\Psi\left(S^Z(t)\right)}{S^Z(t)}\right)^2 d\Lambda(t).$$

Consequently, for $y \to T_Z$ in view of the condition (C4) and the arbitrariness $\eta > 0$ converge to zero, i.e. (37) for $k = 1$ rightly. Since the empirical process $\varepsilon_n(t)$ converges weakly in $D[0, T_Z]$ to a Brownian bridge $B\left(1 - S^Z(t)\right)$ by the theorem of Doob-Donsker and in view of condition (C3) and presentation

$$B_{2n}(x) - B_{2n}(y) = \int_{y\wedge Z^{(n)}}^{x\wedge Z^{(n)}} \Psi'\left(S^Z(t)\right)\varepsilon_n(t)d\Lambda(t)$$

verify the validity of (37) and in the case for $k = 2$. Now the density $B_n(x)$ follows from (37) by the triangle inequality. Thus, (30) holds. To prove (29) it suffices to note that under condition (C1)

$$\begin{gathered} q_n(x) = \mathbb{D}_n(x)\left[\varphi'\left(S^X\left(x \wedge Z^{(n)}\right)\right)\right]^{-1} + o_p(1) = \\ = \mathbb{D}_n(x)\left\{\left[\varphi'\left(S^X(x)\right)\right]^{-1} 1[0, Z^{(n)}) + \left[\varphi'\left(S^X\left(Z^{(n)}\right)\right)\right]^{-1} I\{Z^{(n)}\}\right\} + o_p(1). \end{gathered} \tag{40}$$

Now (29) follows from (40) for $n \to \infty$. The theorem 3 is proved.

Remark. Consider independent censoring model(i.e. $\{X_k\}$ and $\{Y_k\}$ are mutually independent). In this case in (2) $C(u;v) = uv = C^*(u;v)$, $u, v\epsilon[0,1]$ and hence $\varphi(u) = -\log u, u \in [0,1]$ and $\varphi^{[-1]}(t) = \varphi^{-1}(t) == \exp(-t)$, so that

$$S^Z(x) = S^X(x)S^Y(x), x\epsilon\bar{R}^+ \quad (41)$$

It is easy to verify that from (12) and (16) respectively we obtain the exponential-hazard estimator

$$\tilde{S}_n^X(x) = exp\left\{-\int_0^x \frac{I(\mathbb{J}_n(t) > 0)}{\mathbb{J}_n(t)} d\bar{\mathbb{N}}_n(t)\right\} \quad (42)$$

and relative-risk power estimator of Abdushukurov (1998) (see[1]):

$$S_n^X = [S_n^Z(x)]^{R_n(x)}, R_n(x) = \frac{\Lambda_n(x)}{\Lambda_n^Z(x)} \quad (43)$$

Note that the estimator (12) is investigated in [3]. Moreover the Zeng-Klein's (1994) copula-graphic estimator is (see [3,6]):

$$\hat{S}_n^X(x) = \varphi^{[-1]}\left[\int_0^x I(\mathbb{J}_n(t) > 0)\left(\varphi\left(\frac{\mathbb{J}_n(t) - 1}{n}\right) - \varphi\left(\frac{\mathbb{J}_n(t)}{n}\right)\right)\right] d\bar{\mathbb{N}}_n(t) \quad (44)$$

which in independence model (41) is reduced to well - known Kaplan- Meier product - limit estimator (see [9])

$$\hat{S}_n^X(x) = \prod_{1\le x}\left\{1 - \frac{d\bar{\mathbb{N}}_n(t)}{\mathbb{J}_n(t)}\right\} \quad (45)$$

Let $\tilde{S}_n^Y, S_n^Y$ and $\hat{S}_n^Y$ are respectively estimators of S^Y of exponential-hazard, relative-risk power and product-limit structures obtained from formulas (42), (43) and (45) by using events $\delta_k = 0$ instead of $\delta_k = 1$. Then we have:

(a) $\tilde{S}_n^X(x)\tilde{S}_n^Y(x) = exp\{-\Lambda_n^Z(x)\} \neq S_n^Z(x)$ and for $x \geq Z_{(n)} = max\{Z_k, 1 \le k \le n\}, max\{\tilde{S}_n^X(x), \tilde{S}_n^Y(x)\} < 1$;

(b) $S_n^X(x)S_n^Y(x) = S_n^Z(x)$ for all $x\epsilon\bar{R}^+$ and $S_n^X(x) = S_n^Y(x) = 0$, for $x \le Z_{(n)}$;

(c) $\hat{S}_n^X(x)\hat{S}_n^Y(x) \neq S_n^Z(x)$ and for $x \geq Z_{(n)}$ the estimators $\hat{S}_n^X$ and $\hat{S}_n^Y$ are undefined. Moreover the estimators $\hat{S}_n^X$ and $\hat{S}_n^Y$ require also the condition $P(X_k = Y_k) = 0, k = 1,2, ...$, which in many practical situations is not hold. Thus only the relative-risk power estimators have identifiability properties with independence censoring model satisfying empirical analogue of equality (41). Analogously a new estimator (16) is more suitable estimator for $\hat{S}_n^X$ than the estimators (12) and (43).

4 Conclusions

In figures 1 and 2 below we demonstrate plots of estimators (12), (16) and (44) of $\hat{S}_n^X$ using well-known Channing House data of size $n = 97$ (see [1], [7], [8]). Here, thin-solid line stands for $\tilde{S}_n^X$, medium-one for $\hat{S}_n^X$ and thick-solid line stands for a new estimator $\hat{S}_n^X$. Note that estimate $\hat{S}_n^X$ is defined in whole line.

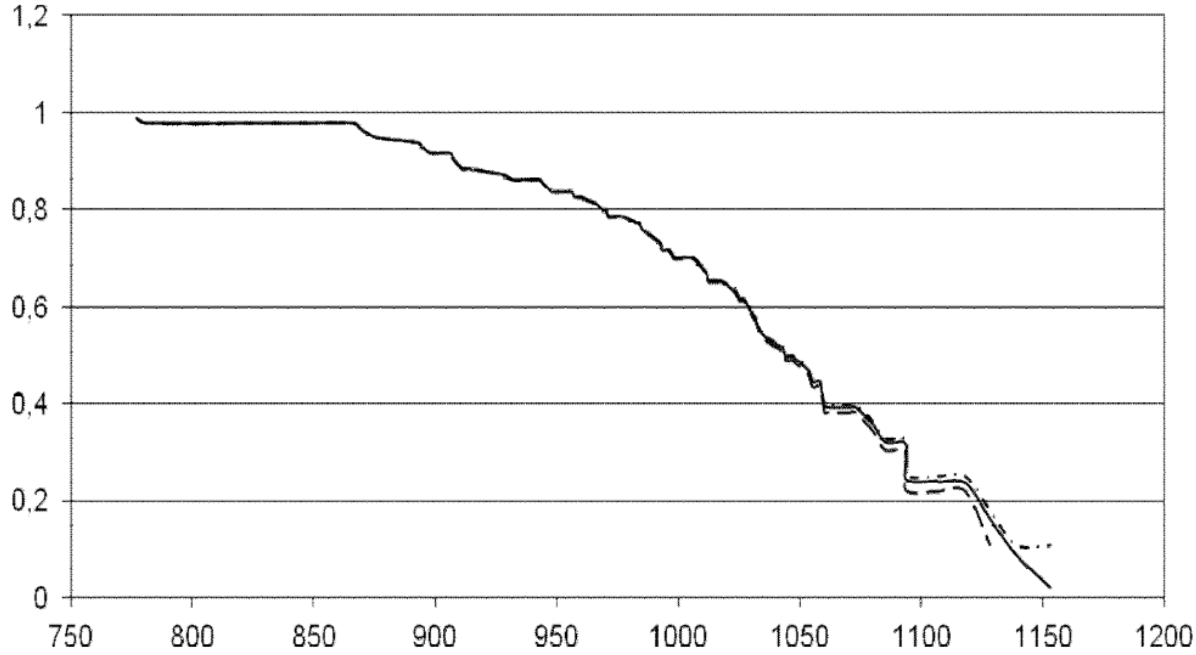

Figure 1. Plots of estimates $\tilde{S}_n^X$ (thin-solid), $\hat{S}_n^X$ (medium one) and S_n^X (thick-solid) for copula generator $\varphi(u) = -\ln u, u \in [0,1]$.

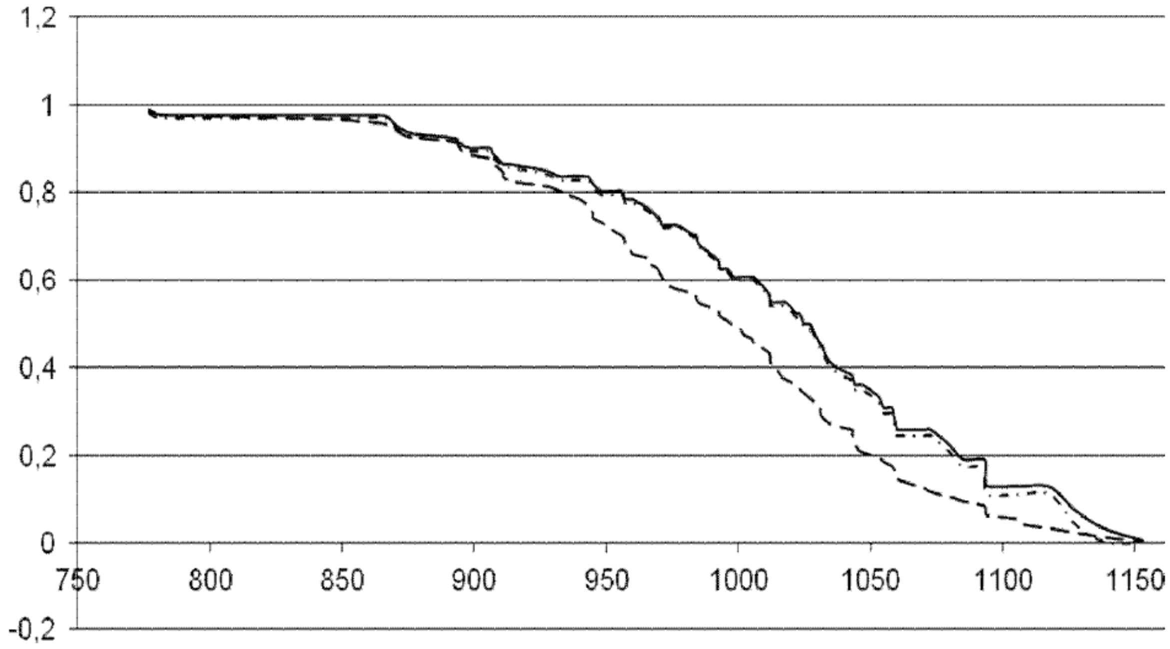

Figure 2. Plots of estimates $\tilde{S}_n^X$ (thin-solid), $\hat{S}_n^X$ (medium one) and S_n^X (thick-solid) for copula generator $\varphi(u) = (-\ln u)^2, u \in [0,1]$.

5 Estimation of mean residual life function

Let $E(x) = E(x; S^X) = E(x; S^X) = E(X_1 - x/X_1 > x) = (S^X(x))^{-1}. \int_x^{+\infty} S^X(t)dt, x \in [0, T_x)$, is mean residual life function of r.v. X_1. Consider estimate of E(x):

$$E_x(x) = \begin{cases} E(x, S_n^x), & x \in [0, \mathbb{Z}^{(n)}), \\ 0, & x \geq \mathbb{Z}^{(n)}. \end{cases}$$

Now we state our result on consistency of E_n with weight function $\varkappa(.): [0,1] \to \bar{R}^+$ We assume following conditions for $\varkappa$:

(C6) Function $\varkappa: [0,1] \to [0,\infty]$ is measurable and, for every $\eta > 0$: $\sup_{u\in[0,1-\eta]}\{\chi(u)\} < \infty$;

(C7) Function $\varkappa(u)/(1-u)$ is no decreasing in a neighborhood of 1:

(C8) $\int_0^{T_x}\left\{\left(S^X(x)\right)^{-1}\int_x^{T_X}\chi\left(F(y)\right)dy\right\}dF(x) < \infty$.

Note that the conditions (C1) - (C8) satisfies, for example, for copula generators of Clayton and Frank (see [5]).

Theorem 4. Let $\mu = EX_1 < \infty$, conditions (C1)-(C3) and (C6)-(C8) are hold. Then for $n \to \infty$,

$$\varepsilon_n(F) \overset{P}{\to} 0.$$

Proof of theorem 4. We have

$$\begin{aligned}&\varepsilon_n(F) \le \sup_{x\in[0;\mathbb{Z}^{(n)})}\chi\left(F(x)\right)|E_n(x)-E(x)| + \\ &+ \sup_{x\in[0;\mathbb{Z}^{(n)})}\chi\left(F(x)\right)|E_n(x)-E(x)| = \varepsilon_{1n}(F)+\varepsilon_{2n}(F).\end{aligned} \tag{46}$$

To prove (34), it is necessary to prove that for $n \to \infty, \varepsilon_{mn}(F) \overset{P}{\to} 0,\ m = 1,2$ since $E_n(x) = 0$ for all $x \ge Z^n$, then by (4) for $n \to \infty$

$$\varepsilon_{2n}(F) = \sup_{x\in[\mathbb{Z}^{(n)},T_Z)}\chi\left(F(x)\right)E(x) \overset{a.s.}{\to} 0. \tag{47}$$

By the other way, for a given number c>1 and almost all of the elementary events ω, we can find a number $n_0 = n_0(\omega: c)$ such that for all $x \in \left[0;\mathbb{Z}^{(n)}\right)$ and $n \ge n_0$:

$$\frac{S_n^x(x)}{S^x(x)} \ge c. \tag{48}$$

According to (10) and for all $x \in \left[0;\mathbb{Z}^{(n)}\right)$:

$$|E_n(x)-E(x)| \le c\Phi_{1n}(x) + c\Phi_{2n}(x), \tag{49}$$

where

$$\Phi_{1n}(x) = \frac{E(x)}{S^x(x)}|S_n^x(x)-S^x(x)|, \Phi_{2n}(x) = \frac{1}{S^x(x)}\int_x^{+\infty}|S_n^x(u)-S^x(u)|du.$$

Since when, we Show that

Since for $n \to \infty, \varepsilon_{1n}(F) \le c(\varepsilon_{3n}(F)+\varepsilon_{4n}(F))$, then we Show that

$$\varepsilon_{3n}(F) = \sup_{x\in[0,\mathbb{Z}^{(n)})}\chi\left(F(x)\right)\Phi_{1n}(x) \overset{P}{\to} 0, \tag{50}$$

$$\varepsilon_{4n}(F) = \sup_{x\in[0,\mathbb{Z}^{(n)})}\chi\left(F(x)\right)\Phi_{2n}(x) \overset{P}{\to} 0. \tag{51}$$

For given number $\eta > 0$ and sufficiently large number n

$$\varepsilon_{3n}(F) \leq \sup_{x\in[0,F^{-1}(1-\eta))} \chi\big(F(x)\big)\Phi_{1n}(x) + \\ + \sup_{x\in[F^{-1}(1-\eta),,\mathbb{Z}^{(n)})} \chi\big(F(x)\big)\Phi_{1n}(x) = \varepsilon_{5n}(F) + \varepsilon_{6n}(F), \tag{52}$$

where for $n \to \infty$, according to (50) and conditions (C6), (C7) we have

$$\varepsilon_{mn}(F) \xrightarrow{P} 0, m = 5,6. \tag{53}$$

From (52) and (53) we obtain (50). Similarly, for sufficiently large n

$$\varepsilon_{4n}(F) \leq \sup_{x\in[0,F^{-1}(1-\eta))} \chi\big(F(x)\big)\Phi_{2n}(x) + \\ + \sup_{x\in[F^{-1}(1-\eta),,\mathbb{Z}^{(n)})} \chi\big(F(x)\big)\Phi_{2n}(x) = \varepsilon_{7n}(F) + \varepsilon_{8n}(F). \tag{54}$$

In view of condition $\mu < \infty$, for $n \to \infty$ exist a number $c_0 > 0$ such that

$$\varepsilon_{7n}(F) \leq c_0 . \int_{x}^{+\infty} |S_n^x(x) - S^x(x)| dx \xrightarrow{P} 0. \tag{55}$$

Let $\chi^*\big(F(x)\big) = \chi(F(x))/S^X(x)$. Than according to the conditions (C6), (C7) and (C8) for $n \to \infty$

$$\varepsilon_{8n}(F) \leq \int_{F^{-1}(1-\eta)}^{T_X} \chi^*\big(F(x)\big)|S_n^x(x) - S^x(x)| dx \xrightarrow{P} 0. \tag{56}$$

From (54)-(56) follow (51). The theorem 4 is proved.

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