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Approximate solutions of boundary value problems of fractional order by using sinc-Galerkin method

Sertan Alkan

Department of Management Information Systems, Bartin University, Bartin-TURKEY

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Abstract: The aim of the present study is to obtain approximate solutions of fractional order linear two-point boundary value problem which are generalizations of classical boundary value problems by using sinc-Galerkin method. The fractional derivatives are defined in the Caputo sense using frequently in fractional calculus. The method is tested on some problems with homogeneous and nonhomogeneous boundary conditions and comparisons are made with the exact solutions and numerical solutions obtained by Haar Wavelet method. Numerical and graphical results show that the sinc-Galerkin method is a very effective and powerful tool in solving such problems.

Keywords: Fractional order boundary value problem, sinc-Galerkin method, Caputo derivative.

1. Introduction

Fractional calculus, which might be considered as an extension of classical calculus, are as old as the classical calculus and fractional differential equations have been frequently used to describe many scientific phenomena in earthquake engineering, biomedical engineering, image processing, signal processing and physics.

Many numerical methods are developed because of the obtaining exact solution of ordinary differential equations of fractional order is more difficult than the obtaining one of ordinary differential equations of integer order. These methods include Haar wavelet method [1], Adomian decomposition method [2], spline collocation method [3], least sequares finite-element method [4], variational iteration method [5], generalized differential transform method [6], extrapolation method [7], Chebyshev wavelet method [8] and Legendre wavelet method [9].

In this paper, we use sinc-Galerkin method to obtain approximate solutions of fractional linear two-point boundary value problem

$$ {}_{a}^{C}D_{x}^{\alpha} y + y = f(x), \;\; a < x < b, \;\; 1 < \alpha < 2 $$

with boundary conditions

$$ y(a) = y(b) = 0 $$

where ${}_{a}^{C}D_{x}$ is Caputo fractional derivative operator.

The paper is organized as follows. Section 2 presents basic theorems for fractional calculus and sinc-Galerkin method. In Section 3, we use the sinc-Galerkin method to obtain an approximate solution of a general fractional two-point boundary value problem. In Section 4, we present two examples in order to illustrate the effective and accuracy of the present

* Corresponding author e-mail: mathselpi@gmail.com

method. The obtained results are compared with the Haar wavelet method in the table and graphical forms.

2. Preliminaries

2.1 Fractional Calculus

In this section, firstly we present the definitions of the Riemann-Liouville and the Caputo of fractional derivative. Later, we give the definition of the integration by parts of fractional order by using these definitions.

Definition 2.1[10] Let $f:[a,b]\to R$be a function, αa positive real number, $n$the integer satisfying $n-1\leq\alpha<n$, and $\Gamma$the Euler gamma function. Then,

1.The left and right Riemann-Liouville fractional integrals of order α of a function $f(x)$are given as

$$ {}_aI_x^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\,dt, $$

and

$$ {}_xI_b^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)\,dt, $$

respectively,

2.The left and right Riemann-Liouville fractional derivatives of order $\alpha$of $f(x)$are given as

$$ {}_aD_x^\alpha f(x)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x (x-t)^{n-\alpha-1}f(t)\,dt, \tag{2.1} $$

and

$$ {}_xD_b^\alpha f(x)=\frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_x^b (t-x)^{n-\alpha-1}f(t)\,dt, \tag{2.2} $$

respectively.

3.The left and right Caputo fractional derivatives of order $\alpha$of $f(x)$are given as

$$ {}_a^CD_x^\alpha f(x)=\frac{1}{\Gamma(n-\alpha)}\int_a^x (x-t)^{n-\alpha-1}f^{(n)}(t)\,dt, \tag{2.3} $$

and

$$ {}_x^CD_b^\alpha f(x)=\frac{1}{\Gamma(n-\alpha)}\int_x^b (-1)^n(t-x)^{n-\alpha-1}f^{(n)}(t)\,dt, \tag{2.4} $$

respectively.

In private, if $f(a)=f'(a)==f^{(n-1)}(a)=0$,then

$$ {}_a^CD_x^\alpha f(x)={}_aD_x^\alpha f(x) \tag{2.5} $$

and if $f(b)=f'(b)==f^{(n-1)}(b)=0$,then

$$ {}_x^CD_b^\alpha f(x)={}_xD_b^\alpha f(x). $$

Now we can write the definition of integration by parts of fractional order by using the relations given in (2.1)-(2.4).

Definition 2.2 [10] If $0 < \alpha < 1$ and f is a function such that $f(a) = f(b) = 0$, we can write

$$\int_a^b g(x)\,{}_a^C D_x^\alpha f(x)\,dx = \int_a^b f(x)\,{}_x D_b^\alpha g(x)\,dx$$

and

$$\int_a^b g(x)\,{}_x^C D_b^\alpha f(x)\,dx = \int_a^b f(x)\,{}_a D_x^\alpha g(x)\,dx.$$

Teorem 2.1 [3] Let be $\alpha > 0$ and $n \in N$ such that $n-1 < \alpha \leq n$ and $f(x) \in C^n[a,b]$, then

$${}_a^C D_x^\alpha f(x) = {}_a I_x^{n-\alpha} f^{(n)}(x).$$

Teorem 2.2 [3] Let be $\alpha > 0$. If $f$is continious, then

$${}_a D_x^\alpha \,{}_a I_x^\alpha f(x) = f(x).$$

2.2 Sinc basis functions properties and quadrature interpolations

In this section, we recall notations and definitions of the sinc function and derive useful formulas that are important for this paper.

The Sinc basis functions

Definition 2.3 [14] The function defined all $z \in C$ by

$$sinc(z) = \begin{cases} \frac{sin(\pi z)}{\pi z}, & z \neq 0, \\ 1, & z = 0. \end{cases} \tag{2.6}$$

is called the sinc function.

Definition 2.4 [14] Let f be a function defined on R and let $h > 0$. Define the series

$$C(f,h)(x) = \sum_{k=-\infty}^{\infty} f(kh)\, sinc\left(\frac{x-kh}{h}\right) \tag{2.7}$$

where from (2.6)

$$S(k,h)(x) = sinc\left(\frac{x-kh}{h}\right) = \begin{cases} \frac{\sin\left(\pi \frac{x-kh}{h}\right)}{\pi \frac{x-kh}{h}} & x \neq kh \\ 1 & x = kh. \end{cases}$$

Whenever the series in (2.7) converges it is called the Whittaker cardinal function of f. They are based on the infinite strip D_s in the complex plane

$$D_s \equiv \left\{ w = u + iv : |v| < d \leq \frac{\pi}{2} \right\}.$$

In general, approximations can be constructed for infinite, semi-infinite and finite intervals. Define the function

$$w = \phi(z) = \ln\left(\frac{z}{1-z}\right) \tag{2.8}$$

which is a conformal mapping from D_E, the eye-shaped domain in the z-plane, onto the infinite strip D_S, where

$$D_E = z = \left\{ x + iy : \left| \arg\left(\frac{z}{1-z}\right) \right| < d \leq \frac{\pi}{2} \right\}.$$

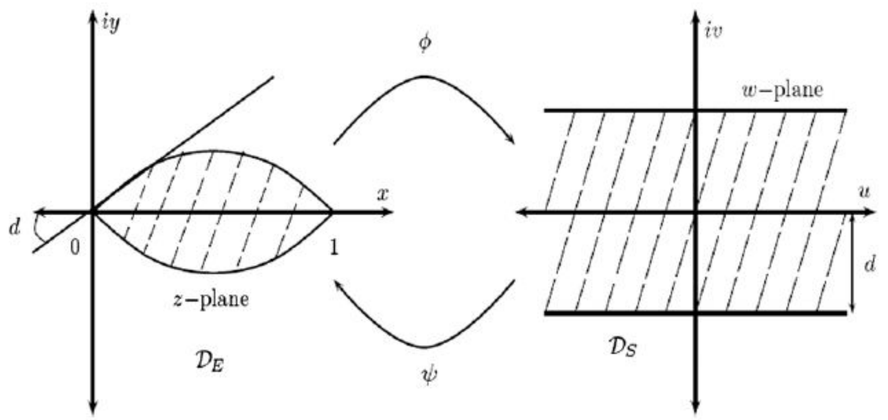

Figure 1. The domains D_E and D_S .

This is shown in Figure 1. For the sinc-Galerkin method, the basis functions are derived from the composite translated sinc functions

$$S_k(z) = S(k,h)(z)\, o\phi(z) = sinc\left(\frac{\phi(z) - kh}{h}\right)$$

for $z \in D_E$.The function $z = \phi^{-1}(w) = \frac{e^w}{1+e^w}$is an inverse mapping of $w = \phi(z)$.We may define the range of ϕ^{-1} on the real line as

$$\Gamma = \{\phi^{-1}(u) \in D_E : -\infty < u < \infty\}$$

the evenly spaced nodes $\{kh\}_{k=-\infty}^{\infty}$ on the real line. The image which corresponds to these nodes is denoted by

$$x_k = \phi^{-1}(kh) = \frac{e^{kh}}{1+e^{kh}}.$$

Sinc function interpolation and quadratures

Definition 2.5 [11] Let D_E be a simply connected domain in the complex plane C, and let ∂D_E denote the boundary of D_E. Let $a,\ b$ be points on ∂D_E and ϕbe a conformal map D_E onto D_S such that $\phi(a) = -\infty$and $\phi(b) = \infty$. If the inverse map of $\phi$is denoted by φ, define

$$\Gamma = \{\phi^{-1}(u) \in D_E : -\infty < u < \infty\}$$

and $z_k = \varphi(kh),\ k = \pm 1, \pm 2,$

Definition 2.6 [11] Let $B(D_E)$be the class of functions Fthat are analytic in D_E and satisfy

$$\int_{\psi(L+u)} |F(z)|\, dz \to 0,\ as\ u = \mp\infty,$$

where

$$L = \left\{iy : |y| < d \leq \frac{\pi}{2}\right\},$$

and those on the boundary of D_E satisfy

$$T(F) = \int_{\partial D_E} |F(z)\, dz| < \infty.$$

Theorem 2.5 [11] Let Γbe $(0,1)$, $F \in B(D_E)$, then for $h > 0$sufficiently small,

$$\int_\Gamma F(z)\, dz - h \sum_{j=-\infty}^{\infty} \frac{F(z_j)}{\phi'(z_j)} = \frac{i}{2} \int_{\partial D} \frac{F(z)\, k(\phi,h)(z)}{sin(\pi\phi(z)/h)} dz \equiv I_F \tag{2.9}$$

where

$$|k(\phi,h)|_{z\in\partial D}=\left|e^{\left[\frac{i\pi\phi(z)}{h}sgn(Im\phi(z))\right]}\right|_{z\in\partial D}=e^{\frac{-\pi d}{h}}.$$

For the sinc-Galerkin method, the infinite quadrature rule must be truncated to a finite sum. The following theorem indicates the conditions under which an exponential convergence results.

Theorem 2.6 [11] If there exist positive constants $\alpha,\ \beta$ and C such that

$$\left|\frac{F(x)}{\phi'(x)}\right|\leq C\begin{cases}e^{-\alpha|\phi(x)|}, & x\in\psi((-\infty,\infty))\\ \\ e^{-\beta|\phi(x)|}, & x\in\psi((0,\infty))\end{cases}\tag{2.10}$$

then the error bound for the quadrature rule (2.9) is

$$\left|\int_{\Gamma}F(x)\,dx-h\sum_{j=-M}^{N}\frac{F(x_j)}{\phi'(x_j)}\right|\leq C\left(\frac{e^{-\alpha Mh}}{\alpha}+\frac{e^{-\beta Nh}}{\beta}\right)+|I_F|.\tag{2.11}$$

The infinite sum in (2.9) is truncated with the use of (2.10) to arrive at the inequality (2.11). Making the selections

$$h=\sqrt{\frac{\pi d}{\alpha M}}$$

$$N\equiv\left[\left[\frac{\alpha M}{\beta}+1\right]\right]$$

where $[\![\cdot]\!]$ is an integer part of the statement and M is the integer value which specifies the grid size, then

$$\int_{\Gamma}F(x)\,dx=h\sum_{j=-M}^{N}\frac{F(x_j)}{\phi'(x_j)}+O\left(e^{-(\pi\alpha dM)^{\frac{1}{2}}}\right).$$

We used these theorems to approximate the integrals that arise in the formulation of the discrete systems corresponding to a second-order boundary value problem.

3 The sinc-Galerkin Method

Consider linear two-point boundary value problem

$${}_a^C D_x^{\alpha}y+y=f(x),\ \ a<x<b,\ \ 1<\alpha<2\tag{3.1}$$

with boundary conditions

$$y(a)=y(b)=0\tag{3.2}$$

where ${}_a^C D_x$ is Caputo fractional derivative operator. According to Theorem 2.1, equation (3.1) is written by

$${}_aI_x^{2-\alpha}y''+y=f(x),\ \ a<x<b,\ \ 0<2-\alpha<1\tag{3.3}$$

If we apply ${}_aD_x^{2-\alpha}$ to both sides of equation (3.3) and use Theorem 2.2, we have

$$y''+{}_aD_x^{\beta}y=g(x),\ \ a<x<b,\ \ 0<\beta<1\tag{3.4}$$

where $g(x)={}_aD_x^{2-\alpha}f(x)$ and $\beta=2-\alpha$. Also, by using boundary conditions and equality (2.5), equation (3.4) can be written as

$$y''+{}_a^C D_x^{\beta}y=g(x),\ \ a<x<b,\ \ 0<\beta<1\tag{3.5}$$

An approximate solution for $y(x)$ is represented by the formula

$$y_n(x)=\sum_{j=-M}^{M}c_jS_j(x),\ \ n=2M+1,\tag{3.6}$$

where $S_j(x)$ is the function $S(j,h)\,o\phi(x)$ for some fixed step size h. The unknown coefficients c_j in (3.6) are determined by orthogonalizing the residual with respect to the basis functions, i.e.

$$< y'', S_k > + < {}_a^C D_x^{\beta} y, S_k > = < g(x), S_k > . \tag{3.7}$$

The inner product used for the sinc-Galerkin method is defined by

$$< f, \eta > = \int_a^b f(x)\,\eta(x)\,w(x)\,dx.$$

where $w(x)$is weight function and it is convenient to take

$$w(x) = \frac{1}{\phi'(x)}$$

for the case of second-order problems.

Lemma 3.1 [15] Let ϕbe the conformal one-to-one mapping of the simply connected domain D_E onto D_S, given by (2.8). Then

$$\delta_{jk}^{(0)} = [S(j,h)\,o\phi(x)]|_{x=x_k} = \begin{cases} 1, & j=k \\ 0, & j \neq k \end{cases}$$

$$\delta_{jk}^{(1)} = h\frac{d}{d\phi}[S(j,h)\,o\phi(x)]\Big|_{x=x_k} = \begin{cases} 0, & j=k \\ \frac{(-1)^{k-j}}{k-j}, & j \neq k \end{cases}$$

$$\delta_{jk}^{(2)} = h^2\frac{d^2}{d\phi^2}[S(j,h)\,o\phi(x)]\Big|_{x=x_k} = \begin{cases} -\frac{\pi^2}{3}, & j=k \\ \frac{-2(-1)^{k-j}}{(k-j)^2}, & j \neq k \end{cases}$$

The method of approximating the integrals in (3.6) begins by integrating by parts to transfer all derivatives from $y$to S_k. The following theorems which can easily prove by using Lemma 3.1 and Definition 2.2 are used to solve Equation (3.1).

Theorem 3.2 [12] The following relations hold

$$< y'', S_k > \cong h \sum_{j=-M}^{M} \sum_{i=0}^{2} \frac{y(x_j)}{\phi'(x_j)\,h^i}\delta_{kj}^{(i)} g_{2,i}, \tag{3.8}$$

and

$$< g, S_k > \cong h\frac{g(x_k)\,w(x_k)}{\phi'(x_k)}, \tag{3.9}$$

where

$$g_{2,2} = w\left(\phi'\right)^2, \;\; g_{2,1} = w\phi'' + 2w'\phi', \;\; g_{2,0} = w'',$$

Theorem 3.3 For $0 < \beta < 1$, the following relations hold:

$$< {}_0^C D_x^{\beta} y(x), S_k > \cong -\frac{h}{\Gamma(1-\beta)} \sum_{j=-M}^{M} \frac{y(x_j)}{\phi'(x_j)} \frac{d}{dx}\left[h_L \sum_{r=-L}^{L} \frac{(x_r - x)^{-\beta} K(x_r)}{\xi'(x_r)}\right]\Bigg|_{x=x_j} \tag{3.10}$$

where $K(x) = S_k(x)\,w(x)$, $\xi(t) = \ln\left(\frac{t-x}{1-t}\right)$and $h_L = \pi/\sqrt{L}$

Proof: See [16].

Replacing each term of (3.7) with the approximation defined in (3.8) – (3.10), replacing $y(x_j)$by c_j and dividing by $h$we obtain the following theorem.

Theorem 3.4 If the assumed approximate solution of the boundary-value problem (3.5) is (3.6), then the discrete sinc–Galerkin system for the determination of the unknown coefficients $\{c_j\}_{j=-M}^{M}$ is given by

$$\sum_{j=-M}^{M}\left\{\sum_{i=0}^{2}\frac{1}{h^i}\delta_{kj}^{(i)}\frac{g_{2,i}(x_j)}{\phi'(x_j)}c_j-\frac{1}{\Gamma(1-\beta)}\frac{c_j}{\phi'(x_j)}\frac{d}{dx}\left[h_L\sum_{r=-L}^{L}\frac{(x_r-x)^{-\beta}K(x_r)}{\xi'(x_r)}\right]\Bigg|_{x=x_j}\right\}=\frac{g(x_k)\,w(x_k)}{\phi'(x_k)},\quad -M\le k\le M.$$

where $K(x)=S_k(x)\,w(x)$, $\xi(t)=\ln\left(\frac{t-x}{1-t}\right)$.

4 Examples

In this section, two problems that have homogeneous and nonhomogeneous boundary conditions will be tested by using the present method. In all the examples, we take $d=\pi/2,\ \ \alpha=\beta=1/2,\ \ N=M$.

Example 4.1 [17] Consider linear fractional boundary value problem

$${}_0^CD_x^{1.2}y(x)+\frac{3}{57}y(x)=f(x),$$

subject to the nonhomogeneous boundary conditions

$$y(0)=0,\ \ y(1)=\frac{1}{\Gamma(3.2)}$$

where $f(x)=x+\frac{3x^{2.2}}{\Gamma(3.2)}$. The exact solution of this problem is $y(x)=\frac{x^{2.2}}{\Gamma(3.2)}$. First we convert the nonhomogeneous boundary conditions to homogeneous conditions by considering the transformation $(x)=y(x)-\frac{x}{\Gamma(3.2)}$. This change of variable yields the following boundary value problem

$${}_0^CD_x^{1.2}u(x)+\frac{3}{57}u(x)=g(x), \tag{4.1}$$

with homogeneous boundary conditions

$$u(0)=0,\ \ u(1)=0$$

where

$$g(x)=x+\frac{3}{57}\frac{x^{2.2}}{\Gamma(3.2)}-\frac{3}{57}\frac{x}{\Gamma(3.2)}$$

According to Teorem 2.1, equation (4.1) can be written as

$${}_0I_x^{0.8}u''(x)+\frac{3}{57}u(x)=g(x)$$

Acting with ${}_0D_x^{0.8}$ on both sides of this equation and using Theorem 2.2, we obtain

$$u''(x)+\frac{3}{57}{}_0D_x^{0.8}u(x)=r(x) \tag{4.2}$$

where $r(x)={}_0D_x^{0.8}g(x)=0.217825\ \left(4.89143\,t^{0.2}+0.194517\,t^{1.4}\right)$

Finally, by using boundary conditions and (2.5), equation (4.2) can be written

$$u''(x)+\frac{3}{57}{}_0^CD_x^{0.8}u(x)=r(x)$$

subject to the homogeneous boundary conditions

$$u(0)=0,\ \ u(1)=0$$

The numerical solutions which are obtained by using the sinc-Galerkin method (SGM) for this problem are presented in Table 1. In addition to, in Table 2, the solutions are compared with the numerical solutions computed by using Haar Wavelet method (HWM). Also the graphs of exact and approximate solutions for different values of $L$and $M$are presented in Figure 2.

Table 1 Numerical results for $L = 5,\ M = 5$

x	Exact Sol.	Num. Sol.	Error
0	0	0	0
0.1	0.002603	0.003024	0.000421008
0.2	0.011960	0.011040	0.000920051
0.3	0.029183	0.028462	0.000721595
0.4	0.054954	0.055164	0.000210129
0.5	0.089785	0.090523	0.000738097
0.6	0.134093	0.134454	0.000361033
0.7	0.188230	0.187585	0.000645323
0.8	0.252506	0.251336	0.001169680
0.9	0.327195	0.327288	0.000092972
1	0	0	0

Table 2 Numerical results for $L = 40,\ M = 100$

x	Exact Sol.	Num. Sol.	Error(SGM)	Error(HWM)
0	0	0	0	0
0.1	0.002603	0.002609	6.29*10^-6	1.53*10^-6
0.2	0.011960	0.011969	8.95*10^-6	1.52 *10^-7
0.3	0.029183	0.029192	8.66*10^-6	8.07*10^-7
0.4	0.054954	0.054961	6.43*10^-6	6.31*10^-7
0.5	0.089785	0.089789	3.23*10^-6	5.19*10^-7
0.6	0.134093	0.134093	2.46*10^-7	1.82*10^-6
0.7	0.188230	0.188227	2.83*10^-6	2.43*10^-6
0.8	0.252506	0.252502	4.40*10^-6	3.11*10^-6
0.9	0.327195	0.327191	4.35*10^-6	3.96*10^-6
1	0	0	0	0

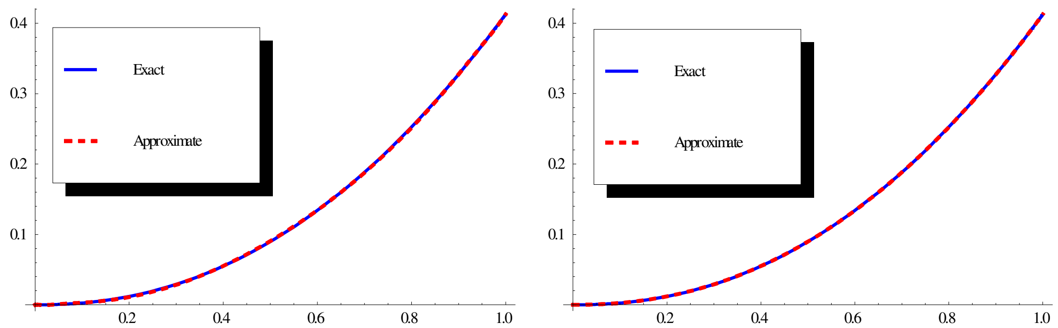

Figure 2. Graphs of exact and approximate solutions for different values of L and M: left figure for $L = 5$ and $M = 5$ and right figure for $L = 40$ and $M = 100$.

Example 4.2 [13] Consider linear fractional boundary value problem

$$ {}_0^C D_x^{1.5} y(x) + y(x) = f(x) \tag{4.3} $$

subject to the homogeneous boundary conditions

$$ y(0) = 0, \;\; y(1) = 0 $$

where $f(x) = x^5 - x^4 + \frac{128}{7\sqrt{\pi}} x^{3.5} - \frac{64}{5\sqrt{\pi}} x^{2.5}$. The exact solution of this problem is $y(x) = x^4 (x-1)$. According to Theorem 2.1, equation (4.3) can be written

$$ {}_0 I_x^{0.5} y''(x) + y(x) = f(x) $$

Acting with ${}_0 D_x^{0.5}$ on both sides of this equation and using Theorem 2.2, we obtain

$$ y''(x) + {}_0 D_x^{0.5} y(x) = g(x) \tag{4.4} $$

where $g(x) = {}_0 D_x^{0.5} f(x) = \frac{256}{63\sqrt{\pi}} x^{4.5} - \frac{128}{35\sqrt{\pi}} x^{3.5} + 20x^3 - 12x^2$.

Finally, by using boundary conditions and (2.5), equation (4.4) can be written

$$ y''(x) + {}_0^C D_x^{0.5} y(x) = g(x) $$

subject to the homogeneous boundary conditions

$$ y(0) = 0, \;\; y(1) = 0 $$

The numerical solutions which are obtained by using the sinc-Galerkin method (SGM) for this problem are presented in Table 3 and Table 4. In addition to, in Figure 3, the graphs of exact and approximate solutions for different values of $L$and M are presented.

Table 3 Numerical results for $L = 5,\ M = 5$x

x	Exact Sol.	Num. Sol.	Error
0	0	0	0
0.1	-0.00009	-0.0007482	0.00065824
0.2	-0.00128	0.00231681	0.00359681
0.3	-0.00567	-0.0005421	0.00512784
0.4	-0.01536	-0.0112397	0.00412032
0.5	-0.03125	-0.0281647	0.00308535
0.6	-0.05184	-0.0488395	0.00300051
0.7	-0.07203	-0.0694385	0.00259152
0.8	-0.08192	-0.0823823	0.00046231
0.9	-0.06561	-0.0689973	0.00338727
1	0	0	0

Table 4 Numerical results for L=40, M=100

x	Exact Sol.	Num. Sol.	Error
0	0	0	0
0.1	-0.00009	-0.0000898	1.49*10^-7
0.2	-0.00128	-0.0012797	2.92*10^-7
0.3	-0.00567	-0.0056695	4.14*10^-7
0.4	-0.01536	-0.0153595	5.01*10^-7
0.5	-0.03125	-0.0312495	5.38*10^-7
0.6	-0.05184	-0.0518395	4.94*10^-7
0.7	-0.07203	-0.0720296	4.02*10^-7
0.8	-0.08192	-0.0819198	2.30*10^-7
0.9	-0.06561	-0.0656100	1.49*10^-11
1	0	0	0

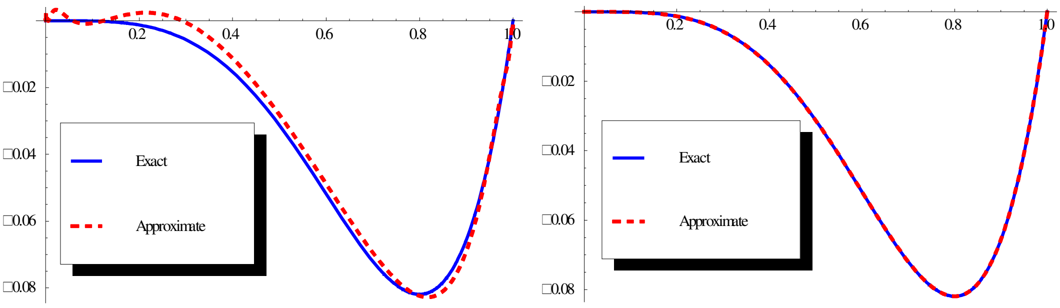

Figure 3. Graphs of exact and approximate solutions for different values of L and M: left figure for $L=5$ and $M=5$ and right figure for $L=40$ and $M=100$.

5 Conclusion

In this study, we use sinc-Galerkin method to obtain approximate solutions of two-point boundary value problems for linear fractional differential equations with constant coefficients. In order to illustrate the effective and accuracy of the present method, it is applied to some special examples in the literature and the obtained results are compared with exact solutions and also with the solutions obtained by the Haar wavelet method. As a result, it is shown that sinc-Galerkin method is very effective and reliable for obtaining approximate solutions of fractional boundary value problems.

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