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# A note on "Exact solutions for nonlinear integral equations by a modified homotopy perturbation method"

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**Abstract:** In the paper "Exact solutions for nonlinear integral equations by a modified homotopy perturbation method" by A. Ghorbani and J. Saberi-Nadjafi, Computers and Mathematics with Applications, 56, (2008) 1032-1039, the authors introduced a new modification of the homotopy perturbation method to solve nonlinear integral equations.We discuss here the restrictions on their method for solving nonlinear integral equations. We also prove analytically that the method given by Ghorbani and Saberi-Nadjafi is equivalent to the series solution method when selective functions are polynomials.

*Keywords:* Modified Homotopy perturbation method; nonlinear integral equations; series solution method.

## 1 Introduction

Recently in [1], Ghorbani and Saberi-Nadjafi proposed a new modification of the homotopy perturbation method for solving nonlinear integral equations.

In this note, we show by an example that this method is not true generally. The purpose of this paper to show that, the new modification of the homotopy perturbation method is applicable for special case of nonlinear integral equations when the exact solution must appear as part of given function in integral equation otherwise this method is equivalence of the series solution method. This paper is organized as follow: The principle of the new modification of the homotopy perturbation method is described in Section 2. Two examples are studied in Section 3. The general remarks are given in Section 4.

## 2 The Principle of the New Modification of the Homotopy Perturbation Method

In [1], Ghorbani and Saberi-Nadjafi consider the following type of nonlinear integral equations:

$$y(x) = g(x) + \int_a^x k(x,t)\,[y(t)]^r dt, \qquad a \le x,\ t \le b,\ r \ge 2 \tag{2.1}$$

$$y(x) = g(x) + \int_a^b k(x,t)\,[y(t)]^r dt, \qquad a \le x,\ t \le b \tag{2.2}$$

Based on the Homotopy perturbation method (HPM) [4,5], they presented a method which called, modified HPM by them. In this regards, they rewrite (2.1) as:

$$y(x) = \sum_{m=0}^{N} \alpha_m\, v_m(x) - \sum_{m=0}^{N} \alpha_m\, v_m(x) + g(x) + \int_a^x k(x,t)\,[y(t)]^r dt \tag{2.3}$$

where $\alpha_m$ and $v_m(x)$, $m = 0,1,2,\cdots,N$ are called by them as the accelerating components of the parameter and selective functions, respectively.

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Based on the HPM, by selecting $F(u) = u - \sum_{m=0}^{N} \alpha_m \, v_m(x)$ they defined the following convex homotopy:

$$H_\alpha(u,p) = u(x) - pg(x) - (p-1)\sum_{m=0}^{N} \alpha_m \, v_m(x) - p\int_a^x k(x,t)\,[y(t)]^r dt = 0 \quad (2.4)$$

where the embedding parameter $p \in (0,1]$ can be considered as an expanding parameter. The HPM uses the embedding parameter $p$ as a " small parameter" , and writes the solution of (2.4) as a power series of $p$, i.e.,

$$u = u_0 + u_1 p + u_2 p^2 + \cdots, \quad (2.5)$$

Setting $p = 1$ results in the approximate solution of (2.4):

$$y = lim_{p\to 1} u = u_0 + u_1 + u_2 + \cdots, \quad (2.6)$$

Substituting Eq. (2.5) into (2.4) and equating the terms with identical powers of $p$, we can obtain a series of equations of the following form:

$$\begin{aligned}
&p^0: \; u_0 - \sum_{m=0}^{N} \alpha_m \, v_m(x) = 0,\\
&p^1: \; u_1 - g(x) - \sum_{m=0}^{N} \alpha_m \, v_m(x) - \int_a^x k(x,t)\,[\,u_0(t)]^r dt = 0,\\
&p^2: \; u_2 - \int_a^x k(x,t)\, H(\,u_0, u_1) dt = 0,\\
&p^3: \; u_3 - \int_a^x k(x,t)\, H(\,u_0, u_1, \quad u_2) dt = 0,\\
&\vdots
\end{aligned} \quad (2.7)$$

where $H(\,u_0, u_1, \cdots, u_j)$ depend upon $u_0, u_1, \cdots, u_j$.The $H\big(\,u_0, u_1, \cdots, u_j\big)$, calculate using Adomian formula [2,6]

$$H\big(\,u_0, u_1, \cdots, u_j\big) = \frac{1}{j!}\frac{\partial^j}{\partial p^j}\Big(\sum_{i=0}^{j} u_i \, p^i\Big)^r\Big|_{p=0}. \quad (2.8)$$

Which is called first time by Ghorbani as He polynomials [3]. However this formula has been used before Ghorbani's definition by first author and others in HPM [6] as Adomian polynomials (For more detail see [7]). It is obvious that the system of nonlinear equations in (2.7) is easy to solve and the components $u_i, i \geq 0$ of the homotopy perturbation method can be completely determined and the series solutions are thus entirely determined.

Remark. We get $\alpha_m, m = 0,1,2,\cdots,N$, and $v_m(x), m = 0,1,2,\cdots,N$ where $v_m(x)$ is form of function $g(x)$ accordingly we will obtain the exact solution, if we could not find $\alpha_m, m = 0,1,2,\cdots,N$ with $v_m(x), m = 0,1,2,\cdots,N$ so this method is not effective and this is a weakness in [1]. But if we increased $N, N \longrightarrow \infty$, we will obtain exact solution by Taylor series method. We discussed an example (2.2) about this case.

## 3 Examples

**Example 1:** [1,8] Consider the following nonlinear Volterra integral equation

$$y(x) = 1 - \frac{3}{2}x^2 - x^3 - \frac{1}{4}x^4 + \int_0^x y^3\,(t) dt, \quad (3.1)$$

with the exact solution $y(x) = 1 + x$.

We apply this new modified HPM. We get $v_0(x) = 1, v_1(x) = x$ then

$$H_\alpha(u,p) = u(x) - p\left(1 - \frac{3}{2}x^2 - x^3 - \frac{1}{4}x^4\right) + (p-1)(\alpha_0 + \alpha_1 x) - p\int_0^x [y(t)]^3\, dt = 0 \quad (3.2)$$

In view of Eq. (2.7) we have

$$p^0: \quad u_0(x) - \alpha_0 - \alpha_1 x = 0 \;\to\; u_0(x) = \alpha_0 + \alpha_1 x,$$
$$p^1: \quad u_1(x) - p\left(1 - \tfrac{3}{2}x^2 - x^3 - \tfrac{1}{4}x^4\right) + \alpha_0 + \alpha_1 x - \int_0^x [u_0(t)]^3\, dt = 0,$$
$$\to \; u_1(x) = 1 - \alpha_0 + (\alpha_0{}^3 + \alpha_1)x + \left(\tfrac{3}{2}\alpha_0{}^2\alpha_1 - \tfrac{3}{2}\right)x^2 + (\alpha_0\alpha_1{}^2 - 1)x^3 + \left(\tfrac{1}{4}\alpha_1{}^3 - \tfrac{1}{4}\right)x^4, \tag{3.3}$$
$$p^{n+1}: \quad u_{n+1}(x) - \int_0^x H_n(t)\, dt = 0 \quad \to \quad u_{n+1}(x) = \int_0^x H_n(t)\, dt \quad n \geq 1.$$

To find $\alpha_m,\ m = 0,1$ in such a way that $u_1 = 0$. If $u_1 = 0$ then $u_2 = u_3 = \cdots = 0$, and the exact solution will be obtained as $y(x) = u_0(x)$. hence for all values of $x$ we have

$$\begin{cases} 1 - \alpha_0 = 0, \\ \alpha_0{}^3 + \alpha_1 = 0, \\ \dfrac{3}{2}\alpha_0{}^2\alpha_1 - \dfrac{3}{2} = 0, \\ \alpha_0\alpha_1{}^2 - 1 = 0, \\ \dfrac{1}{4}\alpha_1{}^3 - \dfrac{1}{4} = 0. \end{cases}$$

Solving the above algebraic equations, we have $\alpha_0 = \alpha_1 = 1$. Therefore the solution will be

$$y(x) = u_0(x) = \alpha_0 + \alpha_1 x = 1 + x$$

which is the same as the exact solution.

**Example 2:** [9] Consider the following nonlinear Volterra integral equation

$$y(x) = x + \int_0^x y^2(t)\, dt = 0 \tag{3.3}$$

with the exact solution $y(x) = tan(x)$ .

We apply this new modified HPM [1] and get $v_0 = 1$ and $v_1(x) = x$ . Then

$$H_\alpha(u,p) = u(x) - px + (p-1)(\alpha_0 + \alpha_1 x) - p\int_0^x [y(t)]^2\, dt = 0, \tag{3.4}$$

In view of Eq. (2.7) we have

$$u_0(x) = \alpha_0 + \alpha_1 x,$$
$$u_1(x) = -\alpha_0 - \alpha_1 x + x + \int_0^x (\alpha_0 + \alpha_1 x)^2\, dt = 0, \tag{3.5}$$
$$u_{n+1}(x) = \int_0^x \sum_{j=0}^{n} u_j\, u_{n-j} dt \qquad n \geq 1.$$

Now we find $\alpha_m,\ m = 0,1$ in such a way that $u_1 = 0$. If $u_1 = 0$ then $u_2 = u_3 = \cdots = 0$, and the exact solution will be obtained as $y(x) = u_0(x)$. hence for all values of $x$ we have

$$\begin{cases} -\alpha_0 = 0, \\ -\alpha_1 + 1 + \alpha_0{}^2 = 0, \\ \alpha_0\alpha_1 = 0, \\ \dfrac{\alpha_1{}^2}{3} = 0. \end{cases}$$

From these algebraic equations we cannot get the value of $\alpha_1$ because of made a counteraction. In fact for this type of nonlinear integral equation this method be failed.

Now, if we increase N and let $v_m(x) = x_m$ for $m = 0,1,2,\cdots$ then in view of Eq. (2.7) we have

$$\begin{aligned} &u_0(x) = \sum_{m=0}^{\infty} \alpha_m\ x^m, \\ &u_1(x) = -\sum_{m=0}^{\infty} \alpha_m\ x^m + x + \int_0^x (\sum_{m=0}^{\infty} \alpha_m\ t^m)^2\, dt\,, \end{aligned} \tag{3.6}$$

Consequently, we have series solution of the form

$$y(x) = \sum_{m=0}^{\infty} \alpha_m\ x^m = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots = \tan(x),$$

which is the same as the exact solution.

**Remark**

We note that in this modified homotopy perturbation method, when algebraic equations cannot be solved and N is taken to be infinity, we obtain the series solution method for solving integral equations.

**Theorem 1**

The new modified homotopy perturbation [1 method] for solving integral equation is the series solution method when $N \longrightarrow \infty$ and $v_m(x) = x_m$ for $m = 0,1,2,\cdots$.

**Proof.** If $N \longrightarrow \infty$ and $v_m(x) = x_m$ for $m = 0,1,2,\cdots$, then in view of Eq. (2.7) we have

$$\begin{aligned} &p^0: \qquad u_0(x) = \sum_{m=0}^{\infty} \alpha_m\ x^m, \\ &p^1: \quad u_1(x) = -\sum_{m=0}^{\infty} \alpha_m\ x^m + g(x) + \int_0^x (\sum_{m=0}^{\infty} \alpha_m\ t^m)^r\ dt \end{aligned} \tag{3.7}$$

According to the modified homotopy perturbation method, we must consider $u_1(x) = 0$ then other components of $u(x)$, $u_2 = u_3 = \cdots = 0$, and the exact solution will be obtained as $y(x) = u_0(x)$. So, if we get $u_1(x) = 0$ then from (16) we have:

$$u_1(x) = 0\ \rightarrow \sum_{m=0}^{\infty} \alpha_m\ x^m = g(x) + \int_0^x (\sum_{m=0}^{\infty} \alpha_m\ t^m)^r\ dt. \tag{3.8}$$

In view of (3.8), it is easy to see this is well known the series solution method. Hence the proof is completed.

**4 Discussion**

In this note, by an example we have shown the new modified HPM presented by Ghorbani and Saberi-Nadjafi for solving nonlinear integral equations is not useful. In fact, when exact solution of integral equation is not appearing as part or a type of given function $g(x)$ this method is failed. In this case $N$ is taken to be infinity, we obtain the series solution method for solving nonlinear integral equations. Another important result, if we select all selective function as xm then this method is equivalence the series solution method.

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