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Pricing Power Options under the Heston Dynamics using the FFT

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Abstract: Numerous studies have presented evidence that certain financial assets may exhibit stochastic volatility or jumps, which cannot be captured within the Black-Scholes environment. This work investigates the valuation of power options when the variance follows the Heston model of stochastic volatility. A closed form representation of the characteristic function of the process is derived from the partial differential equation (PDE) of the replicating portfolio. The characteristic function is essential for the computation of the European power option prices via the Fast Fourier Transform (FFT) technique. Numerical results are presented.

Keywords: Power Option, Partial Differential Equation, Heston Model, Characteristic Function, Fast Fourier Transform

1. Introduction

Since Black & Scholes (1973) introduced the Black-Scholes model for option pricing, many scholars have tried to relax the assumptions made used in accordance to the model. This is because many studies have shown that in reality, certain financial assets may exhibit stochastic volatility or jumps. The evidence of this in option pricing has become an important issue because it gives possibility to model option pricing more accurately. One of the most accepted stochastic volatility models is due to Heston (1993). Such a model relaxes the constant volatility assumption made in the Black-Scholes approach. In this work, the asset price is assumed to follow the log-normal process governed by a single Brownian motion, with the volatility process driven by a second Brownian motion process. Both the asset price process and the volatility process are correlated by a constant correlation coefficient. With the assumption that the market is complete, a replicating portfolio technique is used in obtaining a partial differential equation (PDE). Consequently, using the PDE, the characteristic function of the logarithm of the underlying asset price is derived, which enables the application of the Fast Fourier Transform (FFT) for the computation of the power option prices. The FFT method has been used increasingly since it was first introduced in option pricing by Carr & Madan (1999). It is flexible approach in that it can encapsulate properties such as stochastic volatility, and still maintain its computational efficiency (see Pillay & O'Hara, 2011). Nevertheless, comparison between the FFT approach and Monte Carlo simulation (see Boyle, 1977) is demonstrated numerically to highlight the efficiency of the FFT technique.

2. The Model

Let $(\Omega, F, \mathbb{Q})$ be a probability space on which two Brownian motions, W_t^1 and W_t^2 for $t > 0$, are given. $F_t,\ 0 \le t \le T$ is the filtration generated by the Brownian motions and suppose $\mathbb{Q}$ is a risk-neutral probability. Given the underlying asset price S_t risk-free rate r and a constant factor β, Itô's Lemma implies that S_t^β is also a geometric Brownian motion following

$$dS_t^\beta = \left(\beta r + \frac{1}{2}\beta^2\sigma^2\right) S_t^\beta dt + \beta\sigma S_t^\beta dW_{t,1} \tag{2.1}$$

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We introduce an artificial asset $Z \equiv S_t^{\beta}$. Then Equation (2.1) becomes

$$
\begin{aligned}
dZ &= \left(\beta r + \frac{1}{2}\beta^2\sigma^2 - \frac{1}{2}\beta\sigma^2\right) Zdt + \beta\sigma Z dW_{t,1}, \\
dZ &= (r - q)Zdt + \beta\sigma Z dW_{t,1},
\end{aligned} \tag{2.2}
$$

where $q = (1-\beta)\left(r + \frac{1}{2}\beta\sigma^2\right)$. From Equation (2.2), we observe the volatility is affected by a factor β. Hence, within the Heston environment, we propose the following model that governs the asset price process:

$$
\begin{aligned}
dZ &= \left(\beta r + \frac{1}{2}\beta^2\sigma^2 - \frac{1}{2}\beta\sigma^2\right) Zdt + \beta\sigma Z dW_{t,1}, \\
dv &= \kappa(\theta - Y)dt + \sigma_0\sqrt{Y}dW_{t,2},
\end{aligned} \tag{2.3}
$$

where the variance, $Y = \beta^2 v$. Thus we can represent Equation (2.3) as follows:

$$
dZ = \left(\beta r + \frac{1}{2}\beta^2 v - \frac{1}{2}\beta v\right) Zdt + \beta\sqrt{v} Z dW_{t,1}, \tag{2.4}
$$

$$
dv = \kappa(\theta - \beta^2 v)dt + \sigma_0\beta\sqrt{v}dW_{t,2}, \tag{2.5}
$$

$$
\langle dW_{t,1} dW_{t,2} \rangle = \rho dt, \tag{2.6}
$$

where v follows a square-root mean reverting process, κ is the speed of the mean reversion, θ is the average level of the volatility, and ρ is the correlation coefficient between the two Brownian motions.

3. The Heston PDE for Power Options

Following Gatheral (2006), for a risk-neutral portfolio, we need to hedge the artificial asset and the random changes in the volatility. Assuming the market is complete (Esser, 2003), we consider a portfolio $\prod$ of an option with value f, $-\Delta$ units of Z and $-\phi$ units of another option with value g, to make the net amount equal to zero, which relies on the volatility,

$$
\prod = f - \Delta Z - \phi g = 0 \tag{3.1}
$$

Employing the two-dimensional extension of Itô's Lemma yields

$$
\begin{aligned}
df = \Bigg[\frac{\partial f}{\partial t} &+ \frac{\partial f}{\partial Z}\left(\beta r + \frac{1}{2}\beta^2 v - \frac{1}{2}\beta v\right) Z + \frac{\partial f}{\partial v}\kappa(\theta - \beta^2 v) + \frac{1}{2}\frac{\partial^2 f}{\partial Z^2} + \beta^2 v Z^2 + \frac{1}{2}\frac{\partial^2 f}{\partial v^2}\sigma_0^2\beta^2 v \\
&+ \frac{\partial^2 f}{\partial Z \partial v}\rho\sigma_0\beta^2 v Z\Bigg] dt + \frac{\partial f}{\partial Z}\beta\sqrt{v} Z dW_{t,1} + \frac{\partial f}{\partial v}\sigma_0\beta\sqrt{v}dW_{t,2}
\end{aligned} \tag{3.2}
$$

This is the same for dg that is:

$$
\begin{aligned}
dg = \Bigg[\frac{\partial g}{\partial t} &+ \frac{\partial g}{\partial Z}\left(\beta r + \frac{1}{2}\beta^2 v - \frac{1}{2}\beta v\right) Z + \frac{\partial g}{\partial v}\kappa(\theta - \beta^2 v) + \frac{1}{2}\frac{\partial^2 g}{\partial Z^2} + \beta^2 v Z^2 + \frac{1}{2}\frac{\partial^2 g}{\partial v^2}\sigma_0^2\beta^2 v \\
&+ \frac{\partial^2 g}{\partial Z \partial v}\rho\sigma_0\beta^2 v Z\Bigg] dt + \frac{\partial g}{\partial Z}\beta\sqrt{v} Z dW_{t,1} + \frac{\partial g}{\partial v}\sigma_0\beta\sqrt{v}dW_{t,2}
\end{aligned} \tag{3.3}
$$

The change in the portfolio $\prod$ in time dt is given by $d\prod = df - \Delta dZ - \phi dg$. It follows that by replacing the actual parameters yields,

$$\begin{aligned} d\prod = &\left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial Z}\left(\beta r + \frac{1}{2}\beta^2 v - \frac{1}{2}\beta v\right) Z + \frac{\partial f}{\partial v}\kappa(\theta - \beta^2 v) + \frac{1}{2}\frac{\partial^2 f}{\partial Z^2}\beta^2 v Z^2 + \frac{1}{2}\frac{\partial^2 f}{\partial v^2}\sigma_0^2\beta^2 v \right.\\ &\left. + \frac{1}{2}\frac{\partial^2 f}{\partial Z \partial v}\rho\sigma_0\beta^2 v Z\right] dt + \frac{\partial f}{\partial Z}\beta\sqrt{v} Z dW_{t,1} + \frac{\partial f}{\partial v}\sigma_0\beta\sqrt{v} dW_{t,2} \\ &- \Delta\left[\left(\beta r + \frac{1}{2}\beta^2 v - \frac{1}{2}\beta v\right) Z dt + \beta\sqrt{v} Z dW_{t,1}\right] \\ &- \phi\left[\frac{\partial g}{\partial t} + \frac{\partial g}{\partial Z}\left(\beta r + \frac{1}{2}\beta^2 v - \frac{1}{2}\beta v\right) Z + \frac{\partial g}{\partial v}\kappa(\theta - \beta^2 v) + \frac{1}{2}\frac{\partial^2 g}{\partial Z^2}\beta^2 v Z^2 \right.\\ &\left. + \frac{1}{2}\frac{\partial^2 g}{\partial v^2}\sigma_0^2\beta^2 v + \frac{\partial^2 g}{\partial Z \partial v}\rho\sigma_0\beta^2 v Z\right] dt - \phi\left[\frac{\partial g}{\partial Z}\beta\sqrt{v} Z dW_{t,1} + \frac{\partial g}{\partial v}\sigma_0\beta\sqrt{v} dW_{t,2}\right] \\ &= \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial Z}\left(\beta r + \frac{1}{2}\beta^2 v - \frac{1}{2}\beta v\right) Z + \frac{\partial f}{\partial v}\kappa(\theta - \beta^2 v) + \frac{1}{2}\frac{\partial^2 f}{\partial Z^2}\beta^2 v Z^2 \right.\\ &\left. + \frac{1}{2}\frac{\partial^2 f}{\partial v^2}\sigma_0^2\beta^2 v + \frac{\partial^2 f}{\partial Z \partial v}\rho\sigma_0\beta^2 v Z - \Delta\left(\beta r + \frac{1}{2}\beta^2 v - \frac{1}{2}\beta v\right) Z\right] dt \\ &- \phi\left[\frac{\partial g}{\partial t} + \frac{\partial g}{\partial Z}\left(\beta r + \frac{1}{2}\beta^2 v - \frac{1}{2}\beta v\right) Z + \frac{\partial g}{\partial v}\kappa(\theta - \beta^2 v) + \frac{1}{2}\frac{\partial^2 g}{\partial Z^2}\beta^2 v Z^2 \right.\\ &\left. + \frac{1}{2}\frac{\partial^2 g}{\partial v^2}\sigma_0^2\beta^2 v + \frac{\partial^2 g}{\partial Z \partial v}\rho\sigma_0\beta^2 v Z\right] dt \left[\frac{\partial f}{\partial Z}\beta\sqrt{v} Z - \Delta\beta\sqrt{v} Z - \phi\frac{\partial g}{\partial Z}\beta\sqrt{v} Z\right] dW_{t,1} \\ &+ \left[\frac{\partial f}{\partial v}\sigma_0\beta\sqrt{v} - \phi\frac{\partial g}{\partial v}\sigma_0\beta\sqrt{v}\right] dW_{t,2} \end{aligned} \tag{3.4}$$

Knowing that $\prod = 0$, we have $f = \Delta Z + \phi g$. In order to cancel out the randomness terms $dW_{t,1}$ and $dW_{t,2}$, we use the following:

$$\frac{\partial f}{\partial Z} = \Delta + \phi\frac{\partial g}{\partial Z} \tag{3.5}$$

$$\frac{\partial f}{\partial v} = \phi\frac{\partial g}{\partial v}. \tag{3.6}$$

For that reason,

$$\begin{aligned} d\prod = &\left[\frac{\partial f}{\partial t} + \left(\Delta + \phi\frac{\partial g}{\partial z}\right)\left(\beta r + \frac{1}{2}\beta^2 v - \frac{1}{2}\beta v\right) Z + \phi\frac{\partial g}{\partial z}\kappa(\theta - \beta^2 v) + \frac{1}{2}\frac{\partial^2 f}{\partial Z^2}\beta^2 v Z^2 \right.\\ &\left. + \frac{1}{2}\frac{\partial^2 f}{\partial v^2}\sigma_0^2\beta^2 v + \frac{\partial^2 f}{\partial Z \partial v}\rho\sigma_0\beta^2 v Z - \Delta\left(\beta r + \frac{1}{2}\beta^2 v - \frac{1}{2}\beta v\right) Z\right] dt \\ &- \phi\left[\frac{\partial g}{\partial t} + \frac{\partial g}{\partial Z}\left(\beta r + \frac{1}{2}\beta^2 v - \frac{1}{2}\beta v\right) Z + \frac{\partial g}{\partial v}\kappa(\theta - \beta^2 v) + \frac{1}{2}\frac{\partial^2 g}{\partial Z^2}\beta^2 v Z^2 \right.\\ &\left. + \frac{1}{2}\frac{\partial^2 g}{\partial v^2}\sigma_0^2\beta^2 v + \frac{\partial^2 g}{\partial Z \partial v}\rho\sigma_0\beta^2 v Z\right] dt \\ &+ \left[(\Delta + \phi\frac{\partial g}{\partial Z})\beta\sqrt{v} Z - \Delta\beta\sqrt{v} Z - \phi\frac{\partial g}{\partial Z}\beta\sqrt{v} Z\right] dW_{t,1} \\ &+ \left[\phi\frac{\partial g}{\partial v}\sigma_0\beta\sqrt{v} - \phi\frac{\partial g}{\partial v}\sigma_0\beta\sqrt{v}\right] dW_{t,2} \\ &= \left[\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial Z^2}\beta^2 v Z^2 + \frac{1}{2}\frac{\partial^2 f}{\partial v^2}\sigma_0^2\beta^2 v + \frac{\partial^2 f}{\partial Z \partial v}\rho\sigma_0\beta^2 v Z\right] dt \\ &- \phi\left[\frac{\partial g}{\partial t} + \frac{1}{2}\frac{\partial^2 g}{\partial Z^2}\beta^2 v Z^2 + \frac{1}{2}\frac{\partial^2 g}{\partial v^2}\sigma_0^2\beta^2 v + \frac{\partial^2 g}{\partial Z \partial v}\rho\sigma_0\beta^2 v Z\right] dt. \end{aligned} \tag{3.7}$$

In order to avoid arbitrage opportunities, the portfolio should earn a risk-free rate r. Mathematically, this means

$$d\prod = r\prod dt = r(f - \Delta Z - \phi g)dt.$$

Since $\prod = 0$, then $d\prod = 0$. On that account, using the respective Equation (3.5), Equation (3.6) and Equation (3.7) renders

$$\frac{\frac{\partial f}{\partial t}+\frac{1}{2}\frac{\partial^2 f}{\partial Z^2}\beta^2 vZ^2+\frac{1}{2}\frac{\partial^2 f}{\partial v^2}\sigma_0^2\beta^2 v+\frac{\partial^2 f}{\partial Z\partial v}\rho\sigma_0\beta^2 vZ-rf+rZ\frac{\partial f}{\partial Z}}{\frac{\partial f}{\partial v}} = \frac{\frac{\partial g}{\partial t}+\frac{1}{2}\frac{\partial^2 g}{\partial Z^2}\beta^2 vZ^2+\frac{1}{2}\frac{\partial^2 g}{\partial v^2}\sigma_0^2\beta^2 v+\frac{\partial^2 g}{\partial Z\partial v}\rho\sigma_0\beta^2 vZ-rg+rZ\frac{\partial g}{\partial Z}}{\frac{\partial g}{\partial v}} \tag{3.8}$$

Following Heston (1993), both sides of Equation (3.8) are equal to some function, say h such that $h(Z,v) = -\kappa(\theta-\beta^2 v)+\lambda(Z,v)$, where $\lambda(Z,v)=\lambda\beta^2 v$ is the volatility risk premium. Thence,

$$\frac{\partial f}{\partial t}+\frac{1}{2}\frac{\partial^2 f}{\partial Z^2}\beta^2 vZ^2+\frac{1}{2}\frac{\partial^2 f}{\partial v^2}\sigma_0^2\beta^2 v+\frac{\partial^2 f}{\partial Z\partial v}\rho\sigma_0\beta^2 vZ-rf+rZ\frac{\partial f}{\partial Z}=\frac{\partial f}{\partial v}[-\kappa(\theta-\beta^2 v)+\lambda\beta^2 v]. \tag{3.9}$$

Suppose we have the risk-neutral measures be $\kappa^*=\kappa+\lambda$, and $\theta^*=\frac{\kappa\theta}{\kappa+\lambda}$. This cancels out the volatility risk premium. Consequently, the stochastic process followed by the variance is now

$$dv=\kappa^*(\theta^*-\beta^2 v)dt+\sigma_0\beta\sqrt{v}dW_{t,2}. \tag{3.10}$$

It follows that Equation (3.9) becomes

$$\frac{\partial f}{\partial t}+\frac{1}{2}\frac{\partial^2 f}{\partial Z^2}\beta^2 vZ^2+\frac{1}{2}\frac{\partial^2 f}{\partial v^2}\sigma_0^2\beta^2 v+\frac{\partial^2 f}{\partial Z\partial v}\rho\sigma_0\beta^2 vZ+\frac{\partial f}{\partial v}\kappa^*(\theta^*-\beta^2 v)+rZ\frac{\partial f}{\partial Z}-rf=0, \tag{3.11}$$

which has a similar form to the Heston PDE. Assume that $X\equiv lnZ$. Solving for its partial derivatives, and then substituting the results into Equation (3.9) returns

$$0=\frac{\partial f}{\partial t}+\left(r-\frac{1}{2}\beta^2 v\right)\frac{\partial f}{\partial X}+\frac{1}{2}\frac{\partial^2 f}{\partial X^2}\beta^2 v+\frac{1}{2}\frac{\partial^2 f}{\partial v^2}\sigma_0^2\beta^2 v+\frac{\partial^2 f}{\partial X\partial v}\rho\sigma_2\beta^2 v+\frac{\partial f}{\partial v}\kappa^*(\theta^*-\beta^2 v)-rf. \tag{3.12}$$

4. Deriving the Characteristic Function

So as to solve for the characteristic function, we conjecture that the solution for the PDE (3.12) has the following form:

$$\begin{aligned} f &= Ze^{(\beta-1)\left(r+\frac{1}{2}\beta v\right)\tau}P_1-Ke^{-r\tau}P_2 \\ &= e^{X+(\beta-1)\left(r+\frac{1}{2}\beta v\right)\tau}P_1-Ke^{-r\tau}P_2. \end{aligned} \tag{4.1}$$

It follows that by replacing the form (4.1) into the PDE (3.12) yields

$$\begin{aligned} 0 = e^{X+(\beta-1)(r+\frac{1}{2}\beta v)\tau}\Bigg\{&-(\beta-1)\left(r+\frac{1}{2}\beta v\right)P_1 \\ &+\left(\frac{1}{2}\beta^2\tau-\frac{1}{2}\beta\tau\right)\left[\rho\sigma_0\beta^2 v+\kappa^*(\theta^*-\beta^2 v)+\left(\frac{1}{2}\beta^2\tau-\frac{1}{2}\beta\tau\right)\frac{1}{2}\sigma_0^2\beta^2 v\right]P_1 \\ &+\left[r+\frac{1}{2}\beta^2 v+\rho\sigma_0\beta^2 v\left(\frac{1}{2}\beta^2\tau-\frac{1}{2}\beta\tau\right)\right]\frac{\partial P_1}{\partial X}+\frac{1}{2}\frac{\partial^2 P_1}{\partial X^2}\beta^2 v \\ &+\left[\rho\sigma_0\beta^2 v+\kappa^*(\theta^*-\beta^2 v)+\sigma_0^2\beta^2 v\left(\frac{1}{2}\beta^2\tau-\frac{1}{2}\beta\tau\right)\right]\frac{\partial P_1}{\partial v}+\frac{1}{2}\frac{\partial^2 P_1}{\partial v^2}\sigma_0^2\beta^2 v \\ &+\frac{\partial^2 P_1}{\partial X\partial v}\rho\sigma_0\beta^2 v-\frac{\partial P_1}{\partial \tau}\Bigg\} \\ &-Ke^{-r\tau}\left[\left(r-\frac{1}{2}\beta^2 v\right)\frac{\partial P_2}{\partial X}+\frac{1}{2}\frac{\partial^2 P_2}{\partial X^2}\beta^2 v+\frac{\partial P_2}{\partial v}\kappa^*(\theta^*-\beta^2 v)+\frac{1}{2}\frac{\partial^2 P_2}{\partial v^2}\sigma_0^2\beta^2 v\right. \\ &\left.+\frac{\partial^2 P_2}{\partial X\partial v}\rho\sigma_0\beta^2 v-\frac{\partial P_2}{\partial \tau}\right]. \end{aligned} \tag{4.2}$$

This can be represented as $e^{X+(\beta-1)(r+\frac{1}{2}\beta v)\tau}[f(P_1)]-Ke^{-r\tau}[f(P_2)]=0$, where one possible solution is $f(P_1)=f(P_2)=0$. Alternatively, this implies that P_1 and P_2 must satisfy the PDEs

$$0 = \left[-(\beta-1)\left(r+\frac{1}{2}\beta v\right)k_j + \frac{1}{2}\sigma_0^2\beta^2 v c_j^2 + (a - b_j\beta^2 v)\mathrm{c}_j\right]P_j + \left(r + l_j\beta^2 v + \rho\sigma_0\beta^2 v \mathrm{c}_j\right)\frac{\partial P_j}{\partial X}$$
$$+ \frac{1}{2}\frac{\partial^2 P_j}{\partial X^2}\beta^2 v + \frac{1}{2}\frac{\partial^2 P_j}{\partial v^2}\sigma_0^2\beta^2 v + \left[(a - b_j\beta^2 v) + \sigma_0^2\beta^2 v \mathrm{c}_j\right]\frac{\partial P_j}{\partial v} + \frac{\partial^2 P_j}{\partial X \partial v}\rho\sigma_0\beta^2 v \tag{4.3}$$
$$- \frac{\partial P_j}{\partial \tau},$$

for $j = 1{,}2$, where $k_1 = 1, k_2 = 0, c_1 = \frac{1}{2}\beta^2\tau - \frac{1}{2}\beta\tau, c_2 = 0,\ l_1 = \frac{1}{2},\ l_2 = -\frac{1}{2}, b_1 = \kappa^* - \rho\sigma_0, b_2 = \kappa^*,$ and $a = \kappa^*\theta^*$. We now investigate the characteristic function within the Heston framework for power options. We suggest that the characteristic function has the following form:

$$\varphi_j(X, v, t; u) = \exp\left(C_j + D_j\beta^2 v + iuX\right). \tag{4.4}$$

Accordingly, we substitute the characteristic function (4.4) into PDE (4.3),

$$0 = \left[-(\beta-1)\left(r+\frac{1}{2}\beta v\right)k_j + \frac{1}{2}\sigma_0^2\beta^2 v c_j^2 + (a - b_j\beta^2 v)c_j\right]\varphi_j + \left(r + l_j\beta^2 v + \rho\sigma_0\beta^2 v c_j\right)iu\varphi_j$$
$$+ \frac{1}{2}\left(-u^2\varphi_j\right)\beta^2 v + \frac{1}{2}\left(\beta^4 D_j^2\varphi_j\right)\sigma_0\beta^2 v + \left[(a - b_j\beta^2 v) + \sigma_0^2\beta^2 v c_j\right]B^2 D_j\varphi_j$$
$$+ \rho\sigma_0\beta^2 v\left(\beta^2 D_j iu\varphi_j\right) - \frac{\partial C_j}{\partial \tau}\varphi_j - \beta^2 v\frac{\partial D_j}{\partial \tau}\varphi_j \tag{4.5}$$
$$= \beta^2 v\left[\frac{1}{2}\sigma_0^2 c_j^2 - b_j c_j + l_j iu + \rho\sigma_0 c_j iu - \frac{1}{2}u^2 + \frac{1}{2}\sigma_0^2\beta^4 D_j^2 - b_j\beta^2 D_j + \sigma_0^2 c_j\beta^2 D_j\right.$$
$$\left. + \rho\sigma_0 iu\beta^2 D_j - \frac{\partial D_j}{\partial \tau}\right] + \left[-(\beta-1)\left(r+\frac{1}{2}\beta v\right)k_j + ac_j + riu + a\beta^2 D_j - \frac{\partial C_j}{\partial \tau}\right]$$

subject to the following boundary conditions: $C_j(0) = 0$, and $D_j(0) = 0$. This reduces to solving two ordinary differential equations (ODE),

$$\frac{\partial D_j}{\partial \tau} = \left(\frac{1}{2}\sigma_0^2 c_j^2 - b_j c_j + l_j iu + \rho\sigma_0 c_j iu - \frac{1}{2}u^2\right) + \left(\sigma_0^2 c_j\beta^2 + \rho\sigma_0 iu\beta^2 - \beta^2 b_j\right)D_j + \frac{1}{2}\sigma_0^2\beta^4 D_j^2, \tag{4.6}$$

$$\frac{\partial C_j}{\partial \tau} = -(\beta-1)\left(r+\frac{1}{2}\beta v\right)k_j + ac_j + riu + a\beta^2 D_j. \tag{4.7}$$

Equation (4.6) is nonlinear and is of the form of a Riccati equation. Any equation of the Riccati type can always be transformed to the following second order linear homogeneous ordinary differential equation (Bastami et al., 2010) using a substitution $D_j = -\frac{Y'}{YR}$:

$$Y'' - \left(Q + \frac{R'}{R}\right)Y' + PRY = 0, \tag{4.8}$$

where $P = \frac{1}{2}\sigma_0^2 c_j^2 - b_j c_j + l_j iu + \rho\sigma_0 c_j iu - \frac{1}{2}u^2, Q = \sigma_0^2 c_j\beta^2 + \rho\sigma_0 iu\beta^2 - \beta^2 b_j$ and $R = \frac{1}{2}\sigma_0^2\beta^4$. Making further substitutions, $a = 1, b = -(Q + \frac{R'}{R})$ and $c = PR$, the ODE (4.8) is now,

$$\begin{aligned} &a\frac{\partial^2 Y}{\partial \tau^2} + b\frac{\partial Y}{\partial \tau} + cY = 0, \\ &aY'' + bY' + cY = 0. \end{aligned} \tag{4.9}$$

Besides, the characteristic equation of the ODE (4.9) is $ar^2 + br + c = 0$ which is a quadratic equation with roots,

$$r_{1,2} = \frac{\beta^2\left(\sigma_0^2 c_j + \rho\sigma_0 iu - b_j\right) \pm \beta^2\sqrt{\left(\sigma_0^2 c_j + \rho\sigma_0 iu - b_j\right)^2 - \sigma_0^2\left(\sigma_0^2 c_j^2 - 2b_j c_j + 2l_j iu + 2\rho\sigma_0 c_j iu - u^2\right)}}{2}. \tag{4.10}$$

Suppose that r_1 and r_2 are distinct real numbers, then the general solution is of the form, $Y = A_1 e^{r_1\tau} + A_2 e^{r_2\tau}$, where $Y' = \frac{\partial Y}{\partial \tau} = r_1 A_1 e^{r_1\tau} + r_2 A_2 e^{r_2\tau}$. Replacing this back into $D_j = -\frac{Y'}{YR}$ yields,

$$D_j = \frac{-r_1 A_1 e^{r_1 \tau} - r_2 A_2 e^{r_2 \tau}}{R(A_1 e^{r_1 \tau} + A_2 e^{r_2 \tau})}. \tag{4.11}$$

Recall the terminal condition $D(0) = 0$. It follows that

$$-\frac{r_1}{r_2} = \frac{A_2}{A_1}.$$

Carrying on with the calculation,

$$D_j = \frac{-\left\{e^{r_1 \tau}\left[r_1 c_1 + r_2\left(-\frac{c_1 r_1}{r_2}\right)\frac{e^{r2\tau}}{e^{r_1\tau}}\right]\right\}}{R\left\{e^{r_1\tau}\left[c_1 + \left(-\frac{c_1 r_1}{r_2}\right)\frac{e^{r2\tau}}{e^{r_1\tau}}\right]\right\}} = -\frac{1}{R}\left[\frac{r_1 - r_1(e^{r_2\tau})(e^{-r_1\tau})}{1 - \frac{r_1}{r_2}(e^{r_2\tau})(e^{-r_1\tau})}\right] = -\frac{r_1}{R}\left[\frac{1 - e^{(r_2 - r_1)\tau}}{1 - \frac{r_1}{r_2} e^{(r_2 - r_1)\tau}}\right]. \tag{4.12}$$

We now define the following,

$$d = r_1 - r_2 = \beta^2 \sqrt{(\sigma_0^2 c_j + \rho\sigma_0 iu - b_j)^2 - \sigma_0^2(\sigma_0^2 c_j^2 - 2b_j c_j + 2l_j iu + 2\rho\sigma_0 c_j iu - u^2)},$$

$$g = \frac{r_1}{r_2} = \frac{\beta^2\left(\sigma_0^2 c_j + \rho\sigma_0 iu - b_j\right) + d}{\beta^2\left(\sigma_0^2 c_j + \rho\sigma_0 iu - b_j\right) - d}.$$

Thus, continuing to solve (4.12),

$$D_j = \frac{\beta^2\left(b_j - \rho\sigma_0 iu - \sigma_0^2 c_j\right) - d}{\sigma_0^2 \beta^4}\left(\frac{1 - e^{-d\tau}}{1 - g e^{-d\tau}}\right). \tag{4.13}$$

Given the solution in (4.13), we can now solve the ODE (4.7) by first integrating both sides of the ODE.

$$C_j = \left[-(\beta - 1)\left(r + \frac{1}{2}\beta v\right)k_j + a c_j + riu\right]\tau + \frac{a}{\sigma_0^2 \beta^2}\left[\left[\beta^2\left(b_j - \rho\sigma_0 iu - \sigma_0^2 c_j\right) - d\right]\tau - 2ln\left(\frac{1 - g e^{-d\tau}}{1 - g}\right)\right]. \tag{4.14}$$

We have now obtained solutions for the ODE as given by (4.6) and (4.7), which are shown in (4.13) and (4.14), respectively. Choosing $j = 2$, and replacing the solutions into Equation (4.4) results to the following,

$$\begin{aligned} \varphi\beta &= \exp(iuX) \\ &\times exp\left\{riu\tau + \frac{\kappa^*\theta^*}{\sigma_0^2\beta^2}\left[\left[\beta^2(\kappa^* - \rho\sigma_0 iu) - d\right]\tau - 2ln\left(\frac{1 - g e^{-d\tau}}{1 - g}\right)\right]\right\} \\ &\times exp\left\{\frac{v}{\sigma_0^2\beta^2}\left[\beta^2(\kappa^* - \rho\sigma_0 iu) - d\left(\frac{1 - e^{-d\tau}}{1 - g e^{-d\tau}}\right)\right]\right\}, \end{aligned} \tag{4.15}$$

where

$$d = \beta^2\sqrt{(\rho\sigma_0 iu - \kappa^*)^2 + \sigma_0^2(iu + u^2)},$$

$$g = \frac{\beta^2(\rho\sigma_0 iu - \kappa^*) + d}{\beta^2(\rho\sigma_0 iu - \kappa^*) + d}.$$

Using the result in (4.15), we can now apply the Fast Fourier Transform technique to price the power option when the volatility is stochastic.

5. Power Option Pricing using the Fast Fourier Transform

The essence behind the FFT approach is the characteristic function of the stochastic process. Provided that this is obtained analytically, we can use this approach to price the options. The characteristic function is defined as follows:

Definition 5.1: (Characteristic Function). *For a one-dimensional stochastic process* $X_t, 0 \leq t \leq T$, *the characteristic function is the Fourier transform of the probability density function* $q_T(X_T)$ *given as follows:*

$$\varphi(v) = \mathbb{E}_{\mathbb{Q}}(e^{ivX_T}) = \int_{-\infty}^{\infty} e^{ivX_T} q_T(X_T) dX_T. \tag{5.1}$$

Let K be the strike price and T the maturity of a power option with terminal asset price S_T^{β}, which is governed by the dynamics (2.1). The price of a power call option is computed as the discounted risk-neutral conditional expectation of the terminal payoff $(S_T^{\beta} - K)^+ = \max(S_T^{\beta} - K, 0)$:

$$PC(S_T) = e^{-rT} \mathbb{E}_{\mathbb{Q}} \left[\left(S_T^{\beta} - K \right)^+ | \mathcal{F}_t \right], \tag{5.2}$$

where r is a constant interest rate. We define $X_t = lnS_t$ and $k = \frac{lnK}{\beta}$. Moreover, we express the option pricing function (5.2) as a function of the log strike k instead of the terminal log asset price X_T,

$$PC_T(k) e^{-rT} \int_k^{\infty} (e^{X_T} - e^k) f_T(X_T) d(X_T), \tag{5.3}$$

where $f_T(X_T)$ is the density function of the process X_T. Following Carr & Madan (1999), for $\alpha > 0$, we define a modified power call price,

$$\hat{P}C_T(k) = e^{\alpha k} PC_T(k), \tag{5.4}$$

where the Fourier Transform (FT) of $\hat{P}C_T(k)$ is given by:

$$\mathcal{F}_{\beta}(v) = \int_{-\infty}^{\infty} e^{ivk} \hat{P}C_T(k) dk. \tag{5.5}$$

Applying the inverse FT to (5.5), then substituting (5.4) with (5.3) into (5.5), and also by the definition of the characteristic function (5.1), we obtain the price of a power call option as follows:

$$PC_T(k) = \frac{e^{-\alpha k}}{2\pi} \left[\int_{-\infty}^{\infty} e^{ivk} e^{-rT} \frac{\varphi_{\beta}[v - i(\alpha+1)]}{(\alpha + iv)(\alpha + iv + 1)} dv \right], \tag{5.6}$$

where

$$\mathcal{F}_{\beta}(v) = e^{-r\tau} \frac{\varphi_{\beta}[v - i(\alpha+1)]}{(\alpha + iv)(\alpha + iv + 1)}. \tag{5.7}$$

Thus for an efficient implementation of the FFT, a closed-form representation of the characteristic function $\varphi_{\beta}(v)$ is needed, which we have shown earlier, has the form of (4.15). Given the pricing function (5.6), we can price the power call option as follows:

$$PC_T(k_u) = \frac{e^{-\alpha k_u}}{\pi} \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(u-1)} e^{ibv_j} \mathcal{F}_{\beta}(v_j) \frac{\eta}{3} \left[3 + (-1)^j - \delta_{j-1} \right], \tag{5.8}$$

where $v_j = \eta(j-1), k_u = -b + \omega(u-1), b = \frac{N\omega}{2}, \omega\eta = \frac{2\pi}{N}$, and δ_n is the Kronecker delta function which is unity for $n = 0$ and zero otherwise. The choice of ω and η is essential because it governs this approach. A small ω gives us a range of prices across a wide range of strike prices; while a large value of η can give inaccurate prices. Moreover, the FFT is an algorithm that evaluate the summations of the following form efficiently:

$$X(k) = \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(u-1)} x(j), \quad k = 1, \dots, N \tag{5.9}$$

with $x(j) = e^{ibv_j} \mathcal{F}_{\beta}(v_j) \frac{\eta}{3} \left[3 + (-1)^j - \delta_{j-1} \right]$. Hence, the presentation of the power call price in the form (5.8) is a special case of (5.9) which enables the use of the FFT.

6. Power Option Pricing using Monte Carlo Simulation

Consider the problem of pricing a power call option of the form (5.2), as exhibited in Section 5. For application of the Monte Carlo simulation, we apply the fully truncated Euler scheme by Lord et al. (2010).

Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a probability space on which is defined two standard Wiener processes W_t^S with respect to the underlying, and W_t^v, with respect to the volatility. Let $\mathcal{F}_t, 0 \leq t \leq T$, be the filtration generated by these Brownian motion. Suppose $\mathbb{Q}$ is a risk-neutral probability under which the asset price process $S_t, 0 \leq t \leq T$ is governed by dynamics given in (2.4) and (2.5). To facilitate the discretization, we consider the log-asset price $X_t = lnS_t^{\beta}$. Applying Itô's Lemma to this function yields the following log-asset price Dynamics

$$dX_t = \beta\left(r - \frac{1}{2}v\right)dt + \beta\sqrt{v}dW_t^S. \tag{6.1}$$

Suppose we approximate the paths of the log asset price process (6.1) and the stochastic volatility process (2.5), on a discrete time grid via Euler discretization. Let $[t = t_0 < t_1 < \cdots < t_M = T]$ be a partition of the time interval $[t, T]$ into M equal segments of length Δt_i that is $t_i = \frac{iT}{M}$ for each $i = 0,1, \ldots, M$. The fully truncated Euler discretization of the log asset price process is

$$\hat{X}_i = \hat{X}_{i-1} + \beta\left(r - \frac{1}{2}\hat{v}_t^+\right)\Delta t_i + \beta\sqrt{\hat{v}_{i-1}^+ \Delta t_i} Z_S, \tag{6.2}$$

$$\hat{v}_i = \hat{v}_{i-1}^+ + \kappa(\theta - \beta^2 \hat{v}_{i-1}^+)\Delta t_i + \beta\sigma\sqrt{\hat{v}_{i-1}^+ \Delta t_i} Z_v, \tag{6.3}$$

where $\hat{v}_i^+ = \max(\hat{v}_i, 0)\,, Z_v \sim N(0,1)$ and $Z_S = \rho Z_v + \sqrt{1-\rho^2}Z$, where $Z \sim N(0,1)$. Using the Milstein scheme, the discretization of the volatility process (6.3) is:

$$\hat{v}_i = \hat{v}_{i-1}^+ + \kappa(\theta - \beta^2 \hat{v}_{i-1}^+)\Delta t_i + \beta\sigma\sqrt{\hat{v}_{i-1}^+ \Delta t_i} Z_v + \frac{1}{4}\sigma^2 \Delta t_i (Z_v^2 - 1). \tag{6.4}$$

We simulate the diffusion part of the log asset price by drawing a random sample from a normal distribution with mean 0 and standard deviation 1 for both Z_S and Z_v for each $i = 0,1, \ldots, M$, and obtain a log asset price for the maturity date of the option, $\hat{X}_M = \hat{X}_T$. By repeating this procedure, many paths can be generated. The price of a power call option (5.2) can be estimated by Monte Carlo simulation using

$$PC\big(t, \hat{X}_T\big) = \frac{e^{-r(T-t)}}{n}\sum_{i=1}^{n} \max\,(e^{\hat{X}_T^i} - K, 0), \tag{6.5}$$

where n is the number of sample paths used in simulation and $\hat{X}_T$ denotes the simulated value of X_t over each sample path using M time steps. This Monte Carlo estimator converges to the correct price $PC(t, S_T)$ as the number of time steps M and the number of samples n become large.

7. Numerical Results

In this section, we present a numerical comparison between the Fast Fourier Transform (FFT) approach and the Monte Carlo simulation technique. We apply the two approaches for the pricing of a power call option with stochastic volatility with a view to comparing the performance of the two techniques.[1]

We employed the FFT scheme with $N = 2^{10}, \delta = 1.25$, and α between [0.28, 0.32] to minimize the relative error between the results obtained from both techniques. Linear interpolation is applied to obtain a single option price corresponding to the respective strike price. For the Monte Carlo simulation, we employed the Milstein scheme because this produce better result than the Euler discretization. We take N = 500, 000 sample paths, and partition the time interval $[0, T]$ into $m = 200$ equal segments. Since we are only considering the comparison of the accuracy and efficiency of the models, we do not calibrate the model parameters, and rather we use the following hypothetical parameters of $S = 2, r = 0.08, T = 1, \kappa = 2, \theta = 0.04, \sigma_0 = 0.1, \rho = -0.5\ and\ v_0 = 0.05$.

[1] The codes were written in MATLAB, and the computations were conducted on an Intel Core 2 Duo processor P8400 @2.26 GHz machine running under Windows Vista Service Pack 2 with 2 GB RAM.

Table 1 compares the pricing accuracy between the two techniques across a range of strike prices, as well as the relative error (in percentage) between the two prices. Using the Monte Carlo simulation as the benchmark, it demonstrates the efficiency of the FFT technique over the Monte Carlo simulation technique.

Strike, K	FFT	Computation Time (seconds)	Monte Carlo	Computation Time (seconds)	% Difference
0.5	3.9480	0.003092	3.9346	26.196885	0.340568292
1.0	3.4626	0.003568	3.4742	27.867434	0.333889816
1.5	3.0064	0.003087	3.0130	28.040573	0.219050780
2.0	2.5508	0.003033	2.5526	24.529332	0.070516336
2.5	2.1001	0.003479	2.0904	30.537946	0.464026024
3.0	1.6456	0.003223	1.6427	24.736727	0.176538625
3.5	1.2163	0.003309	1.2196	24.399634	0.270580518
4.0	0.8450	0.003407	0.8476	24.733935	0.306748466

Table1: Comparison of prices for the power call option with stochastic volatility computed by FFT and Monte Carlo simulation

8. Conclusion

In this paper, we provide a valuation of power options under the Heston dynamics using the fast Fourier transform (FFT) technique. We present an analytical form of the characteristic function which is derived from the partial differential equation (PDE) of the replicating portfolio. The numerical results show that the FFT technique is more efficient than the Monte Carlo simulation.

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