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THE GROUP OF UNITS OF GROUP ALGEBRAS OF ABELIAN GROUPS OF ORDER 36 AND $C_3 \times A_4$ OVER ANY FINITE FIELD

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Abstract. Let $\mathcal{FG}$ be the group algebra of the group $\mathcal{G}$ over the field $\mathcal{F}$ having characteristic $p > 0$ and $q = p^n$ elements and $U(\mathcal{FG})$ be the unit group of $\mathcal{FG}$. In this paper, we are proceeding to determine the structure of unit group of group algebra of all four non isomorphic abelian groups and one non abelian group $C_3 \times A_4$ of order 36, for any prime $p > 0$.

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1. Introduction

The study of a group of units is one of the classical topics in group rings because of their topological applications and then again, after the description of simple groups as special finite p-groups. It started with the papers of Higman [7,8] and later the study of the unit group of modular group algebras was reported in the papers of S. A. Jennings [9]. In general, group of units are involved in the study of homological algebra and algebraic number theory. Recently, we have found their applications in algebraic coding theory. Therefore, the study of group of units provide a topic where many branches of algebra have a rich interplay. For arbitrary primes p, A. Bovdi and Szakacs [3] provided a technique for finding the generators for the Sylow-p subgroup of the unitary units of F_pG where G is an abelian group. Later, this technique was used to find a generating set of the unitary units and hence to generate codes.

2. Preliminaries

Let $\mathcal{FG}$ be the group algebra of group $\mathcal{G}$ over field $\mathcal{F}$ and $U(\mathcal{FG})$ be the multiplicative group of all invertible elements of the group algebra $\mathcal{FG}$. The ring homomorphism $\epsilon : \mathcal{FG} \to \mathcal{F}$ is defined by $\epsilon(\sum_{g\in\mathcal{G}} r_g\, g) = \sum_{g\in\mathcal{G}} r_g$ is known as augmentation mapping of $\mathcal{FG}$ and its kernel is called augmentation ideal, denoted by $\omega(\mathcal{G})$. The Annihilator of w is defined as $Ann(w) = \{\alpha \in \mathcal{FG}|\ \alpha w = w\alpha = 0\}$. Let

$V(\mathcal{FG})$ be the normalized unit group and $\mathcal{J}(\mathcal{FG})$ be the Jacobson radical of $\mathcal{FG}$ then, $V = 1 + \mathcal{J}(\mathcal{FG})$. For other basic details see [13]. In recent years, we have seen a lot of papers that characterize the structure of unit groups of group algebras and can be easily found in [2,5,6,12,14,17,18,19,20].

Most recently, Sahai and Ansari, in [1,16] completely characterized the unit groups of group algebras of group of order 16 and 20. In this paper, we will determine the structure of unit groups of group algebras of all four non-isomorphic abelian group $C_{36}, C_6^2, C_2 \times C_{18}, C_3 \times C_{12}$ and one non abelian group $C_3 \times A_4$ of order 36. Our notations are same as in [2,16].

Following are the useful results that we use repeatedly in our proof.

Lemma 2.1. [13] *Let $\mathcal{G}$ be a group and $\mathcal{R}$ be a commutative ring. Then the set of all finite class sums forms an $\mathcal{R}$-basis of $\mathcal{Z}(\mathcal{RG})$, the center of $\mathcal{RG}$.*

Lemma 2.2. [13] *Let $\mathcal{FG}$ be a semisimple group algebra. If $\mathcal{G}'$ denotes the commutator subgroup of $\mathcal{G}$, then*

$$\mathcal{FG} = \mathcal{FG}_{e_{\mathcal{G}'}} \oplus \Delta(\mathcal{G}, \mathcal{G}'),$$

where $\mathcal{FG}_{e_{\mathcal{G}'}} \cong \mathcal{F}(\mathcal{G}/\mathcal{G}')$ is the sum of all commutative simple components of $\mathcal{FG}$ and $\Delta(\mathcal{G}, \mathcal{G}')$ is the sum of all the others.

Lemma 2.3. [10, Lemma 1.17] *Let $\mathcal{G}$ be a locally finite p-group and let $\mathcal{F}$ be a field of characteristic p. Then $\mathcal{J}(\mathcal{FG}) = \omega(\mathcal{FG})$.*

Theorem 2.4. [11] *Let N be a normal subgroup of $\mathcal{G}$ such that $\mathcal{G}/N$ is p-solvable. If $|\mathcal{G}/N| = np^a$ where $(n,p) = 1$, then*

$$\mathcal{J}(\mathcal{FG})^{p^a} \subseteq \mathcal{FG}.\mathcal{J}(\mathcal{F}N) \subseteq \mathcal{J}(\mathcal{FG}),$$

where $\mathcal{F}$ is a field of characteristic $p > 0$. In particular, if $\mathcal{G}$ is p-solvable of order np^a where $(n,p) = 1$, then $\mathcal{J}(\mathcal{FG})^{p^a} = 0$.

3. Main results

This section provides the structure of unit groups of group algebras of all four non isomorphic abelian groups and one non abelian group $C_3 \times A_4$ over the finite field $\mathcal{F}$ of positive characteristics $p > 0$. Theorem 3.1 contains the structure of $U(\mathcal{F}C_{36})$ for both the semisimple (when $p > 3$) and non-semisimple (when $p = 2$ and 3) cases. Again, Theorem 3.2, Theorem 3.3 and Theorem 3.4 provides the structure of $U(\mathcal{F}C_6^2)$, $U(\mathcal{F}[C_2 \times C_{18}])$ and $U(\mathcal{F}[C_3 \times C_{12}])$, respectively. Theorem 3.5 gives the structure of $U(\mathcal{F}[C_3 \times A_4])$ for both semisimple and non-semisimple cases. We have characterized the unit group structure by finding the cyclotomic $\mathcal{F}$-classes $S_{\mathcal{F}}(\gamma_g)$ of $g \in \mathcal{G}$ in semisimple cases ($p \nmid o(\mathcal{G})$) and for non-semisimple cases ($p \big| o(\mathcal{G})$), we have determined the Jacobson radical $\mathcal{J}(\mathcal{FG})$ for the structure of normalised unit group $V(\mathcal{FG})$, which gives the structure of $U(\mathcal{FG})$ by using the result $U(\mathcal{FG}) \cong V(\mathcal{FG}) \times \mathcal{F}^*$.

Theorem 3.1. *Let $\mathcal{G} \cong C_{36}$ and $\mathcal{F}$ be a finite field of characteristic $p > 0$ having $q = p^n$ elements.*

(1) *If $p = 2$, then*

$$U(\mathcal{FG}) \cong \begin{cases} C_2^{9n} \times C_4^{9n} \times C_{2^n-1}^9, & if q \equiv 1 \pmod 9; \\ C_2^{9n} \times C_4^{9n} \times C_{2^n-1} \times C_{2^{2n}-1}^4, & if q \equiv -1 \pmod 9; \\ C_2^{9n} \times C_4^{9n} \times C_{2^n-1} \times C_{2^{2n}-1} \times C_{2^{6n}-1}, & if q \equiv 2, -4 \pmod 9; \\ C_2^{9n} \times C_4^{9n} \times C_{2^n-1}^3 \times C_{2^{3n}-1}^2, & if q \equiv -2, 4 \pmod 9. \end{cases}$$

(2) *If $p = 3$, then*

$$U(\mathcal{FG}) \cong \begin{cases} C_3^{16n} \times C_9^{8n} \times C_{3^n-1}^4, & if q \equiv 1 \pmod 4; \\ C_3^{16n} \times C_9^{8n} \times C_{3^n-1}^2 \times C_{3^{2n}-1}, & if q \equiv -1 \pmod 4. \end{cases}$$

(3) *If $p > 3$, then*

$$U(\mathcal{FG}) \cong \begin{cases} C_{p^n-1}^{36}, & if q \equiv 1 \pmod{36}; \\ C_{p^n-1}^2 \times C_{p^{2n}-1}^{17}, & if q \equiv -1 \pmod{36}; \\ C_{p^n-1}^4 \times C_{p^{2n}-1}^4 \times C_{p^{6n}-1}^4, & if q \equiv 5, -7 \pmod{36}; \\ C_{p^n-1}^6 \times C_{p^{2n}-1}^3 \times C_{p^{3n}-1}^4 \times C_{p^{6n}-1}^2, & if q \equiv -5, 7 \pmod{36}; \\ C_{p^n-1}^{12} \times C_{p^{3n}-1}^8, & if q \equiv -11, 13 \pmod{36}; \\ C_{p^n-1}^2 \times C_{p^{2n}-1}^5 \times C_{p^{6n}-1}^4, & if q \equiv 11, -13 \pmod{36}; \\ C_{p^n-1}^4 \times C_{p^{2n}-1}^{16}, & if q \equiv 17 \pmod{36}; \\ C_{p^n-1}^{18} \times C_{p^{2n}-1}^9, & if q \equiv -17 \pmod{36}. \end{cases}$$

Proof. Let $C_{36} =< x \,|\, x^{36} = 1 >$.

(1) Let $Char\mathcal{F} = 2$ with $|\mathcal{F}| = 2^n$ elements. Since $C_4 =< x : x^4 = 1 >$ is a normal subgroup of $\mathcal{G}$, therefore $\mathcal{G}/C_4 \cong C_9$ and $|\mathcal{G} : C_4| \neq 0$ in $\mathcal{F}$. As C_4 is a 2-group, therefore by Lemma 2.3 and [15, Theorem 7.2.7] , $\mathcal{J}(\mathcal{F}C_{36}) = \omega(C_4)$. So, $\mathcal{F}C_{36}/\mathcal{J}(\mathcal{F}C_{36}) \cong \mathcal{F}C_9$. Hence, from the ring epimorphism $\mathcal{F}C_{36} \to \mathcal{F}C_9$, we have a group epimorphism $\theta : U(\mathcal{F}C_{36}) \to U(\mathcal{F}C_9)$ with $Ker\theta = H = 1 + \omega(C_4)$. Also, we have a group homomorphism

$\phi : U(\mathcal{F}C_9) \to U(\mathcal{F}C_{36})$. It can be easily seen that $\theta o \phi = 1_{U(FC_9)}$. Thus, $U(FC_{36})$ is a split extension of $U(\mathcal{F}C_9)$ by H and hence,

$$U(\mathcal{F}C_{36}) \cong H \times U(\mathcal{F}C_9).$$

Now,

$\omega(C_4) = \{\alpha = \sum_{i=0}^{35} a_i x^i | \sum_{i=0}^{3} a_{9i+j} = 0, j = 0,1,2,3,4,5,6,7,8;\ a_i \in \mathcal{F}\}$ and $\alpha^4 = 0$ for any $\alpha \in \omega(C_4)$. It is clear that $dim_{\mathcal{F}} \mathcal{J}(\mathcal{F}C_{36}) = 27$ and $exp(H) = 4$. Thus, $|H| = 2^{27n}$ and $H \cong C_2^{k_1} \times C_4^{k_2}$. So, $2^{k_1} \times 4^{k_2} = 2^{27n}$. Next, we will calculate k_1 and k_2. Set

$$S = \{\alpha \in \omega(C_4) : \alpha^2 = 0 \text{ and there exists}\, \beta \in \omega(C_4) \text{ such that}\, \alpha = \beta^2\}.$$

By direct computation, we have

$S = \{ \sum_{i=0}^{8} a_{2i}(x^{2i} + x^{18+2i}),\ a_{2i} \in \mathcal{F}\}$. Therefore, $|S| = 2^{9n}$ which implies that $k_2 = 9n$ and then $k_1 = 9n$. Hence $H \cong C_2^{9n} \times C_4^{9n}$ and

$$U(\mathcal{F}C_{36}) \cong C_2^{9n} \times C_4^{9n} \times U(\mathcal{F}C_9)$$

and for the structure of $U(\mathcal{F}C_9)$ see, [16].

(2) Let $Char(\mathcal{F}) = 3$ with $|\mathcal{F}| = 3^n$ elements. Let $C_9 =< x : x^9 = 1 >$ be a normal subgroup of $\mathcal{G}$ such that $\mathcal{G}/C_9 \cong C_4$ and $|\mathcal{G} : C_9| \neq 0$ in $\mathcal{F}$. Since C_9 is a 3-group, therefore by Lemma 2.3 and [15, Theorem 7.2.7], $\mathcal{J}(\mathcal{F}C_{36}) = \omega(C_9)$. So, $\mathcal{F}C_{36}/\mathcal{J}(\mathcal{F}C_{36}) \cong \mathcal{F}C_4$ and hence

$$U(\mathcal{F}C_{36}) \cong H \times U(\mathcal{F}C_4),$$

where $H = 1 + \omega(C_9)$. Also, we have

$$\omega(C_9) = \{\alpha = \sum_{i=0}^{35} a_i x^i | \sum_{i=0}^{8} a_{4i+j} = 0, j = 0,1,2,3; a_i \in \mathcal{F}\}$$

and $\alpha^9 = 0$ for any $\alpha \in \omega(C_9)$. It is clear that $dim_{\mathcal{F}} \mathcal{J}(\mathcal{F}C_{36}) = 32$ and $exp(H) = 9$. Thus, $|H| = 3^{32n}$, $H \cong C_3^{k_1} \times C_9^{k_2}$ and $3^{k_1} \times 9^{k_2} = 3^{32n}$. Set

$$S = \{\alpha \in \omega(C_9) : \alpha^3 = 0 \text{ and there exists}\, \beta \in \omega(C_9) \text{ such that}\, \alpha = \beta^3\}.$$

By direct computation, we have

$$S = \{ \sum_{i=0}^{3} a_{3i}(x^{3i} + x^{24+3i}) + a_{12}(x^{12} + x^{24}) + a_{15}(x^{15} + x^{27}) + a_{18}(x^{18} + x^{30}) + a_{21}(x^{21} + x^{33});\ a_{3i} \in \mathcal{F}\}.$$

Therefore, $|S| = 3^{8n}$ which implies that $k_2 = 8n$ and then $k_1 = 16n$. Hence, $H \cong C_3^{16n} \times C_9^{8n}$ and

$$U(\mathcal{F}C_{36}) \cong C_3^{16n} \times C_9^{8n} \times U(\mathcal{F}C_4)$$

and the structure of $U(\mathcal{F}C_4)$ follows from [16].

(3) Since $p > 3$, therefore by Maschke's theorem $\mathcal{F}\mathcal{G}$ is semisimple as p does not divide $|\mathcal{G}|$. It is clear that all elements are p-regular and $m = 36$. Now we have the cases:

(a) If $q \equiv 1 \pmod{36}$, then $T = \{1\} \pmod{36}$. Thus, $|S_{\mathcal{F}}(\gamma_g)| = 1$ for every $g \in \mathcal{G}$. Thus by [4, Proposition 1.2 and Theorem 1.3],
$$\mathcal{F}C_{36} \cong \mathcal{F}^{36}.$$

(b) If $q \equiv -1 \pmod{36}$, then $T = \{1, -1\} \pmod{36}$. Thus, $|S_{\mathcal{F}}(\gamma_g)| = 1$ for $g = 1, x^{18}$; $|S_{\mathcal{F}}(\gamma_g)| = 2$ for $g = x^i; 1 \leq i \leq 17$. Thus by [4, Proposition 1.2 and Theorem 1.3],
$$\mathcal{F}C_{36} \cong \mathcal{F}^2 \oplus \mathcal{F}_2^{17}.$$

(c) If $q \equiv 5, -7 \pmod{36}$, then $T = \{1, 5, 13, 17, 25, 29\} \pmod{36}$. Thus, $|S_{\mathcal{F}}(\gamma_g)| = 1$ for $g = 1, x^{\pm 9}, x^{18}$; $|S_{\mathcal{F}}(\gamma_g)| = 2$ for $g = x^{\pm 3}, x^6, x^{12}$ and $|S_{\mathcal{F}}(\gamma_g)| = 6$ for $g = x^{\pm 1}, x^2, x^4$. Thus by [4, Proposition 1.2 and Theorem 1.3],
$$\mathcal{F}C_{36} \cong \mathcal{F}^4 \oplus \mathcal{F}_2^4 \oplus \mathcal{F}_6^4.$$

(d) If $q \equiv 7, -5 \pmod{36}$, then $T = \{1, 7, 13, 19, 25, 31\} \pmod{36}$. Thus, $|S_{\mathcal{F}}(\gamma_g)| = 1$ for $g = 1, x^{\pm 6}, x^{\pm 12}, x^{18}$; $|S_{\mathcal{F}}(\gamma_g)| = 2$ for $g = x^{\pm 3}, x^9$; $|S_{\mathcal{F}}(\gamma_g)| = 3$ for $g = x^{\pm 8}, x^{\pm 10}$ and $|S_{\mathcal{F}}(\gamma_g)| = 6$ for $g = x^{\pm 1}$. Hence,
$$\mathcal{F}C_{36} \cong \mathcal{F}^6 \oplus \mathcal{F}_2^3 \oplus \mathcal{F}_3^4 \oplus \mathcal{F}_6^2.$$

(e) If $q \equiv -11, 13 \pmod{36}$, then $T = \{1, 13, 25\} \pmod{36}$. Thus, $|S_{\mathcal{F}}(\gamma_g)| = 1$ for $g = 1, x^{\pm 3}, x^{\pm 6}, x^{\pm 9}, x^{\pm 12}, x^{\pm 15}, x^{18}$; $|S_{\mathcal{F}}(\gamma_g)| = 3$ for $g = x^{\pm 1}, x^{\pm 2}, x^{\pm 4}, x^{\pm 5}$. Hence,
$$\mathcal{F}C_{36} \cong \mathcal{F}^{12} \oplus \mathcal{F}_3^8.$$

(f) If $q \equiv 11, -13 \pmod{36}$, then $T = \{1, 11, 13, 23, 25, 35\} \pmod{36}$. Thus, $|S_{\mathcal{F}}(\gamma_g)| = 1$ for $g = 1, x^{18}$; $|S_{\mathcal{F}}(\gamma_g)| = 2$ for $g = x^3, x^6, x^9, x^{12}, x^{15}$; $|S_{\mathcal{F}}(\gamma_g)| = 6$ for $g = x, x^2, x^4, x^5$. Hence,
$$\mathcal{F}C_{36} \cong \mathcal{F}^2 \oplus \mathcal{F}_2^5 \oplus \mathcal{F}_6^4.$$

(g) If $q \equiv 17 \pmod{36}$, then $T = \{1, 17\} \pmod{36}$. Thus, $|S_{\mathcal{F}}(\gamma_g)| = 1$ for $g = 1, x^{\pm 9}, x^{18}$; $|S_{\mathcal{F}}(\gamma_g)| = 2$ for $g = x^{\pm 1}, x^{\pm 3}, x^{\pm 5}, x^{\pm 7}$, $x^2, x^4, x^6, x^8, x^{10}, x^{12}, x^{14}, x^{16}$. Hence,
$$\mathcal{F}C_{36} \cong \mathcal{F}^4 \oplus \mathcal{F}_2^{16}.$$

(h) If $q \equiv -17 \pmod{36}$, then $T = \{1, 19\} \pmod{36}$. Thus, $|S_{\mathcal{F}}(\gamma_g)| = 1$ for $g = 1, x^{\pm 2}, x^{\pm 4}, x^{\pm 6}, x^{\pm 8}, x^{\pm 10}, x^{\pm 12}, x^{\pm 14}, x^{\pm 16}, x^{18}$, $|S_{\mathcal{F}}(\gamma_g)| = 2$ for $g = x^{\pm 1}, x^{\pm 3}, x^{\pm 5}, x^{\pm 7}, x^9$. Hence,

$$\mathcal{F}C_{36} \cong \mathcal{F}^{18} \oplus \mathcal{F}_2^9.$$

□

Theorem 3.2. *Let $\mathcal{G} \cong C_6^2$ and $\mathcal{F}$ be a finite field of characteristic $p > 0$ having $q = p^n$ elements.*

(1) *If $p = 2$, then*

$$U(\mathcal{F}\mathcal{G}) \cong \begin{cases} C_2^{27n} \times C_{2^n-1}^9, & if q \equiv 1 \pmod{3}; \\ C_2^{27n} \times C_{2^n-1} \times C_{2^{2n}-1}^4, & if q \equiv -1 \pmod{3}. \end{cases}$$

(2) *If $p = 3$, then $U(\mathcal{F}\mathcal{G}) \cong C_3^{32n} \times C_{3^n-1}^4$.*

(3) *If $p > 3$, then*

$$U(\mathcal{F}\mathcal{G}) \cong \begin{cases} C_{p^n-1}^{36}, & if q \equiv 1 \pmod{6}; \\ C_{p^n-1}^4 \times C_{p^{2n}-1}^{16}, & if q \equiv -1 \pmod{6}. \end{cases}$$

Proof. Let $C_6^2 = < x, y \,|\, x^6 = y^6 = 1, xy = yx >$.

(1) Let $Char(\mathcal{F}) = 2$ with $|\mathcal{F}| = 2^n$ elements and $\mathcal{G} \cong C_6^2 \cong C_3 \times C_3 \times K_4$. Let K_4 be a normal subgroup of $\mathcal{G}$ such that $\mathcal{G}/K_4 \cong C_3 \times C_3$ and $|\mathcal{G} : K_4| \neq 0 \in \mathcal{F}$. Since K_4 is 2-group, therefore by Lemma 2.3 and [15, Theorem 7.2.7], $\mathcal{J}(\mathcal{F}C_6^2) = \omega(K_4)$. So, $FC_6^2/\mathcal{J}(\mathcal{F}C_6^2) \cong \mathcal{F}C_3^2$ and hence,

$$U(\mathcal{F}C_6^2) \cong H \times U(\mathcal{F}C_3^2)$$

where $H = 1 + \omega(K_4)$. It is clear that $\alpha^2 = 0$ for any $\alpha \in \omega(K_4)$ and $dim_{\mathcal{F}}\mathcal{J}(\mathcal{F}G) = 27$. So, exponent of H is 2. Thus, $|H| = 2^{27n}$ and $H \cong C_2^{27n}$. Hence,

$$U(\mathcal{F}\mathcal{G}) \cong C_2^{27n} \times U(FC_3^2)$$

and the structure of $U(\mathcal{F}C_3^2)$ is given in [16, Theorem 3.7].

(2) Let $Char(\mathcal{F}) = 3$ with $|\mathcal{F}| = 3^n$ elements and $\mathcal{G} \cong C_6 \times C_6 \cong C_3 \times C_3 \times K_4$. Let C_3^2 be a normal subgroup of $\mathcal{G}$ such that $\mathcal{G}/C_3^2 \cong C_2 \times C_2$ and $|\mathcal{G} : C_3^2| \neq 0 \in \mathcal{F}$. Since C_3^2 is 3-group, therefore by Lemma 2.3 and [15, Theorem 7.2.7], $\mathcal{J}(\mathcal{F}C_6^2) = \omega(C_3^2)$. So, $\mathcal{F}C_6^2/\mathcal{J}(\mathcal{F}C_6^2) \cong \mathcal{F}C_2^2$ and hence,

$$U(\mathcal{F}C_6^2) \cong H \times U(\mathcal{F}C_2^2)$$

where $H = 1+\omega(C_3^2)$. It is clear that $\alpha^3 = 0$ for any $\alpha \in \omega(C_3^2)$ and $dim_{\mathcal{F}}\mathcal{J}(\mathcal{FG}) = 32$, so the exponent of H is 3. Thus, $|H| = 3^{32n}$ and $H \cong C_3^{32n}$. Hence,

$$U(\mathcal{FG}) \cong C_3^{32n} \times U(\mathcal{F}C_2^2)$$

and the structure of $U(\mathcal{F}C_2^2)$ is given in [16, Theorem 3.2].

(3) Since $p > 3$, then $\mathcal{FG}$ is semisimple as p does not divide $|\mathcal{G}|$. It is clear all elements are p-regular and $m = 6$. Now we have the cases:

(a) If $q \equiv 1 \pmod 6$, then $T = \{1\} \pmod 6$. Thus, $|S_{\mathcal{F}}(\gamma_g)| = 1$ for all $g \in \mathcal{G}$. Thus by [4, Proposition 1.2 and Theorem 1.3],

$$\mathcal{F}(C_6 \times C_6) \cong \mathcal{F}^{36}.$$

(b) If $q \equiv -1 \pmod 6$, then $T = \{1, -1\} \pmod 6$. Thus, $|S_{\mathcal{F}}(\gamma_g)| = 1$ for all $g = 1, x^3, y^3, (xy)^3$ and $|S_{\mathcal{F}}(\gamma_g)| = 2$ for all $g = x, x^2, y, y^2, xy^{\pm 1}, x^2y^{\pm 2}$ $, xy^{\pm 2}, xy^3, x^2y^{\pm 1}, x^2y^3, x^3y, x^3y^2$. Hence,

$$\mathcal{F}(C_6 \times C_6) \cong \mathcal{F}^4 \oplus \mathcal{F}_2^{16}.$$

□

Theorem 3.3. *Let $\mathcal{G} \cong C_2 \times C_{18}$ and $\mathcal{F}$ be a finite field of characteristic $p > 0$ having $q = p^n$ elements.*

(1) *If $p = 2$, then*

$$U(\mathcal{FG}) \cong \begin{cases} C_2^{27n} \times C_{2^n-1}^9, & if q \equiv 1 \pmod 9; \\ C_2^{27n} \times C_{2^n-1} \times C_{2^{2n}-1}^4, & if q \equiv -1 \pmod 9; \\ C_2^{27n} \times C_{2^n-1} \times C_{2^{2n}-1} \times C_{2^{6n}-1}, & if q \equiv 2, -4 \pmod 9; \\ C_2^{27n} \times C_{2^n-1}^3 \times C_{2^{3n}-1}^2, & if q \equiv -2, 4 \pmod 9. \end{cases}$$

(2) *If $p = 3$, then*

$$U(\mathcal{FG}) \cong C_3^{16n} \times C_9^{8n} \times C_{3^n-1}^4.$$

(3) *If $p > 3$, then*

$$U(\mathcal{FG}) \cong \begin{cases} C_{p^n-1}^{36}, & if q \equiv 1 \pmod{18}; \\ C_{p^n-1}^4 \times C_{p^{2n}-1}^{16}, & if q \equiv -1 \pmod{18}; \\ C_{p^n-1}^4 \times C_{p^{2n}-1}^4 \times C_{p^{6n}-1}^4, & if q \equiv 5, -7 \pmod{18}; \\ C_{p^n-1}^{12} \times C_{p^{3n}-1}^8, & if q \equiv -5, 7 \pmod{18}. \end{cases}$$

Proof. Let $C_2 \times C_{18} =< x, y \,|\, x^2 = y^{18} = 1, xy = yx >$.

(1) Let $Char(\mathcal{F}) = 2$ with $|\mathcal{F}| = 2^n$ elements and $\mathcal{G} \cong C_2 \times C_{18} \cong C_9 \times K_4$. Let K_4 be a normal subgroup of $\mathcal{G}$ such that $\mathcal{G}/K_4 \cong C_9$ and $|\mathcal{G} : K_4| \neq 0 \in \mathcal{F}$. Since K_4 is 2-group, therefore by Lemma 2.3 and [15, Theorem 7.2.7], $\mathcal{J}(\mathcal{FG}) = \omega(K_4)$. So, $\mathcal{FG}/\mathcal{J}(\mathcal{FG}) \cong \mathcal{F}C_9$ and hence,

$$U(\mathcal{FG}) \cong H \times U(\mathcal{F}C_9)$$

where $H = 1 + \omega(K_4)$. It is clear that $\alpha^2 = 0$ for any $\alpha \in \omega(K_4)$ and $dim_{\mathcal{F}}\mathcal{J}(\mathcal{FG}) = 27$; so, exponent of H is 2. Thus, $|H| = 2^{27n}$ and $H \cong C_2^{27n}$. Hence,

$$U(\mathcal{FG}) \cong C_2^{27n} \times U(\mathcal{F}C_9),$$

and the structure of $U(\mathcal{F}C_9)$ is given in [16, Theorem 3.6].

(2) Let $Char(\mathcal{F}) = 3$ with $|\mathcal{F}| = 3^n$ elements. Let C_9 be a normal subgroup of $\mathcal{G}$ such that $\mathcal{G}/C_9 \cong C_2^2$ and $|\mathcal{G} : C_9| \neq 0 \in \mathcal{F}$. Since C_9 is 3-group, therefore by Lemma 2.3 and [15, Theorem 7.2.7], $\mathcal{J}(\mathcal{FG}) = \omega(C_9)$. So, $\mathcal{FG}/\mathcal{J}(\mathcal{FG}) \cong \mathcal{F}C_2^2$ and hence,

$$U(\mathcal{F}(C_2 \times C_{18})) \cong H \times U(\mathcal{F}C_2^2)$$

where $H = 1 + \omega(C_9)$. Now,

$$\omega(C_9) = \Big\{\alpha = \sum_{j=0}^{1}\sum_{i=0}^{17} a_{18j+i}x^j y^i \,|\, \sum_{i=0}^{8} a_{4i+j} = 0, j = 0, 1, 2, 3;\ a_i \in \mathcal{F}\Big\}$$

and $\alpha^9 = 0$ for any $\alpha \in \omega(C_9)$. It is clear that $dim_{\mathcal{F}}\mathcal{J}(\mathcal{FG}) = 32$ and $exp(H) = 9$. Thus $|H| = 3^{32n}$ and $H \cong C_3^{k_1} \times C_9^{k_2}$; so, $3^{k_1} \times 9^{k_2} = 3^{32n}$. Set

$$S = \big\{\alpha \in \omega(C_9) : \alpha^3 = 0 \text{ and there exists}\, \beta \in \omega(C_9) \text{ such that}\, \alpha = \beta^3\big\}.$$

By direct computation, we have

$$S = \Bigg\{ \sum_{i=0}^{3} a_{3i}(y^{3i} + xy^{3i+6}) + a_{12}(y^{12} + xy^6) + a_{15}(y^{15} + xy^9) + a_{18}(x + xy^{12}) + a_{21}(xy^3 + xy^{15}), \text{ for all}\, a_i \in \mathcal{F}\Bigg\}.$$

Therefore, $|S| = 3^{8n}$ which implies that $k_2 = 8n$ and then $k_1 = 16n$. Hence, $H \cong C_3^{16n} \times C_9^{8n}$ and

$$U(\mathcal{F}(C_2 \times C_{18})) \cong C_3^{16n} \times C_9^{8n} \times U(\mathcal{F}C_2^2),$$

and the structure of $U(\mathcal{F}C_2^2)$ is given in [16, Theorem 3.2].

(3) Let $p > 3$, then $\mathcal{FG}$ is semisimple as p does not divide $|\mathcal{G}|$. It is clear that all elements are p-regular and $m = 18$. Now we have the cases:

(a) If $q \equiv 1 \pmod{18}$, then $T = \{1\} \pmod{18}$. Thus, $|S_{\mathcal{F}}(\gamma_g)| = 1$ for all $g \in \mathcal{G}$. Thus by [4, Proposition 1.2 and Theorem 1.3],
$$\mathcal{F}(C_2 \times C_{18}) \cong \mathcal{F}^{36}.$$
(b) If $q \equiv -1 \pmod{18}$,, then $T = \{1, 5, 7, 11, 13, 17\} \pmod{18}$. Thus, $|S_{\mathcal{F}}(\gamma_g)| = 1$ for $g = 1, x, y^9, xy^9$ and $|S_{\mathcal{F}}(\gamma_g)| = 2$ for $g = y^i, xy^i; 1 \leq i \leq 8$. Hence,
$$\mathcal{F}(C_2 \times C_{18}) \cong \mathcal{F}^4 \oplus \mathcal{F}_2^{16}.$$
(c) If $q \equiv 5, -7 \pmod{18}$, then $T = \{1, -1\} \pmod{18}$. Thus, $|S_{\mathcal{F}}(\gamma_g)| = 1$ for $g = 1, x, y^9, xy^9$; $|S_{\mathcal{F}}(\gamma_g)| = 2$ for $g = y^3, xy^3, y^6, xy^6$ and $|S_{\mathcal{F}}(\gamma_g)| = 6$ for $g = y, xy, y^2, xy^2$. Hence,
$$\mathcal{F}(C_2 \times C_{18}) \cong \mathcal{F}^4 \oplus \mathcal{F}_2^4 \oplus \mathcal{F}_6^4.$$
(d) If $q \equiv 7, -5 \pmod{18}$, then $T = \{1, 7, 13\} \pmod{18}$. Thus, $|S_{\mathcal{F}}(\gamma_g)| = 1$ for $g = 1, x, y^{\pm 3}, y^{\pm 6}, y^9, xy^{\pm 3}, xy^{\pm 6}, xy^9$, and $|S_{\mathcal{F}}(\gamma_g)| = 3$ for $g = y^{\pm 1}, xy^{\pm 1}, y^{\pm 2}, xy^{\pm 2}$. Hence,
$$\mathcal{F}(C_2 \times C_{18}) \cong \mathcal{F}^{12} \oplus \mathcal{F}_3^8. \qquad \square$$

Theorem 3.4. *Let $\mathcal{G} \cong C_3 \times C_{12}$ and $\mathcal{F}$ be a finite field of characteristic $p > 0$ having $q = p^n$ elements.*

(1) *If $p = 2$, then*
$$U(\mathcal{FG}) \cong \begin{cases} C_2^{9n} \times C_4^{9n} \times C_{2^n-1}^9, & if q \equiv 1 \pmod{3}; \\ C_2^{9n} \times C_4^{9n} \times C_{2^n-1} \times C_{2^{2n}-1}^4, & if q \equiv -1 \pmod{3}. \end{cases}$$
(2) *If $p = 3$, then*
$$U(\mathcal{FG}) \cong \begin{cases} C_3^{32n} \times C_{3^n-1}^4, & if q \equiv 1 \pmod{4}; \\ C_3^{32n} \times C_{3^n-1}^2 \times C_{3^{2n}-1}, & if q \equiv -1 \pmod{4}. \end{cases}$$
(3) *If $p > 3$, then*
$$U((\mathcal{FG})) \cong \begin{cases} C_{p^n-1}^{36}, & if q \equiv 1 \pmod{12}; \\ C_{p^n-1}^2 \times C_{p^{2n}-1}^{17}, & if q \equiv -1 \pmod{12}; \\ C_{p^n-1}^4 \times C_{p^{2n}-1}^{16}, & if q \equiv 5 \pmod{12}; \\ C_{p^n-1}^{18} \times C_{p^{2n}-1}^9, & if q \equiv -5 \pmod{12}. \end{cases}$$

Proof. Let $C_3 \times C_{12} =< x, y \,|\, x^3 = y^{12} = 1, xy = yx >$.

(1) Let $Char(\mathcal{F}) = 2$ with $|\mathcal{F}| = 2^n$ elements and $\mathcal{G} \cong C_3 \times C_{12} \cong C_3 \times C_3 \times C_4$. Let C_4 be a normal subgroup of $\mathcal{G}$ such that $\mathcal{G}/C_4 \cong C_3^2$ and $|\mathcal{G} : C_4| \neq 0 \in \mathcal{F}$. Since C_4 is 2-group, therefore by Lemma 2.3 and [15, Theorem 7.2.7], $\mathcal{J}(\mathcal{F}G) = \omega(C_4)$. So, $\mathcal{F}G/\mathcal{J}(\mathcal{F}\mathcal{G}) \cong \mathcal{F}C_3^2$. Now,

$$\omega(C_4) = \left\{\alpha = \sum_{j=0}^{2}\sum_{i=0}^{11} a_{12j+i}x^jy^i \,|\, \sum_{i=0}^{3} a_{9i+j} = 0, j = 0, 1, 2, 3, 4, , 5, 6, 7, 8;\ a_i \in \mathcal{F}\right\}$$

and $\alpha^4 = 0$ for any $\alpha \in \omega(C_4)$. It is clear that $dim_{\mathcal{F}}\mathcal{J}(\mathcal{F}G) = 27$ and $exp(H) = 4$. Thus, $|H| = 2^{27n}$ and $H \cong C_2^{k_1} \times C_4^{k_2}$ so $2^{k_1} \times 4^{k_2} = 2^{27n}$. Set

$$S = \left\{\alpha \in \omega(C_4) : \alpha^2 = 0 \text{ and there exists}\, \beta \in \omega(C_4) \text{ such that}\, \alpha = \beta^2\right\}.$$

By direct computation, we have

$$S = \left\{a_0(1 + xy^6) + a_2(y^2 + xy^8) + a_4(y^4 + xy^{10}) + a_6(x^2 + y^6) + a_8(x^2y^2 + y^8) + a_{10}(x^2y^4 + y^{10}) + a_{12}(x + x^2y^6) + a_{14}(xy^2 + x^2y^8) + a_{16}(xy^4 + x^2y^{10}); \text{ for all}\, a_i \in \mathcal{F}\right\}.$$

Therefore, $|S| = 2^{9n}$ which implies that $k_2 = 9n$ and then $k_1 = 9n$. Hence, $H \cong C_2^{9n} \times C_4^{9n}$ and

$$U(\mathcal{F}(C_3 \times C_{12})) \cong C_2^{9n} \times C_4^{9n} \times U(\mathcal{F}C_3^2),$$

and the structure of $U(\mathcal{F}C_3^2)$ follows from [16, Lemma 3.3].

(2) Let $char(\mathcal{F}) = 3$ with $|\mathcal{F}| = 3^n$ elements and $\mathcal{G} \cong C_3 \times C_{12} \cong C_3 \times C_3 \times C_4$. Let C_3^2 be a normal subgroup of $\mathcal{G}$ such that $\mathcal{G}/C_3^2 \cong C_4$ and $|\mathcal{G} : C_3^2| \neq 0 \in \mathcal{F}$. Since C_3^2 is 3-group, therefore by Lemma 2.3 and [15, Theorem 7.2.7] , $\mathcal{J}(\mathcal{F}\mathcal{G}) = \omega(C_3^2)$. So, $\mathcal{F}\mathcal{G}/\mathcal{J}(\mathcal{F}\mathcal{G}) \cong \mathcal{F}C_4$. Now, $\alpha^3 = 0$ for any $\alpha \in \omega(C_3^2)$ and $dim_F J(FG) = 32$ (modify notation of F and G). Thus, exponent of H is 3 and $|H| = 3^{32n}$ and $H \cong C_3^{32n}$. Hence,

$$U(\mathcal{F}\mathcal{G}) \cong C_3^{32n} \times U(\mathcal{F}C_4),$$

and the structure of $U(\mathcal{F}C_4)$ is given in [16, Theorem 3.1].

(3) Since $p > 3$, then $\mathcal{F}\mathcal{G}$ is semisimple as p does not divide $|\mathcal{G}|$. It is clear all elements are p-regular and $m = 12$. Now we have the cases:

(a) If $q \equiv 1 \pmod{12}$, then $T = \{1\} \pmod{12}$. Thus, $|S_{\mathcal{F}}(\gamma_g)| = 1$ for all $g \in \mathcal{G}$. Thus by [4, Proposition 1.2 and Theorem 1.3],

$$\mathcal{F}(C_3 \times C_{12}) \cong \mathcal{F}^{36}.$$

(b) If $q \equiv -1 \pmod{12}$, then $T = \{1, -1\} \pmod{12}$. Thus, $|S_{\mathcal{F}}(\gamma_g)| = 1$ for $g = 1, y^6$ and $|S_{\mathcal{F}}(\gamma_g)| = 2$ for $g = x, xy^6, y^i, x^{\pm 1}y^i; 1 \leq i \leq 5$. Hence,

$$\mathcal{F}(C_3 \times C_{12}) \cong \mathcal{F}^2 \oplus \mathcal{F}_2^{17}.$$

(c) If $q \equiv 5 \pmod{12}$, then $T = \{1, 5\} \pmod{12}$. Thus, $|S_{\mathcal{F}}(\gamma_g)| = 1$ for $g = 1, y^6, y^{\pm 3}$ and $|S_{\mathcal{F}}(\gamma_g)| = 2$ for $g = x, y^2, y^4, y^{\pm 1}$, $xy^{\pm 1}, xy^2, xy^{\pm 3}, xy^4, xy^6, (xy)^{-1}, x^{-1}y, x^{-1}y^2, x^{-1}y^4$. Hence,

$$\mathcal{F}(C_3 \times C_{12}) \cong \mathcal{F}^4 \oplus \mathcal{F}_2^{16}.$$

(d) If $q \equiv -5 \pmod{12}$, then $T = \{1, 7\} \pmod{12}$. Thus, $|S_{\mathcal{F}}(\gamma_g)| = 1$ for $g = 1, y^6, y^{\pm 4}, x^{\pm 1}, xy^6, xy^{\pm 4}, x^{-1}y^{\pm 4}, x^{-1}y^6, y^{\pm 2}$, $x^{-1}y^{\pm 2}, xy^{\pm 2}$ and $|S_{\mathcal{F}}(\gamma_g)| = 2$ for $g = y, y^3, y^5, xy, xy^3, xy^5$, $x^{-1}y, x^{-1}y^3, x^{-1}y^5$. Hence,

$$\mathcal{F}(C_3 \times C_{12}) \cong \mathcal{F}^{18} \oplus \mathcal{F}_2^{9}.$$

□

Theorem 3.5. *Let $\mathcal{G} \cong C_3 \times A_4$ and $\mathcal{F}$ be a finite field of characteristic $p > 0$ having $q = p^n$ elements.*

(1) *If $p = 2$, then*

$$U(\mathcal{F}(C_3 \times A_4)) \cong \begin{cases} V \rtimes C_{2^n-1}^9, & \text{if } q \equiv 1 \pmod{3}; \\ V \rtimes C_{2^n-1} \times C_{2^{2n}-1}^4, & \text{if } q \equiv -1 \pmod{3}. \end{cases}$$

where $V = 1 + \mathcal{J}(\mathcal{FG})$, is a 2-group of order 2^{27n} of exponent 4 and $U(\mathcal{F}(C_3 \times A_4))$ is non metabelian group.

(2) *If $p = 3$, then $V_1 = 1 + \mathcal{J}(\mathcal{FG})$, where $\mathcal{J}(\mathcal{FG})$ denotes the Jacobson radical of the group algebra $\mathcal{FG}$. Then,*

(a) $\frac{U(\mathcal{F}(C_3 \times A_4))}{V_1} \cong C_{2^n-1} \times GL(3, \mathcal{F})$.

(b) V_1 is 3-group of order 3^{26n} and exponent 9.

(c) Nilpotency class of V_1 is 4.

(3) *If $p > 3$, then*

$$U(\mathcal{F}(C_3 \times A_4)) \cong \begin{cases} C_{p^n-1}^9 \times GL(3, \mathcal{F})^3, & \text{if } q \equiv 1 \pmod{6}; \\ C_{p^n-1} \times C_{p^{2n}-1}^4 \times GL(3, \mathcal{F}) \times GL(3, \mathcal{F}_2), & \text{if } q \equiv -1 \pmod{6}. \end{cases}$$

Proof. Let $\mathcal{G} \cong C_3 \times A_4 = < a, b, c, d \,|\, a^3 = b^2 = c^2 = d^3 = 1, ab = ba, ac = ca, ad = da, dbd^{-1} = bc = cb, dcd^{-1} = b >$.

TABLE 1. Conjugacy class Description of $C_3 \times A_4$

Symbolic	Elements in the class	Order of the elements
C_0	$\{1\}$	1
C_1	$\{a\}$	3
C_2	$\{a^{-1}\}$	3
C_3	$\{b, c, bc\}$	2
C_4	$\{ab, ac, abc\}$	6
C_5	$\{a^{-1}b, a^{-1}c, a^{-1}bc\}$	6
C_6	$\{d, cd, bd, bcd\}$	3
C_7	$\{d^{-1}, bcd^{-1}, bd^{-1}, cd^{-1}\}$	3
C_8	$\{ad, abd, abcd, acd\}$	3
C_9	$\{a^{-1}d, a^{-1}cd, a^{-1}bcd, a^{-1}bd\}$	3
C_{10}	$\{acd^{-1}, abd^{-1}, abcd^{-1}, ad^{-1}\}$	3
C_{11}	$\{a^{-1}cd^{-1}, a^{-1}bd^{-1}, a^{-1}d^{-1}, a^{-1}bcd^{-1}\}$	3

(1) Let $p = 2$. We define w= sum of all p elements including 1. Clearly, w= $1 + b + c + bc$. Let $\alpha = \sum_{i=0}^{35} a_i g_i$, $g_i \in \mathcal{G} \in Ann_{\mathcal{FG}}(w)$. We rewrite $\alpha = \sum_{i=0}^{11} \alpha_i$ such that $supp(\alpha_i) \subseteq C_i$ for $i = 0, 1, 2, \ldots, 11$. i.e.,

$$\alpha_0 = a_0,\ \alpha_1 = a_1 a,\ \alpha_2 = a_2 a^{-1},\ \alpha_3 = a_3 b + a_4 c + a_5 bc,$$
$$\alpha_4 = a_6 ab + a_7 ac + a_8 abc,\ \alpha_5 = a_9 a^{-1} b + a_{10} a^{-1} c + a_{11} a^{-1} bc,$$
$$\alpha_6 = a_{12} d + a_{13} cd + a_{14} bd + a_{15} bcd,$$
$$\alpha_7 = a_{16} d^{-1} + a_{17} bcd^{-1} + a_{18} bd^{-1} + a_{19} cd^{-1},$$
$$\alpha_8 = a_{20} ad + a_{21} abd + a_{22} abcd + a_{23} acd,$$
$$\alpha_9 = a_{24} a^{-1} d + a_{25} a^{-1} cd + a_{26} a^{-1} bcd + a_{27} a^{-1} bd,$$
$$\alpha_{10} = a_{28} acd^{-1} + a_{29} abd^{-1} + a_{30} abcd^{-1} + a_{31} ad^{-1},$$
$$\alpha_{11} = a_{32} a^{-1} cd^{-1} + a_{33} a^{-1} bd^{-1} + a_{34} a^{-1} d^{-1} + a_{35} a^{-1} bcd^{-1}.$$

Since $\alpha w = 0$, hence we have

$$a_0 + a_3 + a_4 + a_5 = 0;$$

$$a_1 + a_6 + a_7 + a_8 = 0;$$

$$a_2 + a_9 + a_{10} + a_{11} = 0;$$

$$\sum_{j=0}^{3} a_{4i+j} = 0, \quad for\ \ i = 3, 4, 5, 6, 7.$$

Thus, $Ann_{\mathcal{FG}}(w) = \{(c_0 + c_1 a + c_2 a^{-1})(1 + bc) + (c_3 + c_5 a + c_7 a^{-1})(b + bc) + (c_4 + c_6 a + c_8 a^{-1})(c + bc) + [c_9(1 + bc) + c_{10}(c + bc) + c_{11}(b + bc)]d +$

$[c_{12}(1+c)+c_{13}(c+bc)+c_{14}(b+c)]d^{-1}+c_{15}[(1+c)+c_{16}(b+c)+c_{17}(bc+c)]d+c_{18}[(1+b)+c_{19}(c+b)+c_{20}(bc+b)]a^{-1}d+c_{21}[(1+c)+c_{22}(1+b)+c_{23}(1+bc)]ad^{-1}+c_{24}[(c+bc)+c_{25}(b+bc)+c_{26}(1+bc)]a^{-1}d^{-1}\}$.

Obviously, $\beta_1\beta_2=\beta_2\beta_1$, for every $\beta_1,\beta_2\in Ann_{\mathcal{FG}}(w)$ and $Ann_{\mathcal{FG}}(w)^4=0$, which gives $Ann_{\mathcal{FG}}(w)\subseteq\mathcal{J}(\mathcal{FG})$. Hence, by [20, Lemma 2.2], it is clear that $\mathcal{J}(\mathcal{FG})=Ann_{\mathcal{FG}}(w)$ and $dim_F\mathcal{J}(\mathcal{FG})=27$. Hence $dim_{\mathcal{F}}\big(\mathcal{FG}/\mathcal{J}(\mathcal{FG})\big)=9$. We determine the structure of $\mathcal{FG}/\mathcal{J}(\mathcal{FG})$. The conjugacy classes $\{1\}$, $\{a\}$, $\{a^{-1}\}$, $\{d,cd,bd,bcd\}$, $\{d^{-1},bcd^{-1},bd^{-1},cd^{-1}\}$, $\{ad,abd,abcd,acd\}$, $\{a^{-1}d,a^{-1}cd,a^{-1}bcd,a^{-1}bd\}$, $\{acd^{-1},abd^{-1},abcd^{-1},ad^{-1}\}$ and $\{a^{-1}cd^{-1},a^{-1}bd^{-1},a^{-1}d^{-1},a^{-1}bcd^{-1}\}$ are 2-regular and $m=3$.

If $q\equiv 1 \pmod 3$, then $T=\{1\} \pmod 3$. By [4, Proposition 1.2 and Theorem 1.3], $\mathcal{FG}/\mathcal{J}(\mathcal{FG})\cong\mathcal{F}^9$. Hence

$$U(\mathcal{F}(C_3\times A_4))\cong V\rtimes C_{2^n-1}^9.$$

If $q\equiv -1 \pmod 3$, then $T=\{-1,1\} \pmod 3$. So 2-regular $\mathcal{F}-$conjugacy classes are $\{1\}$, $\{a,a^{-1}\}$, $\{d,cd,bd,bcd,d^{-1},bcd^{-1},bd^{-1},cd^{-1}\}$, $\{ad,abd,abcd,acd,a^{-1}cd^{-1},a^{-1}bd^{-1},a^{-1}d^{-1},a^{-1}bcd^{-1}\}$, $\{a^{-1}d,a^{-1}cd,a^{-1}bcd,a^{-1}bd,acd^{-1},abd^{-1},abcd^{-1},ad^{-1}\}$. Thus the number of 2-regular $\mathcal{F}-$conjugacy classes are 5, consequently by [4, Proposition 1.2 and Theorem 1.3], $\mathcal{FG}/\mathcal{J}(\mathcal{FG})\cong\mathcal{F}\oplus\mathcal{F}_2^4$. Hence

$$U(\mathcal{F}(C_3\times A_4))\cong V\rtimes C_{2^n-1}\times C_{2^{2n}-1}^4.$$

Since $dim_F\mathcal{J}(\mathcal{FG})=27$, therefore V is a 2-group of order 2^{27n} and by Theorem 2.4, exponent of V is 4. Now from [18, pp. 4], we have $U(\mathcal{F}(C_3\times A_4))$ is either isomorphic to $U(\mathcal{F}A_4)^3$ or $U(\mathcal{F}A_4)\times U(\mathcal{F}_2A_4)$ and it follows that $U(\mathcal{F}(C_3\times A_4))$ is non metabelian group as $U(\mathcal{F}A_4)$ is non metabelian.

(2) If $p=3$, then we have the following cases:

(a) As C_0 and C_3 are the only 3-regular conjugacy classes, so $m=2$ and this gives two $\mathcal{F}$ conjugacy classes for any q. Therefore by [4, Theorem 1.3], we have two components in the Wedderburn decomposition of $\mathcal{FG}/\mathcal{J}(\mathcal{FG})$. As $dim_F(\mathcal{J}(\mathcal{FG}))=26$ and $dim_{\mathcal{F}}(\mathcal{FG}/\mathcal{J}(\mathcal{FG}))=10$. Thus,

$$\mathcal{FG}/\mathcal{J}(\mathcal{FG})\cong\mathcal{F}\oplus M(3,\mathcal{F}).$$

(b) As $V_1=1+\mathcal{J}(\mathcal{FG})$ and $dim_F(\mathcal{J}(\mathcal{FG}))=26$, hence V_1 is a 3-group and $|V_1|=3^{26n}$. Now by Theorem 2.4, we have $\mathcal{J}^9=0$ and $V_1^9=1$. Hence exponent of V_1 is 9.

(c) Now we compute the nilpotency class of V_1. Let $u_1 = (1+u, 1+v)$, where $u, v \in \mathcal{J}(\mathcal{FG})$, then $u_1 \equiv 1 + vu((u,v)-1) \pmod{\mathcal{J}^3}$. Now if $u_2 = (u_1, 1+v)$, then $u_2 \equiv 1 + v^2u(u,v)((u_1,v)-1) \pmod{\mathcal{J}^5}$ and if $u_3 = (u_2, 1+v)$, then $u_3 \equiv 1 + v^3u(u,v)((u_1,v)-1)((u_2,v)-1) \pmod{\mathcal{J}^7}$. Now $u_4 \equiv 1 \pmod{\mathcal{J}^9}$. It can be easily seen that $\mathcal{J}^9 = 0$. Hence $\gamma_5(V_1) = 1$, thus the nilpotency class of V_1 is 4.

(3) Let Char$\mathcal{F} = p > 3$. Since, $\mathcal{F}\big(\frac{\mathcal{G}}{\mathcal{G}'}\big) \cong \mathcal{F}(C_3 \times C_3)$, thus by [Lemma 2.2],

$$\mathcal{FG} \cong \mathcal{F}(C_3 \times C_3) \oplus (\bigoplus_{i=1}^r M(n_i, \mathcal{F}_i))$$

where each $\mathcal{F}_i$ is a finite extension of $\mathcal{F}$. Now, we have the following cases:

Case 1. If $q \equiv 1 \pmod 6$. Then by [16, Theorem 3.3], $\mathcal{F}(C_3 \times C_3) \cong \mathcal{F}^9$. Thus,

$$\mathcal{FG} \cong \mathcal{F}^9 \oplus (\bigoplus_{i=1}^{r} M(n_i, \mathcal{F}_i))$$

and by Lemma 2.1, $dim_{\mathcal{F}}(\mathcal{Z}(\mathcal{FG})) = 12$ and hence $\sum_{i=1}^r [\mathcal{F}_i : \mathcal{F}] = 3$. Now, $\sum_{i=1}^k n_i^2 \, dim_{\mathcal{F}}(\mathcal{F}_i) = 36$ which implies $\sum_{i=1}^r n_i^2 \, dim_{\mathcal{F}}(\mathcal{F}_i) = 36 - 9 = 27$ and $T = \{1\} \pmod 6$. Therefore, the number of $\mathcal{F}$ conjugacy classes are conjugacy classes of $\mathcal{G}$. Let s be the number of $\mathcal{F}$ conjugacy classes. So, $s = 12$ and $r = 3$. By dimension constraints, $n_i = 3,\ 1 \le i \le 3$. Therefore,

$$\mathcal{FG} \cong \mathcal{F}^9 \oplus M(3, \mathcal{F})^3.$$

Case 2. If $q \equiv -1 \pmod 6$. Then by [16, Theorem 3.3], $\mathcal{F}(C_3 \times C_3) \cong \mathcal{F} \oplus \mathcal{F}_2^4$. Thus,

$$\mathcal{FG} \cong \mathcal{F} \oplus \mathcal{F}_2^4 \oplus (\bigoplus_{i=1}^r M(n_i, \mathcal{F}_i))$$

and by Lemma 2.1, $dim_{\mathcal{F}}(\mathcal{Z}(\mathcal{FG})) = 12$ and hence $\sum_{i=1}^r [\mathcal{F}_i : F] = 3$. Now $\sum_{i=1}^k n_i^2 \, dim_{\mathcal{F}}(\mathcal{F}_i) = 36$, which implies $\sum_{i=1}^r n_i^2 \, dim_{\mathcal{F}}(\mathcal{F}_i) = 36 - 9 = 27$ and $T = \{1, -1\} \pmod 6$. Thus, $|S_{\mathcal{F}}(\gamma_g)| = 1$ for $g = 1$; $|S_{\mathcal{F}}(\gamma_g)| = 2$ for $g = a$; $|S_{\mathcal{F}}(\gamma_g)| = 3$ for $g = b$; $|S_{\mathcal{F}}(\gamma_g)| = 6$ for $g = ab$; and $|S_{\mathcal{F}}(\gamma_g)| = 8$ for $g = d, ad, a^2d$. So, $s = 7$ and $r = 2$. By dimension constraints $n_i = 3$, $1 \le i \le 2$. Therefore,

$$\mathcal{FG} \cong \mathcal{F} \oplus \mathcal{F}_2^4 \oplus M(3, \mathcal{F}) \oplus M(3, \mathcal{F}_2).$$

This completes the proof. □

4. Conclusions

We have characterized the unit group structure of the five groups of order 36 including one non-abelian group namely $C_3 \times A_4$. Based on our work in this paper some interesting problems will arise in near future such as the characterization of the unit group structure of other non-abelin groups of order 36 up to an isomorphism. Using the results in this paper, we can deal with the normal complement problems of group algebras.

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