Ordu Üniv. Bil. Tek. Derg., Cilt:6, Sayı:1, 2016,67-82/Ordu Univ. J. Sci. Tech., Vol:6, No:1,2016,67-82

TACHIBANA AND VISHNEVSKII OPERATORS APPLIED TO X^V AND X^C IN ALMOST PARACONTACT STRUCTURE ON TANGENT BUNDLE $T(M)$

Haşim ÇAYIR

Department of Mathematics, Faculty of Arts and Sciences,
Giresun University, Giresun, 28100

ÖZET

Bu çalışmanın temel amacı Tangent demette almost paracontact yapıya göre X^V ve X^C ye uygulanan Tachibana ve Vishnevskii operatörlerini araştırmaktır. Ayrıca, elde edilen bu bağıntılar almost paracontact yapı içerisindeki bazı özel vectör alanları için de incelenilecektir.

Anahtar Kelimeler: Tachibana Operatörü, Vishnevskii Operatörü, Almost Parakontakt Yapı, Komple Lift, Vertikal Lift, Tangent Demet

Mathematics Subject Classification (2000): **15A72, 47B47, 53A45, 53C15**

ABSTRACT

The main aim of this paper is to investigate Tachibana and Vishnevskii Operators applied to X^V and X^C in almost paracontact structure on tangent bundle $T(M)$. In addition, this results which obtained shall be studied for some special vector fields in almost paracontact structure.

Keywords: Tachibana Operators, Vishnevskii Operators, Almost Paracontact Structure, Complete Lift, Vertical Lift, Tangent Bundle

Mathematics Subject Classification (2000): **15A72, 47B47, 53A45, 53C15**[1]

[1] *hasim.cayir@giresun.edu.tr

1. INTRODUCTION

Let M be an $n-$ dimensional differentiable manifold of class C^{∞} and let $T_p(M)$ be the tangent space of M at a point p of $M.$ Then the set (Yano 1973)

(1.1) $$T(M)=\bigcup_{p\in M} T_P(M)$$

is called the tangent bundle over the manifold $M.$ For any point $\tilde{p}$ of $T(M),$ the correspondence $\tilde{p}\rightarrow p$ determines the bundle projection $\pi:T(M)\rightarrow M,$ thus $\pi(\tilde{p})=p,$ where $\pi:T(M)\rightarrow M$ defines the bundle projection of $T(M)$ over $M.$ The set $\pi^{-1}(p)$ is called the fibre over $p\in M$ and M the base space.

Suppose that the base space M is covered by a system of cooordinate neighbourhoods $\{U:x^h\}$, where (x^h) is a system of local coordinates defined in neighbourhood U of $M.$ The open set $\pi^{-1}(U)\subset T(M)$ is naturally differentiably homeomorphic to the direct product $U\times R^n$, R^n being the $n-$ dimensional vektor space over the real field $R,$ in such a way that a point $\tilde{p}\in T_P(M)(p\in U)$ is represented by an ordered pair (P,X) of the point $p\in U$, and a vector $X\in R^n$, whose components are given by the cartesian coordinates (y^h) of $\tilde{p}$ in the tangent space $T_P(M)$ with respect to natural base $\{\partial_h\}$, where $\partial_h=\frac{\partial}{\partial x^h}$. Denoting by (x^h) the coordinates of $p=\pi(\tilde{p})$ in U and establishing the correspondence $(x^h,y^h)\rightarrow\tilde{p}\in\pi^{-1}(U),$ we can introduce a system of local coordinates (x^h,y^h) in the open set $\pi^{-1}(U)\subset T(M)$. Here we call (x^h,y^h) the coordinates in $\pi^{-1}(U)$ induced from (x^h) or simply, the induced coordinates in $\pi^{-1}(U).$

We deneto by $\Im^r_s(M)$ the set of all tensor fields of class C^{∞} and of type (r,s) in $M.$ We now put $\Im(M)=\sum_{r,s=0}^{\infty}\Im^r_s(M),$ which is the set of all tensor fields in $M.$ Similarly, we denote by $\Im^r_s(T(M))$ and $\Im(T(M))$ respectively the corresponding sets of tensor fields in the tangent bundle $T(M).$

1.1. Vertical lifts. If f is a funtion in M, we write f^v for the function in $T(M)$ obtained by forming the composition of $\pi:T(M)\rightarrow M$ and $f:M\rightarrow R$, so that

(1 .2) $$f^V=f\circ\pi$$

Thus, if a point $\tilde{p}\in\pi^{-1}(U)$ has induced coordinates $(x^h,y^h),$ then

(1.3) $$f^V(\tilde{p})=f^V(x,y)=f\circ\pi(\tilde{p})=f(p)=f(x).$$

Thus the value of $f^V(\tilde{p})$ is constant along the each fibre $T_p(M)$ and equal to the value $f(p)$. We call f^V the vertical lift of the function f (Yano et al 1973).

Let $\tilde{X} \in \Im_0^1(T(M))$ be such that $\tilde{X}f^V = 0$ for all $f \in \Im_0^0(M)$. Then we say that $\tilde{X}$ is a vector field. Let $\begin{bmatrix} \tilde{X}^h \\ \tilde{X}^{\bar{h}} \end{bmatrix}$ be components of $\tilde{X}$ with respect to the induced coordinates. The $\tilde{X}$ is vertical if and only if its components in $\pi^{-1}(U)$ satisfy

(1.4) $$\begin{bmatrix} \tilde{X}^h \\ \tilde{X}^{\bar{h}} \end{bmatrix} = \begin{bmatrix} 0 \\ \tilde{X}^{\bar{h}} \end{bmatrix}.$$

Suppose that $X \in \Im_0^1(M)$, so that a vector field in M. We define a vector field X^V in $T(M)$ by

(1.5) $$X^V(\iota\omega) = (\omega X)^V$$

ω be an arbitrary 1-form in M. We call X^V the vertical lift of X (Yano et al 1973).

Let $\tilde{\omega} \in \Im_1^0(T(M))$ be such that $\tilde{\omega}(X)^V = 0$ for all $X \in \Im_0^1(M)$. Then we say that $\tilde{\omega}$ is a vertical 1-form in $T(M)$. We define the vertical lift ω^V of the 1-form ω by

(1.6) $$\omega^V = (\omega_i)^V (dx^i)^V$$

in each open set $\pi^{-1}(U)$, where $(U : x^h)$ is coordinate neighbourhood in M and ω is given by $\omega = \omega_i dx^i$ in U. The vertical lift ω^V of ω with local expression $\omega = \omega_i dx^i$ has components of the form

(1.7) $$\omega^V : (\omega^i, 0)$$

with respect to the induced coordinates in $T(M)$.

Vertical lifts to a unique algebraic isomorphism of the tensor algebra $\Im(M)$ in to the tensor algebra $\Im(T(M))$ with respect to constant coefficients by the conditions (1.8)

$$(P \otimes Q)^V = P^V \otimes Q^V, (P+R)^V = P^V + R^V$$

P, Q and R being arbitrary elements of $\Im(M)$. The vertical lifts F^V of an element $F \in \Im_1^1(M)$ with local components F_i^h has components of the form (Yano et al 1973)

$$F^V : \begin{pmatrix} 0 & 0 \\ F_i^h & 0 \end{pmatrix}.$$

Vertical lifts has the following formulas (Omran et al 1984; Yano et al 1973):

(1.9) $$(fX)^V = f^V X^V, I^V X^V = 0, \eta^V (X^V) = 0$$

$$(f\eta)^V = f^V \eta^V, [X^V, Y^V] = 0, \varphi^V X^V = 0$$

$$X^V f^V = 0,$$

hold good, where $f \in \Im_0^0(M), X, Y \in \Im_0^1(M), \eta \in \Im_1^0(M), \varphi \in \Im_1^1(M), I = id_{M.}$

1.2. Complete lifts. If f is a function in M, we write f^C fort the function in $T(M)$ defined by

(1.10) $$f^C = \iota(df)$$

and call f^C the comple lift of the function f. The complete lift f^C of a function f has the local expression

(1.11) $$f^C = y^i \partial_i f = \partial f$$

with respect to the indicent coordinates in $T(M)$, where ∂f denotes $y^i \partial_i f$.

Suppose that $X \in \Im_0^1(M)$. Then we define a vector field X^C in $T(M)$ by

(1.12) $$X^C f^C = (Xf)^C,$$

f being an arbitrary function in M and call X^C the complete lift of X in $T(M)$ (Das 1993; Yano et al 1973). The complete lift X^C of X with components x^h in M has components

(1.13)
$$X^C = \begin{pmatrix} X^h \\ \partial X^h \end{pmatrix}$$

with respect to the indicent coordinates in $T(M)$.

Suppose that $\omega \in \Im_1^0(M)$. Then a 1-form ω^C in $T(M)$ defined by

(1.14)
$$\omega^C(X^C) = (\omega X)^C$$

X being an arbitrary vector field in M. We call ω^C the complete lift of ω. The complete lift ω^C of ω with components ω_i in M has component of the form

(1.15)
$$\omega^C : (\partial\omega_i, \omega_i)$$

with respect to the indicent coordinates in $T(M)$ [2].

The complete lifts to a unique algebra isomorphism of the tensor algebra in $\Im(M)$ into the tensor algebra $\Im(T(M))$ with respect to constant coefficients, is given by the conditions

(1.16)
$$(P \otimes Q)^C = P^C \otimes Q^V + P^V \otimes Q^C, (P+R)^C = P^C + R^C,$$

where P, Q and R being arbitrary elements of $\Im(M)$. The complete lifts F^C of an element $F \in \Im_1^1(M)$ with local components F_i^h has components of the form

$$F^C : \begin{pmatrix} F_i^h & 0 \\ \partial F_i^h & F_i^h \end{pmatrix}.$$

In addition, we know that complete lifts are defined by (Omran et al 1984; Yano et al 1973):

(1.17)
$$(fX)^C = f^C X^V + f^V X^C = (Xf)^C,$$

$$X^C f^V = (Xf)^V, \eta^V(X^C) = (\eta(X))^V,$$

$$X^V f^C = (Xf)^V, \varphi^V X^C = (\varphi X)^V,$$

$$\varphi^C X^V = (\varphi X)^V, (\varphi X)^C = \varphi^C X^C,$$

$$\eta^V (X^C) = (\eta(X))^C, \eta^C (X^V) = (\eta(X))^V$$

$$[X^V, Y^C] = [X,Y]^V, I^C = I, I^V X^C = X^V, [X^C, Y^C] = [X,Y]^C.$$

Definition 1. Let M be an $n-$ dimensional diferentiable manifold. Diferential transformation $D = L_X$ is called Lie derivation with respect to vector field $X \in \Im_0^1(M)$ if

(1.18) $$L_X f = Xf, \forall f \in \Im_0^0(M),$$

$$L_X Y = [X,Y], \forall X, Y \in \Im_0^1(M).$$

$[X,Y]$ is called by Lie bracked. The derivative $L_X F$ of a tensor field F of type $(1,1)$ with respect to a vector field X is defined by ([8])

(1.19) $$(L_X F)Y = [X, FY] - F[X,Y].$$

Proposition 1. For any $X \in \Im_0^1(M), f \in \Im_0^0(M)$ and L_X is the Lie derivation with respect to vector field X Yano et al (1973)

i) $L_{X^V} f^V = 0,$

ii) $L_{X^V} f^C = (L_X f)^V,$

iii) $L_{X^C} f^V = (L_X f)^V,$

iv) $L_{X^C} f^C = (L_X f)^C.$

Proposition 2. For any $X, Y \in \Im_0^1(M)$ and L_X is the Lie derivation with respect to vector field X Yano et al (1973)

i) $L_{X^V} Y^V = 0,$

ii) $L_{X^V} Y^C = (L_X Y)^V,$

iii) $L_{X^C} Y^V = (L_X Y)^V$,

iv) $L_{X^C} Y^C = (L_X Y)^C$,

Definition 2. Let M be an $n-$ dimensional diferentiable manifold. Differential transformation of algebra $\Im(M)$, defined by

$$D = \nabla_X : \Im(M) \to \Im(M), X \in \Im_0^1(M),$$

is called a covariant derivation with respect to vector field X if

(1.20) $$\nabla_{fX+gY} t = f\nabla_X t + g\nabla_y t,$$

$$\nabla_X f = Xf,$$

where $\forall f, g \in \Im_0^0(M), \forall X, Y \in \Im_0^1(M), \forall t \in \Im(M).$

On the other hand, a transformation, defined by

$$\nabla : \Im_0^1(M) \times \Im_0^1(M) \to \Im_0^1(M),$$

is called an afin connection ([5],[8]). In addition, the complete lift of an afine connection ∇ in M to $T(M)$ is denoted by ∇^C and defined by the conditions of

(1.21) $$\nabla^C_{X^V} f^V = 0, \nabla^C_{X^V} f^C = (\nabla_X f)^V,$$

$$\nabla^C_{X^C} f^V = (\nabla_X f)^V, \nabla^C_{X^C} f^C = (\nabla_X f)^C,$$

$$\nabla^C_{X^V} Y^V = 0, \nabla^C_{X^V} Y^C = (\nabla_X Y)^V,$$

$$\nabla^C_{X^C} Y^V = (\nabla_X Y)^V, \nabla^C_{X^C} Y^C = (\nabla_X Y)^C$$

for any $X, Y \in \Im_0^1(M), f \in \Im_0^0(M)$ (Yano et al 1973).

2. MAIN RESULTS

2.1.Tachibana Operators Applied to X^V and X^C in Almost Paracontact Structure.

Definition 3. Let an $n-$ dimensional differentiable manifold M be endowed with a tensor field φ of type $(1,1)$, a vector field ξ, a 1-form η, I the idendity and let them satisfy

(2.1) $$\varphi^2 = I - \eta \otimes \xi, \qquad \varphi(\xi) = 0, \qquad \eta \circ \varphi = 0, \qquad \eta(\xi) = 1.$$

Then (φ, ξ, η) define almost paracontact structure on M (Omran et al 1984; Salimov & cayır 2013). From (2.1), we get on taking complete and vertical lifts

(2.2) $$(\varphi^C)^2 = I - \eta^V \otimes \xi^C - \eta^C \times \xi^V,$$

$$\varphi^C \xi^V = 0, \varphi^C \xi^C = 0, \eta^V \circ \varphi^C = 0,$$

$$\eta^C \circ \varphi^C = 0, \eta^V(\xi^V) = 0, \eta^V(\xi^C) = 1,$$

$$\eta^C(\xi^V) = 1, \eta^C(\xi^C) = 0.$$

We now define a $(1,1)$ tensor field $\tilde{J}$ on $\Im(M)$ by

(2.3) $$\tilde{J} = \varphi^C - \xi^V \otimes \eta^V - \xi^C \otimes \eta^C.$$

Then it is easy to show that $\tilde{J}^2 X^V = X^V$ and $\tilde{J}^2 X^C = X^C$, which give that $\tilde{J}$ is an almost product structure on $\Im(M)$. We get from (2.3)

$$\tilde{J} X^V = (\varphi X)^V - (\eta(X))^V \xi^C$$

$$\tilde{J} X^C = (\varphi X)^V - (\eta(X))^V \xi^V - (\eta(X))^C \xi^C$$

for any $X \in \Im_0^1(M)$.

Definition 4. Let $\varphi \in \Im_1^1(M)$, and $\Im(M)=\sum_{r,s=0}^{\infty}\Im_s^r(M)$ be an tensor alebra over $R.$ A map

$\phi_\varphi\big|_{r+s\to 0}:\overset{*}{\Im}(M)\to\Im(M)$ is called a Tachibana operator or ϕ_φ operator on M if

a) ϕ_φ is linear with respect to constant coefficient,

b) $\phi_\varphi:\overset{*}{\Im}(M)\to\Im_{s+1}^r(M)$ for all r and s,

c) $\phi_\varphi(K\overset{C}{\otimes}L)=(\phi_\varphi K)\otimes L+K\otimes\phi_\varphi L$ for all $K,L\in\overset{*}{\Im}(M),$

d) $\phi_{\varphi X}Y=-(L_Y\varphi)X$ for all $X,Y\in\Im_0^1(M),$ where L_Y is the Lie derivation with respect to Y

e) $(\phi_{\varphi X}\eta)Y=(d(\iota_Y\eta(\phi X)-(d(\iota_Y(\eta\circ\phi)X+\eta((L_Y\varphi)X)$

$$=(\phi X(\iota Y\eta))(\phi X)-X(\iota_{\varphi Y}\eta)+\eta((L_Y\varphi)X)$$

for all $\eta\in\Im_1^0(M)$ and $X,Y\in\Im_1^0(M)$, where $\iota_Y\eta=\eta(Y)=\eta\overset{C}{\otimes}Y,\overset{*}{\Im}{}_s^r(M)$ the module of all pure tensor fields of type (r,s) on M with respect to the affinor field φ (Salimov 2013).

Theorem 1. For ϕ_φ Tachibana operator on M, L_X the operator Lie derivation with respect to

$X,\tilde{J}\in\Im_1^1(\Im(M))$ defined by (2.3) and $\eta(Y)=0,$ we have

i) $\phi_{\tilde{J}Y^V}X^V=-(L_{X^V}\tilde{J})Y^V=0,$

ii) $\phi_{\tilde{J}Y^C}X^V=-(L_{X^V}\tilde{J})Y^C=-((L_X\varphi)Y)^V+((L_X\eta)Y)^V\xi^C,$

iii) $\phi_{\tilde{J}Y^V}X^C=-(L_{X^C}\tilde{J})Y^V=-((L_X\varphi)Y)^V+((L_X\eta)Y)^V\xi^C,$

iv) $\phi_{\tilde{J}Y^C} X^C = -(L_{X^C} \tilde{J})Y^C = -((L_X \varphi)Y)^C + ((L_X \eta)Y)^V \xi^V + ((L_X \eta)Y)^C \xi^C,$

where $X, Y \in \Im_0^1(M),$ a tensor field $\varphi \in \Im_1^1(M),$ a vector field ξ and a 1-form $\eta \in \Im_1^0(M).$

Proof. For $\tilde{J}$ is defined by (2,3) and $\eta(Y) = 0$, we get

i) $\phi_{\tilde{J}Y^V} X^V = -(L_{X^V} \tilde{J})Y^V$

$$= L_{X^V}(\varphi^C - \xi^V \otimes \eta^V - \xi^C \otimes \eta^C)Y^V + (\varphi^C - \xi^V \otimes \eta^V - \xi^C \otimes \eta^C)L_{X^V} Y^V$$

$$= -L_{X^V}(\varphi Y)^V + L_{X^V}(\eta^V (Y)^V)\xi^V + L_{X^V}(\eta(Y))^V \xi^C$$

$$= L_{X^V}(\varphi Y)^V$$

$$= 0$$

ii) $\phi_{\tilde{J}Y^C} X^V = -(L_{X^V} \tilde{J})Y^C$

$$= L_{X^V}(\varphi^C - \xi^V \otimes \eta^V - \xi^C \otimes \eta^C)Y^C + (\varphi^C - \xi^V \otimes \eta^V - \xi^C \otimes \eta^C)L_{X^V} Y^C$$

$$= -L_{X^V} \varphi^C Y^C + L_{X^V}(\eta Y)^V \xi^V + L_{X^V}(\eta Y)^C \xi^C + \varphi^C (L_X Y)^V$$

$$-\eta^V (L_X Y)^V \xi^V - (\eta(L_X Y))^V \xi^C$$

$$= -(L_{X^V} \varphi^C)Y^C - \varphi^C (L_{X^V} Y^C) + \varphi^C (L_X Y)^V - (L_X (\eta(Y)))^V \xi^C + (L_X (\eta(Y)))^V \xi^C$$

$$= -((L_X \varphi)Y)^V + ((L_X \eta)Y)^V \xi^C,$$

iii) $\phi_{\tilde{J}Y^V} X^C = -(L_{X^C} \tilde{J})Y^V$

$$= L_{X^C}(\varphi^C - \xi^V \otimes \eta^V - \xi^C \otimes \eta^C)Y^V + (\varphi^C - \xi^V \otimes \eta^V - \xi^C \otimes \eta^C)L_{X^C} Y^V$$

$$= -L_{X^C} \varphi^C Y^V + L_{X^C}(\eta^V (Y)^V)\xi^V + L_{X^C}(\eta(Y))^V \xi^C + \varphi^C L_{X^C} Y^V$$

$$-\eta^V (L_X Y)^V \xi^V - (\eta(L_X Y))^V \xi^C$$

$$= -(L_{X^C}\varphi^C)Y^V - \varphi^C(L_{X^C}Y^V) + \varphi^C L_{X^C}Y^V - (L_X(\eta(Y)))^V \xi^C + ((L_X\eta)Y)^V \xi^C$$

$$= -((L_X\varphi)Y)^V + ((L_X\eta)Y)^V \xi^C,$$

iv) $\phi_{\tilde{J}Y^C} X^C = -(L_{X^C}\tilde{J})Y^C$

$$= -L_{X^C}(\varphi^C - \xi^V \otimes \eta^V - \xi^C \otimes \eta^C)Y^C + (\varphi^C - \xi^V \otimes \eta^V - \xi^C \otimes \eta^C)L_{X^C}Y^C$$

$$= -L_{X^C}\varphi^C Y^C + L_{X^C}((\eta Y)^V)\xi^V + L_{X^C}(\eta(Y))^C \xi^C + \varphi^C L_{X^C}Y^C$$

$$-(\eta(L_X Y))^V \xi^V - (\eta(L_X Y))^C \xi^C$$

$$= -(L_{X^C}\varphi^C)Y^C - \varphi^C(L_{X^C}Y^C) + \varphi^C L_{X^C}Y^C - (L_X(\eta(Y)))^V \xi^V + ((L_X\eta)Y)^V \xi^V$$

$$-(L_X(\eta(Y)))^C \xi^C + ((L_X\eta)Y)^C \xi^C$$

$$= -((L_X\varphi)Y)^C + ((L_X\eta)Y)^V \xi^V + ((L_X\eta)Y)^C \xi^C,$$

where $\eta L_X Y = L_X \eta(Y) - (L_X\eta)Y$ and $\varphi Y \in \Im_0^1(M_n)$.

Corallary 1. If we put $Y = \xi,$ i.e. $\eta(\xi) = 1$ and ξ has the condition (2.1), then we have

i) $\phi_{\tilde{J}\xi^V} X^V = -(L_X\xi)^V,$

ii) $\phi_{\tilde{J}\xi^C} X^V = -((L_X\varphi)\xi)^V + ((L_X\eta)\xi)^V \xi^C,$

iii) $\phi_{\tilde{J}\xi^V} X^C = -((L_X\varphi)\xi)^V + (L_X\xi)^C + ((L_X\eta)\xi)^V \xi^C,$

iv) $\phi_{\tilde{J}\xi^C} X^C = -((L_X\varphi)\xi)^C + (L_X\xi)^V + ((L_X\eta)\xi)^V \xi^V + ((L_X\eta)\xi)^C \xi^C.$

2.2 Vishnevskii Operators Appliedto X^V and X^C in Almost Paracontract Structure.

Definition 5. Suppose now that ∇ is a linear connection on M, and let $\varphi \in \Im_1^1(M)$. We can replace the condition d) of defination 4 by

(2.4) $$d')\ \ \psi_{\varphi X} Y = \nabla_{\varphi X} Y - \varphi \nabla_X Y$$

for any $X, Y \in \Im_0^1(M)$. Then we can consider a new operator by a Vishnevskii operator or $\psi_\varphi -$ operator on M, we shall mean a map $\psi_\varphi : \overset{*}{\Im}(M) \to \Im(M)$, which satisfies conditions a), b), c), e) of definition 4 and the condition (d') (Salimov 2013).

Theorem 2. Let ψ_φ Vishnevskii operator on M, ∇^C the complete lift of an affine connection

∇ in M to $T(M)$ and $\tilde{J} \in \Im_1^1(\Im(M))$ defined by (2.3), then we get

i) $\psi_{\tilde{J}X^V} Y^V = -(\eta(X)\nabla_\xi Y)^V$,

ii) $$\psi_{\tilde{J}X^V} Y^C = ((\hat{\nabla}_Y \varphi)X)^V - ((L_Y \varphi)X)^V - (\eta(X))^V (\hat{\nabla}_Y \xi)^C + (\eta(X))^V (L_Y \xi)^C$$
$$+ (\eta(\hat{\nabla}_Y X))^V \xi^C - (\eta L_Y X)^V \xi^C,$$

iii) $$\psi_{\tilde{J}X^C} Y^V = ((\hat{\nabla}_Y \varphi)X)^V - ((L_Y \varphi)X)^V - (\eta(X))^C (\hat{\nabla}_Y \xi)^V + (\eta(X))^C (L_Y \xi)^V$$
$$+ (\eta(\hat{\nabla}_Y X))^V - (\eta L_Y X)^V \xi^C,$$

iv) $$\psi_{\tilde{J}X^C} Y^C = ((\hat{\nabla}_Y \varphi)X)^C - ((L_Y \varphi)X)^C - (\eta(X)(\hat{\nabla}_Y \xi)^V + (\eta(X)(L_Y \xi)^V$$
$$-(\eta(X))^C (\hat{\nabla}_Y \xi)^C + (\eta(X))^C (L_Y \xi)^C + (\eta(\hat{\nabla}_Y X))^V \xi^V - (\eta L_Y X)^V \xi^V$$
$$+ (\eta(\hat{\nabla}_Y X))^C \xi^C - (\eta L_Y X)^C \xi^C,$$

where $X,Y\in\Im_0^1(M)$, a tensor field $\varphi\in\Im_1^1(M)$, a vector field ξ and a 1-form $\eta\in\Im_1^0(M)$.

Proof. Let $\tilde{J}\in\Im_1^1(\Im(M))$ defined by (2.3), then we have

i) $\psi_{\tilde{J}X^V}Y^V=\nabla^C_{\tilde{J}X^V}Y^V-\tilde{J}\nabla^C_{X^V}Y^V$

$$=\nabla^C_{(\varphi X)^V-(\eta(X))^V\xi^C}Y^V(\varphi^C-\xi^V\otimes\eta^V-\xi^C\otimes\eta^C)\nabla^C_{X^V}Y^V$$

$$=\nabla^C_{(\varphi X)^V}Y^V-(\eta(X))^V\nabla^C_{\xi^C}Y^V$$

$$=-(\eta(X))^V(\nabla_\xi Y)^V$$

$$=-(\eta(X)\nabla_\xi Y)^V,$$

ii) $\psi_{\tilde{J}X^V}Y^C=\nabla^C_{\tilde{J}X^V}Y^C-\tilde{J}\nabla^C_{X^V}Y^C$

$$=\nabla^C_{(\varphi X)^V-(\eta(X))^V\xi^C}Y^C-(\varphi^C-\xi^V\otimes\eta^V-\xi^C\otimes\eta^C)\nabla^C_{X^V}Y^C$$

$$=\nabla^C{}_{(\varphi X)^V}Y^C-(\eta(X))^V\nabla^C{}_{\xi^C}Y^C-\varphi^C(\nabla_XY)^V+(\eta^V(\nabla_XY)^V)\xi^V$$

$$+(\eta^C(\nabla_XY)^V)\xi^C$$

$$=(\nabla_{\varphi X}Y)^V-(\eta(X))^V(\nabla_\xi Y)^C-(\varphi\nabla_XY)^V+(\eta(\nabla_XY))^V)\xi^C$$

$$=(\hat{\nabla}_Y\varphi X+[\varphi X,Y])^V-(\eta(X))^V(\hat{\nabla}_Y\xi+[\xi,Y])^C-\varphi^C(\hat{\nabla}_YX+X,Y])^V$$

$$+(\eta(\hat{\nabla}_YX+X,Y]))^V\xi^C$$

$$=(\hat{\nabla}_Y\varphi)X)^V+(\varphi\hat{\nabla}_YX)^V-(L_Y\varphi)X)^V-(\varphi L_YX)^V-(\eta(X))^V(\hat{\nabla}_Y\xi)^C$$

$$+(\eta(X))^V(L_Y\xi)^C-(\varphi(\hat{\nabla}_YX))^V+(\varphi L_YX)^V+(\eta(\hat{\nabla}_YX))^V\xi^C-(\eta L_YX)^V\xi^C,$$

$$=((\hat{\nabla}_Y\varphi)X)^V-((L_Y\varphi)X)^V-(\eta(X))^V(\hat{\nabla}_Y\xi)^C+(\eta(X))^V(L_Y\xi)^C$$

$$+(\eta(\hat{\nabla}_Y X))^V \xi^C - (\eta L_Y X)^V \xi^C,$$

iii) $\psi_{\tilde{J}X^C} Y^V = \nabla^C_{\tilde{J}X^C} Y^V - \tilde{J}\nabla^C_{X^C} Y^V$

$$= \nabla^C_{(\varphi X)^C - (\eta(X))^V \xi^V - (\eta(X))^C \xi^C} Y^V - (\varphi^C - \xi^V \otimes \eta^V - \xi^C \otimes \eta^C)\nabla^C_{X^V} Y^V$$

$$= \nabla^C_{(\varphi X)^C} Y^V - (\eta(X))^V \nabla^C_{\xi^V} Y^V - (\eta(X))^C \nabla^C_{\xi^C} Y^V - \varphi^C (\nabla_X Y)^V$$

$$+\eta^V (\nabla_X Y)^V \xi^V + \eta^C (\nabla_X Y)^V \xi^C$$

$$= (\nabla_{\varphi X} Y)^V - (\eta(X))^C (\nabla_\xi Y)^V - (\varphi \nabla_X Y)^V + (\eta \nabla_X Y)^V \xi^C$$

$$= (\hat{\nabla}_Y \varphi) X)^V + [\varphi X, Y]^V - (\eta(X))^C (\hat{\nabla}_Y \xi + [\xi, Y])^V - (\varphi(\hat{\nabla}_Y X + [X,Y]))^V$$

$$+(\eta(\hat{\nabla}_Y X + X, Y]))^V \xi^C$$

$$= (\hat{\nabla}_Y \varphi) X)^V + (\varphi(\hat{\nabla}_Y X))^V - (L_Y \varphi) X)^V - (\varphi(L_Y X))^V - (\eta(X))^C (\hat{\nabla}_Y \xi)^V$$

$$+(\eta(X))^C (L_Y \xi)^V - (\varphi(\hat{\nabla}_Y X))^V + (\varphi(L_Y X))^V + (\eta(\hat{\nabla}_Y X))^V - (\eta L_Y X)^V \xi^C$$

$$= ((\hat{\nabla}_Y \varphi) X)^V - ((L_Y \varphi) X)^V - (\eta(X))^C (\hat{\nabla}_Y \xi)^V + (\eta(X))^C (L_Y \xi)^V$$

$$+(\eta(\hat{\nabla}_Y X))^V - (\eta L_Y X)^V \xi^C,$$

iv) $\psi_{\tilde{J}X^C} Y^C = \nabla^C_{\tilde{J}X^C} Y^C - \tilde{J}\nabla^C_{X^C} Y^C$

$$= \nabla^C_{(\varphi X)^C - (\eta(X))^V \xi^V - (\eta(X))^C \xi^C} Y^C - (\varphi^C - \xi^V \otimes \eta^V - \xi^C \otimes \eta^C)\nabla^C_{X^C} Y^C$$

$$= (\nabla^C_{\varphi X} Y)^C - (\eta(X))^V \nabla^C_{\xi^V} Y^C - (\eta(X))^C \nabla^C_{\xi^C} Y^C - \varphi^C (\nabla_X Y)^C$$

$$+(\eta^V (\nabla_X Y)^C)\xi^V + (\eta^C (\nabla_X Y)^C)\xi^C$$

$$=(\nabla_{\varphi X}Y)^C-(\eta(X))^V(\nabla_{\xi}Y)^V-(\eta(X))^C(\nabla_{\xi}Y)^C-(\varphi\nabla_X Y)^C$$

$$+((\eta(\nabla_X Y))^V)\xi^V+((\eta(\nabla_X Y))^C)\xi^C$$

$$=(\hat{\nabla}_Y\varphi X)^C+[\varphi X,Y]^C-(\eta(X))^V(\hat{\nabla}_Y\xi+[\xi,Y])^V-(\eta(X))^C(\hat{\nabla}_Y\xi+[\xi,Y])^C$$

$$-(\varphi(\hat{\nabla}_Y X+[X,Y]))^C+(\eta(\hat{\nabla}_Y X+[X,Y]))^V\xi^V+(\eta(\hat{\nabla}_Y X+[X,Y]))^C\xi^C$$

$$=(\hat{\nabla}_Y\varphi)X)^C+(\varphi(\hat{\nabla}_Y X))^C-(L_Y\varphi)X)^C-(\varphi(L_Y X))^C-(\eta(X)(\hat{\nabla}_Y\xi))^V$$

$$+(\eta(X)(L_Y\xi))^V-(\eta(X))^C(\hat{\nabla}_Y\xi))^C+(\eta(X))^C(L_Y\xi)^C-(\varphi(\hat{\nabla}_Y X))^C$$

$$+(\varphi(L_Y X))^C+(\eta(\hat{\nabla}_Y X))^V\xi^V-(\eta L_Y X)^V\xi^V+(\eta(\hat{\nabla}_Y X))^C\xi^C-(\eta L_Y X)^C\xi^C$$

$$=((\hat{\nabla}_Y\varphi)X)^C-((L_Y\varphi)X)^C-(\eta(X)(\hat{\nabla}_Y\xi)^V+(\eta(X)(L_Y\xi)^V$$

$$-(\eta(X))^C(\hat{\nabla}_Y\xi)^C+(\eta(X))^C(L_Y\xi)^C+(\eta(\hat{\nabla}_Y X))^V\xi^V-(\eta L_Y X)^V\xi^V$$

$$+(\eta(\hat{\nabla}_Y X))^C\xi^C-(\eta L_Y X)^C\xi^C$$

Corollary 2. If we put $X=\xi$, i.e. $\eta(\xi)=1$ and ξ has the condition (2.1), then we get

i) $\psi_{\tilde{J}\xi^V}Y^V=-(\nabla_{\xi}Y)^V$,

ii) $\psi_{\tilde{J}\xi^V}Y^C=((\hat{\nabla}_Y\varphi)\xi)^V-((L_Y\varphi)\xi)^V-(\hat{\nabla}_Y\xi)^C+(L_Y\xi)^C-((\hat{\nabla}_Y\eta)\xi)^V\xi^C$

$$+((L_Y\eta)\xi)^V\xi^C$$

iii) $\psi_{\tilde{J}\xi^C}Y^V=((\hat{\nabla}_Y\varphi)\xi)^V-((L_Y\varphi)\xi)^V-((\hat{\nabla}_Y\eta)\xi)^V+((L_Y\eta)\xi)^V\xi^C$

iv) $\psi_{\tilde{J}\xi^C}Y^C=((\hat{\nabla}_Y\varphi)\xi)^C-((L_Y\varphi)\xi)^C-((\hat{\nabla}_Y\eta)\xi)^V+(L_Y\xi)^V-((\hat{\nabla}_Y\eta)\xi)^V\xi^V$

$$+((L_Y\eta)\xi)^V\xi^V-((\hat{\nabla}_Y\eta)\xi)^C\xi^C+((L_Y\eta)\xi)^C\xi^C$$

3. REFERENCES

Blair. D.E., Contact Manifolds in Riemannian Geometry, Lecture Notes in Math, 509, Springer Verlag, New York, 1976.

Das, Lovejoy S., Fiberings on almost r-contac t manifolds, Publicationes Mathematicae, Debrecen, Hungary, 43, 161-167, 1993.

Oproui, V., Some remarkable structures and connexions, defined on the tangent bundle, Rendiconti di Matematica 3, 6 VI. 1973.

Omran, T., Sharffuddin, A. and Husain, S.I., Lift of Structures on Manifolds, Publications De 1'Instıtut Mathematıqe, Nouvelle serie, 360 (50), pp.93-97, 1984.

Salimov, A.A., Tensor Operators and Their applications, Nova Science Publ., New York, 2013.

Sasaki, S., On The Differantial Geometry of Tangent Boundles of Riemannian Mani- folds, Tohoku Math. J., 10, pp. 338-358, 1958.

Salimov, A.A. and Çayır, H., Some Notes On Almost Paracontact Structures, Comptes Rendus de 1' Acedemie Bulgare Des Sciences, 66(3), 331-338, 2013.

Yano, K. and Ishihara, S., Targent and Cotangent Bundles, Marcel Dekker Inc, New York, 1973.