FRACTIONAL HERMITE-HADAMARD TYPE INEQUALITIES FOR QUASI-CONVEX FUNCTIONS

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Abstract

In this paper, by making use of identity proved in Shi et al (2014) for fractional integrals, several inequalities of Hermite-Hadamard type for quasi-convex functions via Riemann-Liouville fractional integrals and the well known Hölder integral inequality are obtained. Some applications for the special means are also given.

Keywords: Quasi-convex function, Hermite-Hadamard inequality, Riemann-Liouville fractional integral.

QUASİ-KONVEKS FONSİYONLAR İÇİN KESİRLİ HERMİTE-HADAMARD TİPLİ EŞİTSİZLİKLER

Özet

Bu makalede, kesirli integraller için Shi ve arkadaşları tarafından (2014)'de elde edilen özdeşlik kullanılarak, Riemann-Liouville kesirli integralleri ve litaretürde iyi bilinen Hölder eşitsizliği yardımıyla quasi konveks fonksiyonlar için Hermite-Hadamard tipli eşitsizlikler elde edilmiştir. Ayrıca özel ortalamalar için bazı uygulamalar verilmiştir.

Anahtar Kelimeler: Quasi konveks fonksiyon, Hermite-Hadamard eşitsizliği, Riemann-Liouville kesirli integrali.

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1. Introduction

One of the most famous inequality for convex functions is so called Hermite-Hadamard inequality as follows:
Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $x, y \in I$ with $x < y$, then

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_x^y f(t)dt \leq \frac{f(x)+f(y)}{2}$$

is known as the Hermite-Hadamard inequality.

Definition 1.1 *A function* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *is said to be convex if the inequality*

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Definition 1.2 *A mapping* $f : I \to \mathbb{R}$ *is called quasi-convex on the convex set* I *if for all* $x, y \in I$ *and* $\lambda \in [0,1]$

$$f(\lambda x+(1-\lambda)y) \leq \max\{f(x), f(y)\}.$$

This class of functions strictly contains the class of convex functions defined on a convex set in a real linear space, see Eberhand & Pearce (2000) and citations therein for an overview.Recent studies have shown that quasi convex functions have quite close resemblances to convex functions see, example Dragomir & Bond (1997), Dragomir & Pearce (1998), Dragomir (1995), Pearce & Rubinov (1999) for quasi convex and even more general extensions of convex functions in the context of Hermite-Hadamard's inequalities. Apart from generalizations to theory, weakening the convexity condition can increase applicability. Thus in Pearce (2004) use is made of quasi-convexity to obtain a global extremum with rather less effort than via convexity.
We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 1.3 *Let* $f \in L_1[a,b]$. *The Riemann-Liouville integrals* $J_{a+}^{\alpha} f$ *and* $J_{b-}^{\alpha} f$ *of order* $\alpha > 0$ *with* $a \geq 0$ *are defined by*

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \ \ x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \ \ x < b$$

respectively where $\Gamma(\alpha) = \int_0^{\infty} e^{-t}u^{\alpha-1}du$. Here is $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$. In the case of

$\alpha = 1$, the fractional integral reduces to the classical integral. Some recent result and properties concerning the operator can be found Belarbi & Dahmani (2009); Dahmani (2010); Gorenflo & Mainardi (1997); Iscan (2013); Miller & Ross (1993); Sarıkaya & Ogunmez (2012); Sarıkaya et al (2013); Set (2012) .

We establish here new Hermite-Hadamard type inequalities for quasi-convex function via Riemann-Liouville fractional integral. An interesting feature of our results is that they provide new estimate, on these types of inequalities for fractional integrals.

2. Main Results

Lemma 2.1 [15] *Assume* $a,b \in \mathbb{R}$ *with* $a < b$ *and* $f:[a,b] \to \mathbb{R}$ *is a differentiable function on* (a,b). *If* $f' \in (L[a,b])$ *then the following equality holds:*

$$\begin{aligned}\Phi_\alpha(a,b) = \frac{b-a}{16}[&\int_0^1 (1-t^\alpha) f'\left(t\frac{3a+b}{4}+(1-t)\frac{a+b}{2}\right)dt \\ &-\int_0^1 t^\alpha f'\left(ta+(1-t)\frac{3a+b}{4}\right)dt \\ &+\int_0^1 (1-t^\alpha) f'\left(t\frac{a+3b}{4}+(1-t)b\right)dt \\ &-\int_0^1 t^\alpha f'\left(t\frac{a+b}{2}+(1-t)\frac{a+3b}{4}\right)dt]\end{aligned}$$

where for $\alpha > 0$

$$\begin{aligned}&\Phi_\alpha(a,b) \\ &= \frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \\ &\times\left[J_{a^+}^\alpha f\left(\frac{3a+b}{4}\right)+J_{\left(\frac{3a+b}{4}\right)^+}^\alpha f\left(\frac{a+b}{2}\right)+J_{\left(\frac{a+b}{2}\right)^+}^\alpha f\left(\frac{a+3b}{4}\right)+J_{\left(\frac{a+3b}{4}\right)^+}^\alpha f(b)\right].\end{aligned}$$

It is easy to see that

$$\Phi_1(a,b) = \frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx.$$

Theorem 2.1 *Let* $f: I \subset \mathbb{R} \to \mathbb{R}$ *be differentiable on* I^o *such that* $f' \in L([a,b])$ *with* $a,b \in I$, $a < b$ *and* $\alpha > 0$. *If* $|f'|^q$ *is quasi-convex function on* $[a,b]$ *and* $q \geq 1$, *then we have the following inequality:*

$$
\begin{aligned}
&\left|\Phi_\alpha(a,b)\right| \\
&\le \frac{b-a}{16}\Bigg[\frac{\left(\max\left\{|f'(a)|^q,\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right\}+\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'\left(\frac{a+3b}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}}}{\alpha+1} \\
&+\frac{\alpha}{\alpha+1}\left(\max\left\{\left|f'\left(\frac{3a+b}{4}\right)\right|^q,\left|f'\left(\frac{a+b}{2}\right)\right|^q\right\}+\max\left\{\left|f'\left(\frac{a+3b}{4}\right)\right|^q,|f'(b)|^q\right\}\right)^{\frac{1}{q}}\Bigg].
\end{aligned}
$$

Proof. Using Lemma 2.1 and well known power mean inequality and the quasi-convexity of $|f'|^q$ on $[a,b]$, we get

$$
\begin{aligned}
&\left|\Phi_\alpha(a,b)\right| \\
&\le \frac{b-a}{16}\Bigg[\int_0^1 t^\alpha\left|f'\left(ta+(1-t)\frac{3a+b}{4}\right)\right|dt+\int_0^1(1-t^\alpha)\left|f'\left(t\frac{3a+b}{4}+(1-t)\frac{a+b}{2}\right)\right|dt \\
&+\int_0^1 t^\alpha\left|f'\left(t\frac{a+b}{2}+(1-t)\frac{a+3b}{4}\right)\right|dt+\int_0^1(1-t^\alpha)\left|f'\left(t\frac{a+3b}{4}+(1-t)b\right)\right|dt\Bigg] \\
&\le \frac{b-a}{16}\Bigg[\left(\int_0^1 t^\alpha dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^\alpha \max\left\{|f'(a)|^q,\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right\}dt\right)^{\frac{1}{q}} \\
&+\left(\int_0^1(1-t^\alpha)dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t^\alpha)\max\left\{\left|f'\left(\frac{3a+b}{4}\right)\right|^q,\left|f'\left(\frac{a+b}{2}\right)\right|^q\right\}dt\right)^{\frac{1}{q}} \\
&+\left(\int_0^1 t^\alpha dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^\alpha \max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'\left(\frac{a+3b}{4}\right)\right|^q\right\}dt\right)^{\frac{1}{q}} \\
&+\left(\int_0^1(1-t^\alpha)dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t^\alpha)\max\left\{\left|f'\left(\frac{a+3b}{4}\right)\right|^q,|f'(b)|^q\right\}dt\right)^{\frac{1}{q}}\Bigg].
\end{aligned}
$$

Substituting

$$\int_0^1 t^\alpha dt=\frac{1}{\alpha+1}$$

and

$$\int_0^1 (1-t^\alpha)dt = \frac{\alpha}{\alpha+1}$$

into the above inequality and simplifying lead to the required inequality. The proof of Theorem 2.1 is complete.

Corollary 2.1 *In Theorem 2.1, if we choose* $q=1$*, we have*

$$|\Phi_\alpha(a,b)|$$
$$\leq \frac{b-a}{16}[\frac{\max\left\{|f'(a)|,\left|f'\left(\frac{3a+b}{4}\right)\right|\right\}+\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|,\left|f'\left(\frac{a+3b}{4}\right)\right|\right\}}{\alpha+1}$$
$$+\frac{\alpha}{\alpha+1}\left(\max\left\{\left|f'\left(\frac{3a+b}{4}\right)\right|,\left|f'\left(\frac{a+b}{2}\right)\right|\right\}+\max\left\{\left|f'\left(\frac{a+3b}{4}\right)\right|,|f'(b)|\right\}\right)].$$

Corollary 2.2 *In Theorem 2.1, if we choose* $\alpha=1$*, we have*

$$|\Phi_1(a,b)|$$
$$\leq \frac{b-a}{32}[\max\left\{|f'(a)|^q,\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right\}^{\frac{1}{q}}+\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'\left(\frac{a+3b}{4}\right)\right|^q\right\}^{\frac{1}{q}}$$
$$+\max\left\{\left|f'\left(\frac{3a+b}{4}\right)\right|^q,\left|f'\left(\frac{a+b}{2}\right)\right|^q\right\}^{\frac{1}{q}}+\max\left\{\left|f'\left(\frac{a+3b}{4}\right)\right|^q,|f'(b)|^q\right\}^{\frac{1}{q}}].$$

Corollary 2.3 *In Theorem 2.1, if we choose* $q=1$ *and* $\alpha=1$*, we have*

$$|\Phi_1(a,b)|\leq \frac{b-a}{32}[\max\left\{|f'(a)|,\left|f'\left(\frac{3a+b}{4}\right)\right|\right\}+\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|,\left|f'\left(\frac{a+3b}{4}\right)\right|\right\}$$
$$+\max\left\{\left|f'\left(\frac{3a+b}{4}\right)\right|,\left|f'\left(\frac{a+b}{2}\right)\right|\right\}+\max\left\{\left|f'\left(\frac{a+3b}{4}\right)\right|,|f'(b)|\right\}].$$

Theorem 2.2 *Let* $f: I\subset \mathbb{R}\to\mathbb{R}$ *be differentiable on* I^o *such that* $f'\in L([a,b])$ *with* $a,b\in I$*,* $a<b$ *and* $\alpha>0$*. If* $|f'|^q$ *is quasi-convex function on* $[a,b]$*,* $q>1$ *and* $q\geq r\geq 0$*, then we have the following:*

$$
\begin{aligned}
&\left|\Phi_\alpha(a,b)\right| \\
&\le \frac{b-a}{16}\left[\left(\frac{q-1}{\alpha(q-r)+q-1}\right)^{1-\frac{1}{q}}\left(\frac{1}{\alpha r+1}\right)^{\frac{1}{q}}\right. \\
&\times\left(\max\left\{|f'(a)|^q,\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right\}+\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'\left(\frac{a+3b}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}} \\
&+\left(\frac{1}{\alpha}\frac{\Gamma\left(\frac{2q-r-1}{q-1}\right)\Gamma\left(\frac{1}{\alpha}\right)}{\Gamma\left(\frac{2q-r-1}{q-1}+\frac{1}{\alpha}\right)}\right)^{1-\frac{1}{q}}\left(\frac{\Gamma\left(1+\frac{1}{\alpha}\right)\Gamma(r+1)}{\Gamma\left(r+\frac{1}{\alpha}+1\right)}\right)^{\frac{1}{q}} \\
&\left.\times\left(\max\left\{\left|f'\left(\frac{3a+b}{4}\right)\right|^q,\left|f'\left(\frac{a+b}{2}\right)\right|^q\right\}+\max\left\{\left|f'\left(\frac{a+3b}{4}\right)\right|^q,|f'(b)|^q\right\}\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

Proof. Using Lemma 2.1 and well known Hölder inequality and the quasi-convexity of $|f'|^q$ on $[a,b]$, we get

$$
\begin{aligned}
&\left|\Phi_\alpha(a,b)\right| \\
&\le \frac{b-a}{16}\left[\int_0^1 t^\alpha\left|f'\left(ta+(1-t)\frac{3a+b}{4}\right)\right|dt+\int_0^1(1-t^\alpha)\left|f'\left(t\frac{3a+b}{4}+(1-t)\frac{a+b}{2}\right)\right|dt\right. \\
&\left.+\int_0^1 t^\alpha\left|f'\left(t\frac{a+b}{2}+(1-t)\frac{a+3b}{4}\right)\right|dt+\int_0^1(1-t^\alpha)\left|f'\left(t\frac{a+3b}{4}+(1-t)b\right)\right|dt\right] \\
&\le \frac{b-a}{16}\left[\left(\int_0^1 t^{\alpha\left(\frac{q-r}{q-1}\right)}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha r}\max\left\{|f'(a)|^q,\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right\}dt\right)^{\frac{1}{q}}\right. \\
&+\left(\int_0^1(1-t^\alpha)^{\left(\frac{q-r}{q-1}\right)}dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t^\alpha)^r\max\left\{\left|f'\left(\frac{3a+b}{4}\right)\right|^q,\left|f'\left(\frac{a+b}{2}\right)\right|^q\right\}dt\right)^{\frac{1}{q}} \\
&+\left(\int_0^1 t^{\alpha\left(\frac{q-r}{q-1}\right)}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha r}\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'\left(\frac{a+3b}{4}\right)\right|^q\right\}dt\right)^{\frac{1}{q}} \\
&+\left(\int_0^1(1-t^\alpha)^{\left(\frac{q-r}{q-1}\right)}dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t^\alpha)^r\max\left\{\left|f'\left(\frac{a+3b}{4}\right)\right|^q,|f'(b)|^q\right\}dt\right)^{\frac{1}{q}}
\end{aligned}
$$

Substituting

$$\int_0^1 t^{\alpha r}\,dt = \frac{1}{\alpha r+1},$$

$$\int_0^1 t^{\alpha\left(\frac{q-r}{q-1}\right)}\,dt = \frac{q-1}{\alpha(q-r)+q-1},$$

$$\int_0^1 (1-t^{\alpha})^{\left(\frac{q-r}{q-1}\right)}\,dt = \frac{1}{\alpha}\frac{\Gamma\left(\frac{2q-r-1}{q-1}\right)\Gamma\left(\frac{1}{\alpha}\right)}{\Gamma\left(\frac{2q-r-1}{q-1}+\frac{1}{\alpha}\right)},$$

and

$$\int_0^1 (1-t^{\alpha})^{r}\,dt = \frac{\Gamma\left(1+\frac{1}{\alpha}\right)\Gamma(r+1)}{\Gamma\left(r+\frac{1}{\alpha}+1\right)}$$

into the above inequality and simplifying lead to the required inequality. The proof of Theorem 2.2 is complete.

Corollary 2.4 *In Theorem 2.2, if we choose* $r=0$*, we have*

$$\begin{aligned}
&\left|\Phi_{\alpha}(a,b)\right| \\
&\le \frac{b-a}{16}[\left(\frac{q-1}{\alpha q+q-1}\right)^{1-\frac{1}{q}} \\
&\times\left(\max\left\{\left|f'(a)\right|^{q},\left|f'\left(\frac{3a+b}{4}\right)\right|^{q}\right\}+\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^{q},\left|f'\left(\frac{a+3b}{4}\right)\right|^{q}\right\}\right)^{\frac{1}{q}} \\
&+\left(\frac{1}{\alpha}\frac{\Gamma\left(\frac{2q-1}{q-1}\right)\Gamma\left(\frac{1}{\alpha}\right)}{\Gamma\left(\frac{2q-1}{q-1}+\frac{1}{\alpha}\right)}\right)^{1-\frac{1}{q}} \\
&\times\left(\max\left\{\left|f'\left(\frac{3a+b}{4}\right)\right|^{q},\left|f'\left(\frac{a+b}{2}\right)\right|^{q}\right\}+\max\left\{\left|f'\left(\frac{a+3b}{4}\right)\right|^{q},\left|f'(b)\right|^{q}\right\}\right)^{\frac{1}{q}}].
\end{aligned}$$

Corollary 2.5 *In Theorem 2.2, if we choose* $r = q$*, we have*

$$\left|\Phi_\alpha(a,b)\right|$$
$$\leq \frac{b-a}{16}[\left(\frac{1}{\alpha q+1}\right)^{\frac{1}{q}}$$
$$\times\left(\max\left\{\left|f'(a)\right|^q, \left|f'\left(\frac{3a+b}{4}\right)\right|^q\right\} + \max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q, \left|f'\left(\frac{a+3b}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{\alpha}\frac{\Gamma\left(\frac{1}{\alpha}\right)}{\Gamma\left(1+\frac{1}{\alpha}\right)}\right)^{1-\frac{1}{q}}\left(\frac{\Gamma\left(1+\frac{1}{\alpha}\right)\Gamma\left(q+1\right)}{\Gamma\left(1+\frac{1}{\alpha}+q\right)}\right)^{\frac{1}{q}}$$
$$\times\left(\max\left\{\left|f'\left(\frac{3a+b}{4}\right)\right|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right\} + \max\left\{\left|f'\left(\frac{a+3b}{4}\right)\right|^q, \left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}].$$

Corollary 2.6 *In Theorem 2.2, if we choose* $\alpha = 1$*, we have*

$$\left|\Phi_1(a,b)\right|$$
$$\leq \frac{b-a}{16}[\left(\frac{q-1}{2q-r-1}\right)^{1-\frac{1}{q}}\left(\frac{1}{r+1}\right)^{\frac{1}{q}}$$
$$\times\left(\max\left\{\left|f'(a)\right|^q, \left|f'\left(\frac{3a+b}{4}\right)\right|^q\right\} + \max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q, \left|f'\left(\frac{a+3b}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}}$$
$$+\left(\frac{\Gamma\left(\frac{2q-r-1}{q-1}\right)}{\Gamma\left(\frac{2q-r-1}{q-1}+1\right)}\right)^{1-\frac{1}{q}}\left(\frac{\Gamma(r+1)}{\Gamma(r+2)}\right)^{\frac{1}{q}}$$
$$\times\left(\max\left\{\left|f'\left(\frac{3a+b}{4}\right)\right|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right\} + \max\left\{\left|f'\left(\frac{a+3b}{4}\right)\right|^q, \left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}].$$

Remark 2.1 *If we choose* $r=1$, *then Theorem 2.2 reduces the Theorem 2.1.*

Theorem 2.3 *Let* $f: I \subset \mathbb{R} \to \mathbb{R}$ *be differentiable on* I^o *such that* $f' \in L([a,b])$ *with* $a,b \in I$, $a<b$ *and* $\alpha>0$. *If* $|f'|^q$ *is quasi-convex function on* $[a,b]$, $q>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$, *then we have the following inequality:*

$$\begin{aligned}
&|\Phi_\alpha(a,b)| \\
&\le \frac{b-a}{16}[\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}} \\
&\times\left(\max\left\{|f'(a)|^q,\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right\}+\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'\left(\frac{a+3b}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}} \\
&+\left(\frac{\Gamma\left(1+\frac{1}{\alpha}\right)\Gamma(p+1)}{\Gamma\left(p+\frac{1}{\alpha}+1\right)}\right)^{\frac{1}{p}} \\
&\times\left(\max\left\{\left|f'\left(\frac{3a+b}{4}\right)\right|^q,\left|f'\left(\frac{a+b}{2}\right)\right|^q\right\}+\max\left\{\left|f'\left(\frac{a+3b}{4}\right)\right|^q,|f'(b)|^q\right\}\right)^{\frac{1}{q}}].
\end{aligned}$$

Proof. Using Lemma 2.1 and well known Hölder inequality and the quasi-convexity of $|f'|^q$ on $[a,b]$, we get

$$\begin{aligned}
&|\Phi_\alpha(a,b)| \\
&\le \frac{b-a}{16}[\int_0^1 t^\alpha\left|f'\left(ta+(1-t)\frac{3a+b}{4}\right)\right|dt+\int_0^1(1-t^\alpha)\left|f'\left(t\frac{3a+b}{4}+(1-t)\frac{a+b}{2}\right)\right|dt \\
&+\int_0^1 t^\alpha\left|f'\left(t\frac{a+b}{2}+(1-t)\frac{a+3b}{4}\right)\right|dt+\int_0^1(1-t^\alpha)\left|f'\left(t\frac{a+3b}{4}+(1-t)b\right)\right|dt] \\
&\le \frac{b-a}{16}[\left(\int_0^1(t^\alpha)^p\,dt\right)^{\frac{1}{p}}\left(\max\left\{|f'(a)|^q,\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right\}dt\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\left(\int_0^1 (1-t^\alpha)^p\,dt\right)^{\frac{1}{p}}\left(\max\left\{\left|f'\left(\frac{3a+b}{4}\right)\right|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right\}dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 (t^\alpha)^p\,dt\right)^{\frac{1}{p}}\left(\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q, \left|f'\left(\frac{a+3b}{4}\right)\right|^q\right\}dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 (1-t^\alpha)^p\,dt\right)^{\frac{1}{p}}\left(\max\left\{\left|f'\left(\frac{a+3b}{4}\right)\right|^q, |f'(b)|^q\right\}dt\right)^{\frac{1}{q}}.$$

Substituting

$$\int_0^1 (t^\alpha)^p\,dt = \frac{1}{\alpha p+1},$$

$$\int_0^1 (1-t^\alpha)^p\,dt = \frac{\Gamma\left(1+\frac{1}{\alpha}\right)\Gamma(p+1)}{\Gamma\left(p+\frac{1}{\alpha}+1\right)},$$

into the above inequality and simplifying lead to the required inequality. The proof of Theorem 2.3 is complete.

Corollary 2.7 *In Theorem 2.3, if we choose* $\alpha = 1$, *we have*

$$|\Phi_1(a,b)|$$
$$\leq \frac{b-a}{16}\Bigg[\left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$
$$\times\left(\max\left\{|f'(a)|^q, \left|f'\left(\frac{3a+b}{4}\right)\right|^q\right\}+\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q, \left|f'\left(\frac{a+3b}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}}$$
$$+\left(\frac{\Gamma(p+1)}{\Gamma(p+2)}\right)^{\frac{1}{p}}$$
$$\times\left(\max\left\{\left|f'\left(\frac{3a+b}{4}\right)\right|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right\}+\max\left\{\left|f'\left(\frac{a+3b}{4}\right)\right|^q, |f'(b)|^q\right\}\right)^{\frac{1}{q}}\Bigg].$$

3. Applications

We shall consider the means for two positive numbers $a>0$ and $b>0$. We take

1. Arithmetic mean:
$$A(a,b)=\frac{a+b}{2}.$$
2. Harmonic mean:
$$H(a,b)=\frac{2ab}{a+b}.$$
3. Generalized log-mean:
$$L_p(a,b)=\begin{cases}\left[\dfrac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & p\neq 0,-1 \; and \; a\neq b,\\ \dfrac{b-a}{\ln b-\ln a}, & p=-1 \; and \; a\neq b,\\ I(a,b), & p=0 \; and \; a\neq b,\\ a, & a=b.\end{cases}$$

Now, using the results of Section 2, we give some applications to special means of numbers.

Theorem 3.1 *Let* $b>a>0$, $q\geq 1$ *and* $p\in\mathbb{R}$.

1. If $p>1$ and $q(p-1)\leq 1$, or $p<0$ and $p\neq -1$ then
$$\left|\frac{A(a^p,b^p)+[A(a,b)]^p}{2}-[L_p(a,b)]^p\right|$$
$$\leq |p|^q\frac{b-a}{32}[\max\left\{a^{q(p-1)},A(a,A(a,b))^{q(p-1)}\right\}^{\frac{1}{q}}+\max\left\{A(a,b)^{q(p-1)},A(A(a,b),b)^{q(p-1)}\right\}^{\frac{1}{q}}$$
$$+\max\left\{A(a,A(a,b))^{q(p-1)},A(a,b)^{q(p-1)}\right\}^{\frac{1}{q}}+\max\left\{A(A(a,b),b)^{q(p-1)},b^{q(p-1)}\right\}^{\frac{1}{q}}].$$
2. If $p=-1$, then
$$\left|\frac{1}{2}\left[\frac{1}{H(a,b)}+\frac{1}{A(a,b)}\right]-\frac{1}{L(a,b)}\right|$$
$$\leq\frac{b-a}{32}[\max\left\{\frac{1}{a^{2q}},\frac{1}{A(a,A(a,b))^{2q}}\right\}^{\frac{1}{q}}+\max\left\{\frac{1}{A(a,b)^{2q}},\frac{1}{A(A(a,b),b)^{2q}}\right\}^{\frac{1}{q}}$$

$$+\max\left\{\frac{1}{A(a,A(a,b))^{2q}},\frac{1}{A(a,b)^{2q}}\right\}^{\frac{1}{q}}+\max\left\{\frac{1}{A(A(a,b),b)^{2q}},\frac{1}{b^{2q}}\right\}^{\frac{1}{q}}].$$

3. If $q=1$ and $p\geq 2$, or $q=1$ and $-1\neq p<0$, then

$$\left|\frac{A(a^p,b^p)+[A(a,b)]^p}{2}-[L_p(a,b)]^p\right|$$

$$\leq |p|\frac{(b-a)}{32}[\max\left\{a^{(p-1)},A(a,A(a,b))^{(p-1)}\right\}++\max\left\{A(a,b)^{(p-1)},A(A(a,b),b)^{(p-1)}\right\}$$
$$+\max\left\{A(a,A(a,b))^{(p-1)},A(a,b)^{(p-1)}\right\}+\max\left\{A(A(a,b),b)^{(p-1)},b^{(p-1)}\right\}].$$

4. If $p=-1$ and $q=1$, then

$$\left|\frac{1}{2}\left[\frac{1}{H(a,b)}+\frac{1}{A(a,b)}\right]-\frac{1}{L(a,b)}\right|$$
$$\leq\frac{b-a}{32}[\max\left\{\frac{1}{a^2},\frac{1}{A(a,A(a,b))^2}\right\}+\max\left\{\frac{1}{A(a,b)^2},\frac{1}{A(A(a,b),b)^2}\right\}$$
$$+\max\left\{\frac{1}{A(a,A(a,b))^2},\frac{1}{A(a,b)^2}\right\}+\max\left\{\frac{1}{A(A(a,b),b)^2},\frac{1}{b^2}\right\}].$$

Proof. The assertion follows from Corollary 2.2 applied to the quasi-convex mapping $f(x)=x^p$, $p\in\mathbb{R}$.

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