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Some infinite sums related to the generalized *k*-Fibonacci numbers

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Abstract

In this study, some infinite sums related to the generalized *k*-Fibonacci numbers have been obtained by using infinite sums related to classic Fibonacci numbers and generalized Fibonacci numbers in literature.

Keywords: Generalized *k*-Fibonacci numbers, *k*-Fibonacci numbers, sums of integer number sequences

Genelleştirilmiş *k*-Fibonacci sayılarıyla ilgili bazı sonsuz toplamlar

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Özet

Bu çalışmada, literatürdeki klasik Fibonacci sayıları ve genelleştirilmiş Fibonacci sayıları kullanılarak genelleştirilmiş *k*-Fibonacci sayılarıyla ilgili bazı sonsuz toplamlar elde edilmiştir.

Anahtar Kelimeler : Genelleştirilmiş *k*-Fibonacci sayıları, *k*-Fibonacci sayıları, tam sayı dizilerinin toplamları

1. Introduction

The well-known Fibonacci sequence and the golden ratio with the many interesting features have been attracted attention of theoretical physics, engineerings, architects, orthodontics as much as mathematicians [1-3-4-6-8-9]. Numerous features of this interesting number sequence have been found over time [2-4-11]. Different number sequences, such as the Pell and Lucas number sequences that relate to Fibonacci sequence, have been discussed along with studies on Fibonacci sequence, and their different generalizations have been mentioned [2-4-5-7-10-11-12]. Similarly, Falcon and Plaza introduced the *k*-Fibonacci sequence, which is a generalization of these number sequences, giving the classic Fibonacci sequence and the classic Pell sequence for k=1 and k=2, respectively. For any integer number $k\geq 1$, the kth Fibonacci sequence $\{F_{k,n}\}_{n\in\mathbb{N}}$ is defined recurrently by

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \quad (n \geq 1) \tag{1.1}$$

where $F_{k,0}=0$, $F_{k,1}=1$. The solution of the equation (1.1) is

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$$F_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}, \tag{1.2}$$

where the roots of characteristic equation of (1.1) are $r_1 = \frac{k+\sqrt{k^2+4}}{2}$, $r_2 = \frac{k-\sqrt{k^2+4}}{2}$ [2]. Later, in [11], for any integer number $k \geq 1$, the generalized kth Fibonacci sequence $\{G_{k,n}\}_{n\in\mathbb{N}}$, which are a generalization of k-Fibonacci sequence is defined recurrently by

$$G_{k,n+1} = kG_{k,n} + G_{k,n-1}, \quad (n \geq 1), \tag{1.3}$$

where $G_{k,0} = a$, $G_{k,1} = b$, a and b are real numbers. The solution of the equation (1.3) is

$$G_{k,n} = \frac{(b-ar_2)r_1^n - (b-ar_1)r_2^n}{r_1 - r_2}, \tag{1.4}$$

where the roots of characteristic equation of (1.3) are $r_1 = \frac{k+\sqrt{k^2+4}}{2}$, $r_2 = \frac{k-\sqrt{k^2+4}}{2}$ and $r_1 + r_2 = k, \quad r_1 r_2 = -1$ [11].

In this study, based on the some infinite sums of the Fibonacci numbers and generalized Fibonacci numbers [3] and [13] is investigated counterparts in the generalized k-Fibonacci numbers.

2. Main Results

In this section, we obtain some results related to the generalized k-Fibonacci numbers by using [3-10-13].

Theorem 2.1. For generalized k-Fibonacci numbers, the equality named Cassini formula

$$G_{k,n-1}.G_{k,n+1} - G_{k,n}^2 = (-1)^n(b^2 - abk - a^2) \tag{2.1}$$

holds.

Proof: From the (1.4), we can write

$$G_{k,n-1}.G_{k,n+1} - G_{k,n}^2 = \left[\frac{(b-ar_2)r_1^{n-1} - (b-ar_1)r_2^{n-1}}{r_1 - r_2}\right]\left[\frac{(b-ar_2)r_1^{n+1} - (b-ar_1)r_2^{n+1}}{r_1 - r_2}\right] - \left[\frac{(b-ar_2)r_1^n - (b-ar_1)r_2^n}{r_1 - r_2}\right]^2$$

$$G_{k,n-1}.G_{k,n+1} - G_{k,n}^2 = \frac{\left[\begin{array}{l} -(b^2 - ab(r_1+r_2) + a^2 r_1 r_2)(r_1 r_2)^{n-1} r_2^2 - (b^2 - ab(r_1+r_2) + a^2 r_1 r_2)(r_1 r_2)^{n-1} r_1^2 + \\ 2(b^2 - ab(r_1+r_2) + a^2 r_1 r_2)(r_1 r_2)^n \end{array}\right]}{(r_1 - r_2)^2}.$$

By using relations $r_1 + r_2 = k, \quad r_1 r_2 = -1$, we have

$$G_{k,n-1}.G_{k,n+1} - G_{k,n}^2 = \frac{\left[-(b^2 - abk - a^2)(-1)^{n-1} r_2^2 - (b^2 - abk - a^2)(-1)^{n-1} r_1^2 + 2(b^2 - abk - a^2)(-1)^n\right]}{(r_1 - r_2)^2}.$$

From the (1.3), we have equations $r_1^2 = kr_1 + 1, \quad r_2^2 = kr_2 + 1$. If we use these relations in the last eqution, then we obtain

$$G_{k,n-1}.G_{k,n+1} - G_{k,n}^2 = \frac{(b^2 - abk - a^2)(-1)^n (k(r_1 + r_2) + 4)}{(r_1 - r_2)^2}.$$

From the relation $(r_1 - r_2) = k^2 + 4$, we get the he following equation

$$G_{k,n-1}.G_{k,n+1} - G_{k,n}^2 = (b^2 - abk - a^2)(-1)^n.$$

Theorem 2.2. For generalized k-Fibonacci numbers, the equality

$$\sum_{n=1}^{\infty}\frac{1}{G_{k,n}}=\left(\frac{k+1}{kb}+\frac{1}{bk^2+ak}\right)+(b^2-abk-a^2)\sum_{n=2}^{\infty}\frac{(-1)^n}{G_{k,n-1}G_{k,n}G_{k,n+1}} \qquad (2.2)$$

holds.

Proof: We can write the equality

$$\sum_{s=2}^{n}\left(\frac{1}{G_{k,s}}-\frac{G_{k,s}}{G_{k,s-1}G_{k,s+1}}\right)=\sum_{s=2}^{n}\left(\frac{G_{k,s-1}G_{k,s+1}-G_{k,s}^2}{G_{k,s-1}G_{k,s}G_{k,s+1}}\right)$$

By using Cassini formula $G_{k,s-1}.G_{k,s+1}-G_{k,s}^2=(b^2-abk-a^2)(-1)^s$ for generalized k-Fibonacci numbers in above equality, we have

$$\sum_{s=2}^{n}\left(\frac{1}{G_{k,s}}-\frac{G_{k,s}}{G_{k,s-1}G_{k,s+1}}\right)=\sum_{s=2}^{n}\left(\frac{(b^2-abk-a^2)(-1)^s}{G_{k,s-1}G_{k,s}G_{k,s+1}}\right) \qquad (2.3)$$

On the other hand, it is obvious from equation (1.3)

$$\sum_{s=2}^{n}\left(\frac{G_{k,s}}{G_{k,s-1}G_{k,s+1}}\right)=\frac{1}{k}\sum_{s=2}^{n}\left(\frac{G_{k,s+1}-G_{k,s-1}}{G_{k,s-1}G_{k,s+1}}\right)=\frac{1}{k}\sum_{s=2}^{n}\left(\frac{1}{G_{k,s-1}}-\frac{1}{G_{k,s+1}}\right)$$

$$\sum_{s=2}^{n}\left(\frac{G_{k,s}}{G_{k,s-1}G_{k,s+1}}\right)=\frac{1}{k}\left[\left(\frac{1}{G_{k,1}}-\frac{1}{G_{k,3}}\right)+\left(\frac{1}{G_{k,2}}-\frac{1}{G_{k,4}}\right)+\cdots+\left(\frac{1}{G_{k,n-1}}-\frac{1}{G_{k,n+1}}\right)\right]$$

$$\sum_{s=2}^{n}\left(\frac{G_{k,s}}{G_{k,s-1}G_{k,s+1}}\right)=\frac{1}{k}\left[\left(\frac{1}{b}+\frac{1}{bk+a}\right)-\left(\frac{1}{G_{k,n}}+\frac{1}{G_{k,n+1}}\right)\right].$$

If the limits of both sides of the last equation are taken for $n\to\infty$, then we have

$$\sum_{s=2}^{\infty}\left(\frac{G_{k,s}}{G_{k,s-1}G_{k,s+1}}\right)=\lim_{n\to\infty}\frac{1}{k}\left[\left(\frac{1}{b}+\frac{1}{bk+a}\right)-\left(\frac{1}{G_{k,n}}+\frac{1}{G_{k,n+1}}\right)\right].$$

From equation (1.4), we write

$$\sum_{s=2}^{\infty}\left(\frac{G_{k,s}}{G_{k,s-1}G_{k,s+1}}\right)=\lim_{n\to\infty}\frac{1}{k}\left[\left(\frac{1}{b}+\frac{1}{bk+a}\right)-\left(\frac{r_1-r_2}{(b-ar_2)r_1^n-(b-ar_1)r_2^n}+\frac{r_1-r_2}{(b-ar_2)r_1^{n+1}-(b-ar_1)r_2^{n+1}}\right)\right].$$

$$\sum_{s=2}^{\infty}\left(\frac{G_{k,s}}{G_{k,s-1}G_{k,s+1}}\right)=\lim_{n\to\infty}\frac{1}{k}\left[\left(\frac{1}{b}+\frac{1}{bk+a}\right)-\left(\frac{r_1-r_2}{r_1^n\left[(b-ar_2)-(b-ar_1)\left(\frac{r_2}{r_1}\right)^n\right]}+\frac{r_1-r_2}{r_1^{n+1}\left[(b-ar_2)-(b-ar_1)\left(\frac{r_2}{r_1}\right)^{n+1}\right]}\right)\right].$$

It is obvious that $\lim_{n\to\infty}\left(\frac{r_2}{r_1}\right)^n=0$ from $r_2<r_1$. Thus we have

$$\sum_{s=2}^{\infty}\left(\frac{G_{k,s}}{G_{k,s-1}G_{k,s+1}}\right)=\frac{1}{k}\left(\frac{1}{b}+\frac{1}{bk+a}\right).$$

If the limits of both sides of equation (2.3) are taken for $n\to\infty$, then we have

$$\sum_{s=2}^{\infty}\left(\frac{1}{G_{k,s}}-\frac{G_{k,s}}{G_{k,s-1}G_{k,s+1}}\right)=\sum_{s=2}^{\infty}\left(\frac{(b^2-abk-a^2)(-1)^s}{G_{k,s-1}G_{k,s}G_{k,s+1}}\right)$$

From the last equation, we can write

$$\sum_{n=2}^{\infty}\left(\frac{1}{G_{k,n}}\right)=\sum_{n=2}^{\infty}\left(\frac{G_{k,n}}{G_{k,n-1}G_{k,n+1}}+\frac{(b^2-abk-a^2)(-1)^n}{G_{k,n-1}G_{k,n}G_{k,n+1}}\right).$$

Thus we obtain

$$\sum_{n=1}^{\infty}\left(\frac{1}{G_{k,n}}\right)=\frac{1}{G_{k,1}}+\sum_{n=2}^{\infty}\left(\frac{G_{k,n}}{G_{k,n-1}G_{k,n+1}}+\frac{(b^2-abk-a^2)(-1)^n}{G_{k,n-1}G_{k,n}G_{k,n+1}}\right)$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{G_{k,n}}\right)=\frac{1}{b}+\frac{1}{k}\left(\frac{1}{b}+\frac{1}{bk+a}\right)+(b^2-abk-a^2)\sum_{n=2}^{\infty}\left(\frac{(-1)^n}{G_{k,n-1}G_{k,n}G_{k,n+1}}\right)$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{G_{k,n}}\right)=\frac{k+1}{bk}+\frac{1}{bk^2+ak}+(b^2-abk-a^2)\sum_{n=2}^{\infty}\left(\frac{(-1)^n}{G_{k,n-1}G_{k,n}G_{k,n+1}}\right).$$

Theorem 2.3. For $n\geq 2$, the equality

$$\sum_{n=2}^{\infty}\left(\frac{1}{G_{k,n-1}G_{k,n+1}}\right)=\frac{1}{(b^2k^2+abk)}$$

holds.

Proof: We can write the equality

$$\sum_{s=2}^{n}\left(\frac{1}{G_{k,s-1}G_{k,s+1}}\right)=\sum_{s=2}^{n}\left(\frac{G_{k,s}}{G_{k,s}G_{k,s-1}G_{k,s+1}}\right).$$

From the equation (1.3), we have

$$\sum_{s=2}^{n}\left(\frac{1}{G_{k,s-1}G_{k,s+1}}\right)=\sum_{s=2}^{n}\frac{1}{k}\left(\frac{G_{k,s+1}-G_{k,s-1}}{G_{k,s}G_{k,s-1}G_{k,s+1}}\right)=\frac{1}{k}\sum_{s=2}^{n}\left(\frac{1}{G_{k,s}G_{k,s-1}}-\frac{1}{G_{k,s}G_{k,s+1}}\right)$$

$$\sum_{s=2}^{n}\left(\frac{1}{G_{k,s-1}G_{k,s+1}}\right)=\frac{1}{k}\left[\left(\frac{1}{G_{k,1}G_{k,2}}-\frac{1}{G_{k,2}G_{k,3}}\right)+\left(\frac{1}{G_{k,2}G_{k,3}}-\frac{1}{G_{k,3}G_{k,4}}\right)+\cdots+\left(\frac{1}{G_{k,n}G_{k,n-1}}-\frac{1}{G_{k,n}G_{k,n+1}}\right)\right]$$

$$\sum_{s=2}^{n}\left(\frac{1}{G_{k,s-1}G_{k,s+1}}\right)=\frac{1}{k}\left[\frac{1}{G_{k,1}G_{k,2}}-\frac{1}{G_{k,n}G_{k,n+1}}\right]=\frac{1}{k}\left[\frac{1}{b(kb+a)}-\frac{1}{G_{k,n}G_{k,n+1}}\right].$$

If the limits of both sides of the last equation are taken for $n\to\infty$, then we can write

$$\lim_{n\to\infty}\sum_{s=2}^{n}\left(\frac{1}{G_{k,s-1}G_{k,s+1}}\right)=\lim_{n\to\infty}\frac{1}{k}\left[\frac{1}{b(kb+a)}-\frac{1}{G_{k,n}G_{k,n+1}}\right].$$

By considering $\lim_{n\to\infty}\frac{1}{G_{k,n}G_{k,n+1}}=0$, we obtain

$$\sum_{s=2}^{\infty}\left(\frac{1}{G_{k,s-1}G_{k,s+1}}\right)=\left[\frac{1}{b^2k^2+abk}\right].$$

Theorem 2.4. For $n\geq 2$, the equality

$$\sum_{n=1}^{\infty}\left(\frac{1}{G_{k,n}G_{k,n+2}^2G_{k,n+3}}\right)+\sum_{n=1}^{\infty}\left(\frac{1}{G_{k,n}G_{k,n+1}^2G_{k,n+3}}\right)=\left[\frac{1}{bk(kb+a)^2(k^2b+ak+b)}\right]$$

holds.

Proof:

$$\sum_{s=1}^{n}\left(\frac{1}{G_{k,s}G_{k,s+2}^2G_{k,s+3}}\right)+\sum_{s=1}^{n}\left(\frac{1}{G_{k,s}G_{k,s+1}^2G_{k,s+3}}\right)$$

$$=\sum_{s=1}^{n}\left(\frac{G_{k,s+1}}{G_{k,s}G_{k,s+1}G_{k,s+2}^2G_{k,s+3}}\right)+\sum_{s=1}^{n}\left(\frac{G_{k,s+2}}{G_{k,s}G_{k,s+1}^2G_{k,s+2}G_{k,s+3}}\right)$$

From the equation (1.3), we can write the following equality

$$=\frac{1}{k}\sum_{s=1}^{n}\left(\frac{G_{k,s+2}-G_{k,s}}{G_{k,s}G_{k,s+1}G_{k,s+2}^2G_{k,s+3}}\right)+\frac{1}{k}\sum_{s=1}^{n}\left(\frac{G_{k,s+3}-G_{k,s+1}}{G_{k,s}G_{k,s+1}^2G_{k,s+2}G_{k,s+3}}\right)$$

$$=\frac{1}{k}\sum_{s=1}^{n}\left(\frac{1}{G_{k,s}G_{k,s+1}^2G_{k,s+2}}-\frac{1}{G_{k,s+1}G_{k,s+2}^2G_{k,s+3}}\right)$$

$$=\frac{1}{k}\left[\left(\frac{1}{G_{k,1}G_{k,2}^2G_{k,3}}-\frac{1}{G_{k,2}G_{k,3}^2G_{k,4}}\right)+\cdots+\left(\frac{1}{G_{k,n}G_{k,n+1}^2G_{k,n+2}}-\frac{1}{G_{k,n+1}G_{k,n+2}^2G_{k,n+3}}\right)\right]$$

Thus we obtain

$$\sum_{s=1}^{n}\left(\frac{1}{G_{k,s}G_{k,s+2}^2G_{k,s+3}}\right)+\sum_{s=1}^{n}\left(\frac{1}{G_{k,s}G_{k,s+1}^2G_{k,s+3}}\right)=\frac{1}{k}\left[\frac{1}{b(kb+a)^2(k^2b+ak+b)}-\frac{1}{G_{k,n+1}G_{k,n+2}^2G_{k,n+3}}\right]$$

If the limits of both sides of the last equation are taken for $n\to\infty$, then we can write

$$\sum_{s=1}^{\infty}\left(\frac{1}{G_{k,s}G_{k,s+2}^2G_{k,s+3}}\right)+\sum_{s=1}^{\infty}\left(\frac{1}{G_{k,s}G_{k,s+1}^2G_{k,s+3}}\right)=\lim_{n\to\infty}\left[\frac{1}{bk(kb+a)^2(k^2b+ak+b)}-\frac{1}{kG_{k,n+1}G_{k,n+2}^2G_{k,n+3}}\right].$$

By considering $\lim_{n\to\infty}\left[\frac{1}{kG_{k,n+1}G_{k,n+2}^2G_{k,n+3}}\right]=0$, we obtain

$$\sum_{s=1}^{\infty}\left(\frac{1}{G_{k,s}G_{k,s+2}^2G_{k,s+3}}\right)+\sum_{s=1}^{\infty}\left(\frac{1}{G_{k,s}G_{k,s+1}^2G_{k,s+3}}\right)=\frac{1}{bk(kb+a)^2(k^2b+ak+b)}.$$

Theorem 2.5. For generalized *k*-Fibonacci numbers, the equalities

a) $G_{k,n+1}=b\prod_{s=1}^{n}\left(k+\frac{G_{k,s-1}}{G_{k,s}}\right)$

b) $\frac{G_{k,n+1}}{G_{k,n}}=\frac{bk+a}{b}+(b^2-abk-a^2)\sum_{s=2}^{n}\left(\frac{(-1)^s}{G_{k,s}G_{k,s-1}}\right)$

hold.

Proof: a)

$$G_{k,n+1}=G_{k,1}\left(\frac{G_{k,n+1}G_{k,n}\ldots G_{k,2}}{G_{k,n}G_{k,n-1}\ldots G_{k,1}}\right)=b\prod_{s=1}^{n}\left(\frac{G_{k,s+1}}{G_{k,s}}\right)$$

From the equation (1.3), we have

$$G_{k,n+1}=b\prod_{s=1}^{n}\left(\frac{kG_{k,s}+G_{k,s-1}}{G_{k,s}}\right)=b\prod_{s=1}^{n}\left(k+\frac{G_{k,s-1}}{G_{k,s}}\right).$$

b)

$$\frac{G_{k,n+1}}{G_{k,n}}=\left(\frac{G_{k,n+1}}{G_{k,n}}-\frac{G_{k,n}}{G_{k,n-1}}\right)+\left(\frac{G_{k,n}}{G_{k,n-1}}-\frac{G_{k,n-1}}{G_{k,n-2}}\right)+\cdots+\left(\frac{G_{k,3}}{G_{k,2}}-\frac{G_{k,2}}{G_{k,1}}\right)+\frac{G_{k,2}}{G_{k,1}}$$

$$\frac{G_{k,n+1}}{G_{k,n}}=\frac{G_{k,2}}{G_{k,1}}+\sum_{s=2}^{n}\left(\frac{G_{k,s+1}}{G_{k,s}}-\frac{G_{k,s}}{G_{k,s-1}}\right)=\frac{bk+a}{b}+\sum_{s=2}^{n}\left(\frac{G_{k,s+1}G_{k,s-1}-G_{k,s}^2}{G_{k,s}G_{k,s-1}}\right)$$

From Cassini formula for generalized *k*-Fibonacci numbers, we obtain

$$\frac{G_{k,n+1}}{G_{k,n}}=\frac{bk+a}{b}+(b^2-abk-a^2)\sum_{s=2}^{n}\left(\frac{(-1)^s}{G_{k,s}G_{k,s-1}}\right).$$

Theorem 2.6. For generalized *k*-Fibonacci numbers, the equality

$$\sum_{n=1}^{\infty}\left(\frac{G_{k,n}}{G_{k,n+1}G_{k,n+2}}\right)=\frac{1}{bk+a}+(1-k)\left[\frac{1}{kb}+\frac{1-k}{bk^2+ak}+(b^2-abk-a^2)\sum_{n=2}^{\infty}\left(\frac{(-1)^n}{G_{k,n-1}G_{k,n}G_{k,n+1}}\right)\right]$$

holds.

Proof: From the equation (1.3), we can write

$$\sum_{s=1}^{n}\left(\frac{G_{k,s}}{G_{k,s+1}G_{k,s+2}}\right)=\sum_{s=1}^{n}\left(\frac{G_{k,s+2}-kG_{k,s+1}}{G_{k,s+1}G_{k,s+2}}\right)=\sum_{s=1}^{n}\left(\frac{1}{G_{k,s+1}}-\frac{k}{G_{k,s+2}}\right)$$

$$\sum_{s=1}^{n}\left(\frac{1}{G_{k,s+1}}-\frac{k}{G_{k,s+2}}\right)=\left(\frac{1}{G_{k,2}}-\frac{k}{G_{k,3}}\right)+\left(\frac{1}{G_{k,3}}-\frac{k}{G_{k,4}}\right)+\left(\frac{1}{G_{k,4}}-\frac{k}{G_{k,5}}\right)+\cdots+\left(\frac{1}{G_{k,n+1}}-\frac{k}{G_{k,n+2}}\right)$$

$$\sum_{s=1}^{n}\left(\frac{1}{G_{k,s+1}}-\frac{k}{G_{k,s+2}}\right)=\frac{1}{G_{k,2}}-\frac{k}{G_{k,n+2}}+(1-k)\left(\frac{1}{G_{k,3}}+\frac{1}{G_{k,4}}+\cdots+\frac{1}{G_{k,n+1}}\right)$$

$$\sum_{s=1}^{n}\left(\frac{G_{k,s}}{G_{k,s+1}G_{k,s+2}}\right)=\frac{1}{G_{k,2}}-\frac{k}{G_{k,n+2}}+(1-k)\left(\frac{1}{G_{k,n+1}}-\frac{1}{G_{k,1}}-\frac{1}{G_{k,2}}+\sum_{s=1}^{n}\frac{1}{G_{k,s}}\right)$$

If the limits of both sides of the last equation for $n\to\infty$ are taken and necessary arrangements are made, then we write

$$\sum_{s=1}^{\infty}\left(\frac{G_{k,s}}{G_{k,s+1}G_{k,s+2}}\right)=\lim_{n\to\infty}\left[\frac{1}{bk+a}-\frac{k}{G_{k,n+2}}+(1-k)\left(\frac{1}{G_{k,n+1}}-\frac{1}{b}-\frac{1}{bk+a}+\sum_{s=1}^{n}\frac{1}{G_{k,s}}\right)\right].$$

By considering $\lim_{n\to\infty}\left[\frac{k}{G_{k,n+2}}\right]=0$ and $\lim_{n\to\infty}\left(\frac{1-k}{G_{k,n+1}}\right)=0$, and from the theorem 2.1, we obtain

$$\sum_{n=1}^{\infty}\left(\frac{G_{k,n}}{G_{k,n+1}G_{k,n+2}}\right)=\frac{1}{bk+a}+(1-k)\left[\frac{1}{kb}+\frac{1-k}{bk^2+ak}+(b^2-abk-a^2)\sum_{n=2}^{\infty}\left(\frac{(-1)^n}{G_{k,n-1}G_{k,n}G_{k,n+1}}\right)\right].$$

Theorem 2.7. For generalized *k*-Fibonacci numbers, the equality

$$\sum_{n=1}^{\infty}\left(\frac{G_{k,n+1}}{G_{k,n}G_{k,n+3}}\right)=\frac{1}{k^2+1}\left[\frac{1}{kb}+\frac{k^2-k+1}{bk^2+ak}+\frac{k}{bk^2+ak+b}+(1-k)\sum_{n=2}^{\infty}\left(\frac{(-1)^n(b^2-abk-a^2)}{G_{k,n-1}G_{k,n}G_{k,n+1}}\right)\right]$$

holds.

Proof: From the equation (1.3), we can write

$$G_{k,s+3} = kG_{k,s+2} + G_{k,s+1}$$
$$G_{k,s+3} = k(kG_{k,s+1} + G_{k,s}) + G_{k,s+1}$$
$$G_{k,s+3} = (k^2+1)G_{k,s+1} + kG_{k,s}.$$

From the last equation, it is obvious

$$G_{k,s+1} = \frac{G_{k,s+3} - kG_{k,s}}{k^2+1}. \qquad (2.4)$$

By using the equaiton (2.4) in the following equation, we have

$$\sum_{s=1}^{n}\left(\frac{G_{k,s+1}}{G_{k,s}G_{k,s+3}}\right) = \frac{1}{k^2+1}\left[\sum_{s=1}^{n}\left(\frac{G_{k,s+3}-kG_{k,s}}{G_{k,s}G_{k,s+3}}\right)\right] = \frac{1}{k^2+1}\left[\sum_{s=1}^{n}\left(\frac{1}{G_{k,s}} - \frac{k}{G_{k,s+3}}\right)\right]$$

From the last sum, we can write

$$\sum_{s=1}^{n}\left(\frac{G_{k,s+1}}{G_{k,s}G_{k,s+3}}\right) = \frac{1}{k^2+1}\left[\left(\frac{1}{G_{k,1}} - \frac{k}{G_{k,4}}\right) + \left(\frac{1}{G_{k,2}} - \frac{k}{G_{k,5}}\right) + \cdots + \left(\frac{1}{G_{k,n}} - \frac{k}{G_{k,n+3}}\right)\right]$$

$$\sum_{s=1}^{n}\left(\frac{G_{k,s+1}}{G_{k,s}G_{k,s+3}}\right) = \frac{1}{k^2+1}\left[\begin{array}{l}\left(\frac{1}{G_{k,1}} + \frac{1}{G_{k,2}} + \frac{1}{G_{k,3}} - \frac{k}{G_{k,n+1}} - \frac{k}{G_{k,n+2}} - \frac{k}{G_{k,n+3}}\right) + \\ +(1-k)\left[\left(\sum_{s=1}^{n}\frac{1}{G_{k,s}}\right) - \left(\frac{1}{G_{k,1}} + \frac{1}{G_{k,2}} + \frac{1}{G_{k,3}}\right)\right]\end{array}\right]$$

$$\sum_{s=1}^{n}\left(\frac{G_{k,s+1}}{G_{k,s}G_{k,s+3}}\right) = \frac{1}{k^2+1}\left[k\left(\frac{1}{G_{k,1}} + \frac{1}{G_{k,2}} + \frac{1}{G_{k,3}} - \frac{1}{G_{k,n+1}} - \frac{1}{G_{k,n+2}} - \frac{1}{G_{k,n+3}}\right) - (k-1)\left(\sum_{s=1}^{n}\frac{1}{G_{k,s}}\right)\right]$$

$$\sum_{s=1}^{n}\left(\frac{G_{k,s+1}}{G_{k,s}G_{k,s+3}}\right) = \frac{1}{k^2+1}\left[k\left(\frac{1}{b} + \frac{1}{bk+a} + \frac{1}{bk^2+ak+b} - \frac{1}{G_{k,n+1}} - \frac{1}{G_{k,n+2}} - \frac{1}{G_{k,n+3}}\right) - (k-1)\left(\sum_{s=1}^{n}\frac{1}{G_{k,s}}\right)\right]$$

If the limits of both sides of the last equation are taken for $n \to \infty$, then we can write

$$\sum_{s=1}^{\infty}\left(\frac{G_{k,s+1}}{G_{k,s}G_{k,s+3}}\right) = \lim_{n\to\infty}\frac{1}{k^2+1}\left[k\left(\frac{1}{b} + \frac{1}{bk+a} + \frac{1}{bk^2+ak+b} - \frac{1}{G_{k,n+1}} - \frac{1}{G_{k,n+2}} - \frac{1}{G_{k,n+3}}\right) - (k-1)\left(\sum_{s=1}^{n}\frac{1}{G_{k,s}}\right)\right].$$

By considering $\lim_{n\to\infty}\frac{1}{G_{k,n+1}} = 0$, $\lim_{n\to\infty}\frac{1}{G_{k,n+2}} = 0$, $\lim_{n\to\infty}\frac{1}{G_{k,n+3}} = 0$ and from the Theorem 2.1, we obtain

$$\sum_{n=1}^{\infty}\left(\frac{G_{k,n+1}}{G_{k,n}G_{k,n+3}}\right) = \frac{1}{k^2+1}\left[\begin{array}{l}\frac{k}{b} + \frac{k}{bk+a} + \frac{k}{bk^2+ak+b} - \\ (k-1)\left(\frac{k+1}{kb} + \frac{1}{bk^2+ak} + \sum_{n=2}^{\infty}\left(\frac{(-1)^n(b^2-abk-a^2)}{G_{k,n-1}G_{k,n}G_{k,n+1}}\right)\right)\end{array}\right]$$

$$\sum_{n=1}^{\infty}\left(\frac{G_{k,n+1}}{G_{k,n}G_{k,n+3}}\right) = \frac{1}{k^2+1}\left[\frac{1}{kb} + \frac{k^2-k+1}{bk^2+ak} + \frac{k}{bk^2+ak+b} + (1-k)\sum_{n=2}^{\infty}\left(\frac{(-1)^n(b^2-abk-a^2)}{G_{k,n-1}G_{k,n}G_{k,n+1}}\right)\right]$$

3. Conclusions

In the presented study, it has been obtained some infinite sums related to generalized *k*-Fibonacci numbers. By using this study, Sums related to Pell and Jacobsthal numbers can be research.

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