MINIMAL BUT INEFFICIENT PRESENTATIONS OF THE SEMI-DIRECT PRODUCT OF FINITE CYCLIC GROUPS

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ABSTRACT

Let G be a semi-direct product of B by A where B and A are both cyclic groups of order n ($n \in$ N) and p (any prime), respectively. As a main result of this paper, we prove that G has an inefficient but minimal presentation. Then, as an application of this result, we show that a metacyclic group satisfy the main result.

ÖZET

B ve A nın her ikiside sırasıyla n ($n \in$ N) ve p (asal) mertebeli devirli gruplar olmak üzere, G grubu B nin A ile yarı-direkt çarpımı olsun. Bu çalışmanın ana sonucu olarak G nin etkili olmayan ancak minimal olan bir sunuşa sahip olduğunu ispatladık. Daha sonra bu sonucun bir uygulaması olarak metadevirli grupların bu sonucu sağladığını gösterdik.

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1. INTRODUCTION

1.1 Efficiency

Let G be a finitely presented group, and let

$$P = \langle \mathbf{x} ; \mathbf{r} \rangle \quad (1)$$

be a finite presentation for G. Then the Euler characteristic of P is defined by $x(P) = 1 - |\mathbf{x}|+|\mathbf{r}|$, where $|.|$ denotes the number of element in the set. Let

$$\delta(G) = 1 - rk_Z (H_1(G)) + d(H_2(G)),$$

where $rk_{\mathbf{Z}}(.)$ denotes the Z-rank of the torsion-free part and $d(.)$ means the minimal number of generators. Then, by [3], [4], [13], for the presentation P, it is always true that $x(P) \geq \delta(G)$. We then define

$$x(G) = min\{x(P) : P \text{ is a finite presentation for } G\}.$$

Thus we have the following definition.

Definition 1.1 *Let G be a group.*

1) *A presentation P_0 for G is called minimal if $x(P_0) \leq x(P)$, for all presentations P of G.*

2) *A presentation P_0 is called efficient if $x(P_0) = \delta(G)$.*

3) *G is called efficient if $x(G) = \delta(G)$.*

We note that if $x(G) \leq 0$ then G must be infinite and if G is finite cyclic then $x(G)$=1.

In [8], there has been given a large part of some known results about efficiency. We remark that there is interest not just in finding efficient presentations, but finding presentations which are efficient on the minimal number of generators (see [25], [27]). For example, in [10], Çevik proved that certain natural presentations of semi-direct products of cyclic groups are efficient on two generators.

However, not all finitely presented groups are efficient, and in this paper we shall be looking for inefficient finitely presented groups G. Since $x(P) \geq \delta(G)$ holds for all presentations P of G, we see that G is inefficient, by definition, if and only if

$$x(P) \geq x(P_0) > \delta(G),$$

for every presentation P and every minimal presentation P_0.

B.H.Neumann [20] asked whether a finite group G with $\delta(G) = 0$ must be efficient. Swan [25] gave examples (of finite metabelian groups) showing this is not the case. Then were the first examples of inefficient groups. In [28], Wiegold produced a different construction to the same end, and then Neumann added a slight modification to reduce the number of generators. In [17], Kovacs generalized both the above constructions, and he showed how to construct more inefficient finite groups (including some perfect groups) whose Schur multilicator is trivial. In [22], Robertson, Thomas and Wotherspoon examined a class of groups, orginally introduced by Coxeter. By using a symmetric presentation, they showed that groups in this class are inefficient. They also proved that every finite simple group can be embedded into a finite inefficient group.

Lustig [18] gave the first example of a torsion-free inefficient group. Other examples were found by Baik (see [1]), using generalized graph products. In [2], Baik and Pride gave sufficient conditions for a Coxeter group to be efficient. They also found a family of innefficient Coxeter groups $G_{n,k}$ ($n \geq 4$, k an odd integer).

1.2 A presentation of the semi-direct product

Let A, B be groups, and let θ be a homomorphism defined by

$$\theta : A \rightarrow Aut(B), \quad a \mapsto \theta_a$$

for all a $\in A$. Then the semi-direct product $G = B \times_\theta A$ of B by A is defined as follows.

The elements of G are all ordered pairs (a, k) ($a \in A$, $k \in B$) and the multiplication is given by

$$(a,k)\,(a',k') = (aa',\,(k\theta_{a'})k').$$

Similiar definitions of a semi-direct product can be found in [23] or [24].

One can find the proof of the following lemma for instance in [15, Proposition 10.1, Corollary 10.1].

Lemma 1.2 *Suppose that* $P_B = \langle \mathbf{y}\,;\mathbf{s}\rangle$ *and* $P_A = \langle \mathbf{x}\,;\mathbf{r}\rangle$ *are presentations for the groups B and A respectively under the maps*

$$y \mapsto k_y \in B \quad \text{and} \quad x \mapsto a_x \in A.$$

Then we have a presentation

$$P = \langle \mathbf{y}, \mathbf{x}\,; \mathbf{s}, \mathbf{r}, \mathbf{t}\rangle,$$

for $G = B\mathrm{x}_\theta A$, *where* $\mathbf{t} = \{yx\lambda_{\mathbf{yx}}^{-1}x^{-1} \mid y\in \mathbf{y},\ x\in \mathbf{x}\ \}$ *and* λ_{yx} *is a word on* $\mathbf{y}$ *representing the element* $(k_y)\theta_{\mathrm{a}_x}$ *of B* ($a\in A,\ k\in B,\ x\in \mathbf{x},\ y\in \mathbf{y}$).

1.3 The main theorem

Let B be a cyclic group of order n ($n\in$ N) with a presentation $P_B = \langle y; y^n\rangle$, and let A be a cyclic group of order p (p is a prime) with a presentation $P_A = \langle x; x^p\rangle$. Then, by Lemma 1.2, a presentation for $G = B \mathrm{x}_\theta A$ is given by

$$P = \langle y, x; y^n = 1, x^p = 1, x^{-1}yx = y^r\rangle, \tag{2}$$

where

(i) $(r,n) = 1$,

(ii) $(r\text{-}1, nt) = t$ with $t = (r - 1, n)$,

(iii) $r^p \equiv 1 \pmod{nt}$ for $r,t \in$ N.

Now let us take $r = 2$ and $n = 2^p\text{-}1$ in conditions (i), (ii) and (iii). (So that t=1 in (ii) and (iii)). Then, by substituting these values in (2), we get

$$P_G = \langle y, x; y^{2^p-1} = 1, x^p = 1, x^{-1}yx = y^2\rangle, \tag{3}$$

as a presentation for the group G.

Thus we have the following theorem as a main result of this paper.

Theorem 1.3 *Let* P_G, *as in* (3), *be a presentation of the semi-direct product of B by A. Then* P_G *is an inefficient but minimal presentation for the group G.*

2. PRELİMİNARY MATERİAL

In this section we will consider some material for helping to prove Theorem 1.3.

2.1 Spherical pictures for groups

Let us assume that G is a finitely presented group and P, as in (1), is a presentation of G. If we regard P as a 2-complex with one 0-cell, a 1-cell for each $x\in\mathbf{x}$, and a 2-cell for each $R\in\mathbf{r}$ in the standard way, then G is just the fundamental group of P. There is also, of course, the second homotopy module $\pi_2(P)$ of P, which is a left ZG-module. The elements of $\pi_2(P)$ can be reprsented by geometric configurations called *spherical pictures* which are usually labeled by P. We recall that a picture P is called non-spherical if some arcs meet the boundary of P. These are described in detail in [21], and we refer the reader there for details.

In this paper we need only one basepoint on each disc of our pictures (so we will actually use *-pictures, as described in Section 2.4 of [21]). Also, as described in [21], there are certain operations on spherical pictures.

There is an embedding μ of $\pi_2(P)$ into the free module $\underset{R\in r}{\oplus} ZGe_R$ defined as follows (see [6], [7], [21] for the details). Let <P > $\in \pi_2(P)$ and suppose that P has discs $\Delta_1, \Delta_2, \ldots, \Delta_n$ with the label $R_1^{\varepsilon_1}, R_2^{\varepsilon_2}, \ldots, R_n^{\varepsilon_n}$ respectively ($R_i \in$ **r**, $\varepsilon_i = \pm 1$, $i = 1,2,\ldots,n$). Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be a spray for P and let W(γ_i) be the label on each γ_i which represents an element of G. Then

$$\mu(<\mathrm{P}>) = \sum_{i=1}^{n} \varepsilon_i \overline{W(\gamma_i)}\, e_{R_i}.$$

For each spherical picture P over P and for each $R \in$ **r**, let $\lambda_{\mathbf{P},R}$ be the coefficients of e_R in μ (<P >). Let I_2 (P) be the 2-sided ideal in ZG generated by te set

$$\{\lambda_{\mathbf{P},R} : \mathrm{P} \text{ is a spherical picture}, R \in \mathbf{r}\}.$$

This ideal is called the *second Fox ideal* of P. The concept of Fox ideals has been discussed in [18]. In fact we need this concept for Theorem 2.1 below (due to Lustig [18] but see also [16]) which is a test of minimality of group presentations.

Theorem 2.1 ([18]) *Let G be a group with the presentation P as in* (1). *If there is a ring homomorphism ϕ from ZG into the matrix ring of all k* x *k-matrices (k≥1) over some commutative ring L with* 1, *such that* $\phi(1) = 1$, *and if ϕ maps the second Fox ideal $I_2(P)$ to 0, then P is minimal.*

Suppose **X** is a collection of spherical pictures over P. Then, by [21], one can define the additional operation on spherical pictures. Allowing this additional operation leads to the notion of *equivalence (rel* **X**) *of spherical pictures.* Then, by [21], *the elements* <P> (P $\in$ **X**) *generate $\pi_2(P)$ as a module if and only if every spherical picture is equivalent (rel* **X**) *to the empty picture.* If the elements <P> (P $\in$**X**) generate $\pi_2(P)$ then we say that **X** *generates $\pi_2(P)$.* By [21], it can be be shown that if **X** is a set of generating pictures, then *$I_2(P)$ is generated by* $\{\lambda_{\mathbf{P},R} : \mathrm{P} \in \mathbf{X}, R \in \mathbf{r}\}$.

2.2 The *p*-Cockcroft property

Let P be a presentation as given (1). For any picture P over P and for any $R \in$**r**, the *exponent sum* of R in P, denoted by exp_R(P) is the number of discs of P labelled by R, minus the number of discs labelled by R^{-1}. We remark that if any two pictures P_1 and P_2 are equivalent, then $exp_R(\mathrm{P}_1) = exp_R(\mathrm{P}_2)$ for all $R \in$**r**.

For a non-negative integer n, the presentation P is said to be n-**Cockcroft** if $exp_R(\mathrm{P}) \equiv 0$ (mod n), (where congruence (mod 0) is taken to be equality) for all $R \in$**r** and for all spherical pictures P over P. A group G is said to be *n-Cockcroft* if it admits an n-Cockcroft presentation.

To verify that n-Cockcroft property holds, it is enough to check for pictures P $\in$**X**, where **X** is a set of generating pictures.

The 0-Cockcroft property is usually just called Cockcroft. In practice, we usually take n to be 0 or a prime p. The Cockcroft property has received considerable attention in [11], [12], [14] and [16]. The p-Cockcroft property has been discussed, for example, in [9], [16].

The following result which is essentially due to Epstein [13] can also be found in [16, Theorem 2.1].

Theorem 2.2 *Let P be as in* (1). *Then P is efficient if and only if it is p-Cockcroft for some prime p.*

3. PROOF OF THE MAİN THEOREM

Throughout this section B, A will denote finite cyclic groups of order n and p (p is a prime), respectively. Now let us assume that P_G is a presentation, as in (3), for the group $G = B \times_\theta A$.

By using the generating pictures (see below Figure 1) of P_G, we will show that P_G is not p-Cockcroft for any prime p while, by Theorem 2.1, it is minimal. Thus, by definition, we will conclude that G is inefficient.

By [3], the set of generating pictures over P_G can be given as in Figure 1.

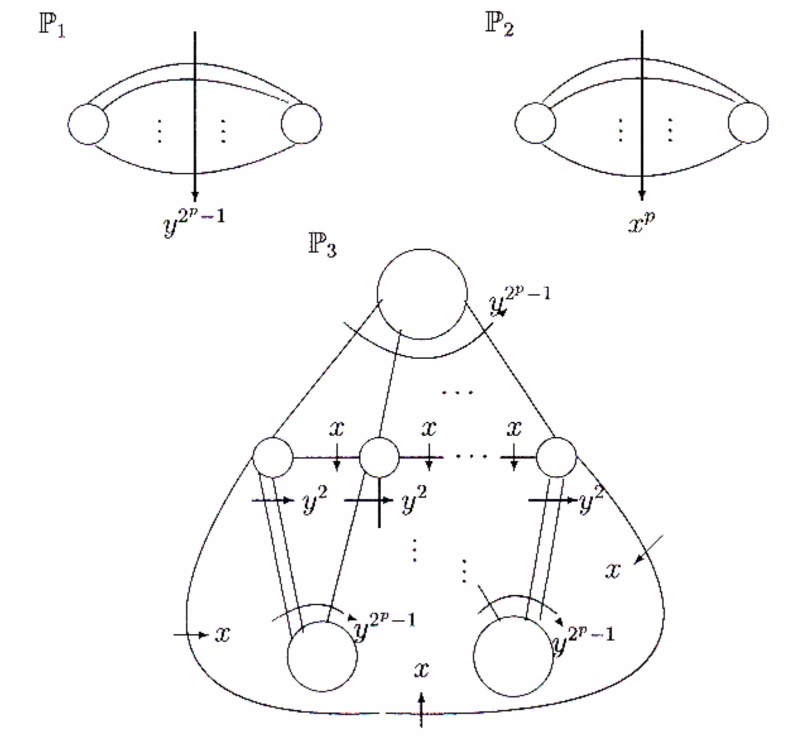

Figure 1.

Now let $R = y^{2^p-1}$, $S = x^p$ and $T = x^{-1}yxy^{-2}$. For the pictures P$_1$ and P$_2$, we have

$$exp_R(\text{P}_1) = 1 - 1 = 0 \quad \text{and} \quad exp_S(\text{P}_2) = 1 - 1 = 0.$$

Also for the picture $\mathbb{P}_3$, we have

$$exp_R(\mathbb{P}_3) = -1 + 2 = 1 \quad \text{and} \quad exp_T(\mathbb{P}_3) = 2^p - 1.$$

Therefore, by definition, we can conclude that P_G is not p-Cockcroft for any prime p and then, by Theorem 2.2, P_G is not efficient.

Now our aim is to show that P_G is minimal and so there could not be an efficient presentation which defines the group G.

By using the sprays on the generating pictures $\mathbb{P}_1$, $\mathbb{P}_2$ and $\mathbb{P}_3$, one can show that the second Fox ideal $I_2(P_G)$ is generated by the elements

$$1-\overline{y},\quad 1-\overline{x},\quad 2\overline{x}-1,\quad \overline{x}(1+\overline{y}+\overline{y}^{2}+....+\overline{y}^{2^p-2}).$$

If we consider a ring homomorphism

$$\phi : \mathbb{Z}G \rightarrow \mathbb{Z}_{2^p-1}$$

defined by $x \mapsto 1$, $y \mapsto 1$ and sending all integer coefficients to their congruence modulo 2^p-1 then ϕ sends the generators of $I_2(P_G)$ to 0 and 1 to 1. Hence, by Theorem 2.1, P_G is minimal. That is, G is not an efficient group.

4. SOME EXAMPLES

In this secion we will investigate some applications of Theorem 1.3.

Of course the first example of the main theorem would be the obvious presentations which are obtained by substituting any prime p in the presentation (3).

Other examples can be given on metacyclic groups. So suppose that G is a finite metacyclic group with a presentation (see [15])

$$P_0 = \langle y, x\,;\, y^n = 1, x^m = y^s, x^{-1}yx = y^r \rangle \qquad (4)$$

where $n,m,s,r \in \mathbb{N}$ such that

$$r,\ s \leq n, \qquad r^m \equiv 1 \ (\text{mod}\ n) \quad \text{and} \quad rs \equiv s \ (\text{mod}\ n). \qquad (5)$$

By taking $r = 2$, $s = n = 2^p-1$ and $m = p$ in P_0, we get the presentation

$$P_1 = \langle y, x\,;\, y^{2^p-1} = 1, x^p = y^{2^p-1}, x^{-1}yx = y^2 \rangle .$$

It is easy to see that the conditions given in equation (5) and the congruences

$$2^p \equiv 1 \ (\text{mod}\ 2^p-1) \quad \text{and} \quad 2(2^p-1) \equiv (2^p-1) \ (\text{mod}\ (2^p-1))$$

hold for the presentation P_1. Therefore, by [15], the metacyclic group G is still presented by the presentation P_1. Moreover by applying Tietze transformations (see[19]) on P_1, we can get the presentation

$$P_2 = \langle y, x\,;\, y^{2^p-1} = 1, x^p = 1, x^{-1}yx = y^2 \rangle ,$$

which is exactly the same with the presentation P_G as given in (3).

Therefore, as an application of Theorem 1.3, we have the following result.

Corollary 4.1 *let G be a metacyclic group presented by P_0 as in* (4). *Then G is an inefficient group.*

Question. Let P be a presentation for the group $G = B \times_\theta A$, as in (2), and let t=1 in conditions (i), (ii), and (iii). Are there any minimal presentations for $r \neq 2$?

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