TENSOR PRODUCT IMMERSIONS WITH TOTALLY REDUCIBLE FOCAL SET

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ABSTRACT

In [1], Carter and the author introduced the idea of an immersion $f: M \to R^n$ with *totally reducible focal set* (TRFS). Such an immersion has the property that, for all $p \in M$, the focal set with base p is a union of hyperplanes in the normal plane to $f(M)$ at $f(p)$. In this study we show that if $f: S^1 \to R^2$ and $g: S^1 \to R^3$ are two isometric immersions then the tensor product immersions $f \otimes f$ and $f \otimes g$ have TRFS property.

Keywords: Immersions, Focal set, Totally reducible focal set, Tensor product immersion

ÖZET

[1] de Carter ve yazar, $f: M \to R^n$ tamamen indirgenebilen focal cümleye (TRFS) sahip immersiyon tanımını verdi. Bu immersiyon, her $p \in M$ için, p ye bağlı focal cümle, $f(p)$ de $f(M)$ ye normal düzlemdeki hiperyüzeylerin bir birleşimidir. Bu çalışmada, eğer $f: S^1 \to R^2$ ve $g: S^1 \to R^3$ iki izometrik immersiyon ise $f \otimes f$ ve $f \otimes g$ tensor çarpım immersiyonlarınında TRFS şartını sağladıkları gösterildi.

Anahtar Kelimeler: İmmersiyonlar, Focal Cümle, Tamamen indirgenebilen focal cümle, Tensor çarpım immersiyonu

1. INTRODUCTION

Let $f:M\to R^n$ be a smooth immersion of connected smooth m-dimensional manifold without boundary into Euclidean n-space. For each $p\in M$, the focal set of f with base p is an algebraic variety. In this study we consider immersions for which this variety is a union of hyperplanes.

For $p\in M$, let U be a neighborhood of p in M such that $f|_U : U\to R^n$ is an embedding. Let $\upsilon_f(p)$ denote the (n-m)-plane which is normal to $f(U)$ at $f(p)$. Then the total space of normal bundle is $N_f=\{(p,x)\in M\times R^n \mid x\in\upsilon_f(p)\}$. The projection map $\eta_f : N_f\to R^n$ is defined by $\eta_f(p,x)=x$ and the set of focal points with base p is $\Gamma_f(p)=\{p\in R^n \mid (p,x)\text{ is a singularity of }\eta_f\}$. The focal set of f which denoted by $\Gamma_f=\bigcup_{p\in M}\Gamma_f(p)$ is the image by η_f. For each $p\in M$, $\Gamma_f(p)$ is a real algebraic variety in $\upsilon_f(p)$ which can be defined as the zeros of polynomial on $\upsilon_f(p)$ of degree $\le m$. The focal point of f has weight (multiplicity) k if $rank(Jac\,\eta_f)=n-k$ [3].

Definition 1. The immersion $f:M\to R^n$ has totally reducible focal set (TRFS) property if for all $p\in M$, $\Gamma_f(p)$ can be defined as the zeros of real polynomial which is a product of real linear factors [1].

Thus each irreducible component of $\Gamma_f(p)$ is an affine in $\upsilon_f(p)$, and $\Gamma_f(p)$ is a union of $(n-m-1)$-planes (possible $\Gamma_f(p)=\Phi$). There are other ways of describing this property. It is shown in ([5], [7], and [8]) that f has TRFS property if and only if f has flat normal bundle, where M is thought of as a Riemannian manifold with metric g induced from R^n. We will give explicit ways of constructing immersions with TRFS property.

In calculating focal sets it is often easiest to work with distance functions. For $x\in R^n$ the distance function $L_x : M\to R$ is defined by $L_x(p)=\|x-f(p)\|^2$. Then $x\in R^n$ is a focal point of f with base p if and only if p is a degenerate critical point of L_x, where at p, $\frac{\partial L_x}{\partial p_i}=0$ and $\left[\frac{\partial^2 L_x}{\partial p_i\partial p_j}\right]$ is singular for $i,j=1,2,\cdots,m$, ([6]).

In this study it has been shown that if $f:S^1\to R^2$ and $g:S^1\to R^3$ are two isometric immersions then the tensor product immersions $f\otimes f$ and $f\otimes g$ have TRFS property.

2. TENSOR PRODUCT IMMERSIONS

Let us recall definitions and results of [2]. Let M and N be two differentiable manifolds and $f:M\to R^n$, $g:N\to R^d$ two immersions. The direct sum and tensor product maps

$$f\oplus g : M\times N\to R^{n+d},$$
$$f\otimes g : M\times N\to R^{nd}$$

are defined by

$$\left(f \oplus g\right)(p,q) = \left(f(p), g(p)\right),$$
$$\left(f \otimes g\right)(p,q) = f(p) \otimes g(p).$$

The necessary and sufficient conditions for $f \otimes g$ to be an immersion were obtained in [3]. It is also proved there that the pairing $(\oplus, \otimes)$ determines a structure of a semiring on the set of classes of differentiable manifolds transversally immersed in Euclidean spaces, modulo orthogonal transformations. Some subsemirings were studied in [4].

If $n = m+1$, $G_f(p)$ consists of a finite number of points so, trivially, any immersion $f: M^m \to R^{m+1}$ has TRFS property. Thus especially an immersion $f: S^1 \to R^2$ has TRFS property. Also every immersions $f: S^1 \to R^n$, $n \geq 3$, has TRFS property [1].

The following results are well known.

Theorem 1. [1] Let $f: M \to R^n$ and $g: N \to R^d$ be immersions with TRFS property. Then $f \times g: M \times N \to R^{n+d}$ defined by $\left(f \times g\right)(p,q) = \left(f(p), g(p)\right)$ has TRFS property.

Theorem 2. [1] If $f: M \to R^n$ has TRFS property and $g: M \to R^{n+k}$ is defined by $g(p) = \left(f(p), t\right) \in R^n x R^k$. Then g has TRFS property.

We prove the following results.

Theorem 3. If $f: S^1 \to R^2$ is an isometric immersion then the tensor product immersion $f \otimes f: S^1 \times S^1 \to R^4$ has TRFS property.

Proof. The tensor product immersion $h = f \otimes f: S^1 \times S^1 \to R^4$ is defined by

$$h(\theta,\varphi) = \left(f \otimes f\right)(\theta,\varphi) = \left(\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta\cos\varphi, \sin\theta\sin\varphi\right),$$

$\left(\theta, \varphi \in R \bmod 2\pi\right)$. Let $x \in R^4$ and $L_x(\theta,\varphi) = \sum_{i=1}^{4}(x_i - h_i(\theta,\varphi))^2$. Then

$$\tfrac{\partial L_x}{\partial\theta} = x_1\sin\theta\cos\varphi + x_2\sin\theta\sin\varphi - x_3\cos\theta\cos\varphi - x_4\cos\theta\sin\varphi = 0, \qquad (1)$$
$$\tfrac{\partial L_x}{\partial\varphi} = x_1\cos\theta\sin\varphi - x_2\cos\theta\cos\varphi + x_3\sin\theta\sin\varphi - x_4\sin\theta\cos\varphi = 0, \qquad (2)$$

and

$$A = \tfrac{\partial^2 L_x}{\partial\theta^2} = \tfrac{\partial^2 L_x}{\partial\varphi^2} = x_1\cos\theta\cos\varphi + x_2\cos\theta\sin\varphi + x_3\sin\theta\cos\varphi + x_4\sin\theta\sin\varphi,$$
$$B = \tfrac{\partial^2 L_x}{\partial\theta\partial\varphi} = -x_1\sin\theta\sin\varphi + x_2\sin\theta\cos\varphi + x_3\cos\theta\sin\varphi - x_4\cos\theta\cos\varphi,$$

and

$$\det H = A^2 - B^2 = 0. \qquad (3)$$

Thus $A^2 - B^2 = \left(A - B\right)\left(A + B\right) = 0$

If $A - B = 0$ then

$$\begin{aligned} &x_1\left(\cos\theta\cos\varphi + \sin\theta\sin\varphi\right) + x_2\left(\cos\theta\sin\varphi - \sin\theta\cos\varphi\right) \\ &+x_3\left(\sin\theta\cos\varphi - \cos\theta\sin\varphi\right) + x_4\left(\sin\theta\sin\varphi + \cos\theta\cos\varphi\right) = 0. \end{aligned} \qquad (4)$$

If $A + B = 0$ then

$$x_1\left(\cos\theta\cos\varphi-\sin\theta\sin\varphi\right)+x_2\left(\cos\theta\sin\varphi+\sin\theta\cos\varphi\right)$$
$$+x_3\left(\sin\theta\cos\varphi+\cos\theta\sin\varphi\right)+x_4\left(\sin\theta\sin\varphi-\cos\theta\cos\varphi\right)=0. \quad (5)$$

Therefore using (1), (2) and (4) we get

$$\Gamma^1{}_h(\theta,\varphi)=\left\{\left(x_1=\lambda x_4, x_2=-\lambda x_4, x_3=x_4, x_4\right)\middle|\ \lambda=\tfrac{(\tan\theta\tan\varphi-1)}{\tan\theta+\tan\varphi}, \tan\theta\neq-\tan\varphi\right\} \quad (6)$$

and using (1), (2) and (5) we get

$$\Gamma^2{}_h(\theta,\varphi)=\left\{\left(x_1=-\mu x_4, x_2=\mu x_4, x_3=-x_4, x_4\right)\middle|\ \mu=\tfrac{(\tan\theta\tan\varphi-1)}{\tan\theta-\tan\varphi}, \tan\theta\neq\tan\varphi\right\} \quad (7)$$

Thus from (6) and (7) we get

$$\Gamma_h=\Gamma^1{}_h(\theta,\varphi)\cup\Gamma^2{}_h(\theta,\varphi).$$

So h has TRFS property.

Remark. If $f:S^1\to R^2$ then, by Theorem 1, $f\times f:S^1\times S^1\to R^4$ has TRFS property. But in this case $\Gamma_{f\times f}=\left\{(0,0,a,b)\mid a,b\in R\right\}\cup\left\{(c,d,0,0)\mid c,d\in R\right\}$.

Theorem 4. If $f:S^1\to R^2$ and $g:S^1\to R^3$ are two isometric immersions then the tensor product immersion $f\otimes g:S^1\times S^1\to R^6$ has TRFS property.

Proof. Let $f:S^1\to R^2$ and $g:S^1\to R^3$ be defined by $f(q)=(\cos q,\sin q)$ and $g(j)=(\cos j,\sin j,k), k Î R$, respectively. The tensor product immersion $h=f\otimes g:S^1\times S^1\to R^6$ is defined by

$$h(\theta,\varphi)=\left(f\otimes g\right)(\theta,\varphi)=\left(\cos\theta\cos\varphi,\cos\theta\sin\varphi,\sin\theta\cos\varphi,\sin\theta\sin\varphi,k\cos\varphi,k\sin\varphi\right),$$

$\left(\theta,\varphi\in R \bmod 2\pi\right)$. Let $x\in R^6$ and $L_x(\theta,\varphi)=\sum_{i=1}^{6}(x_i-h_i(\theta,\varphi))^2$. Then

$$\tfrac{\partial L_x}{\partial\theta}=x_1\sin\theta\cos\varphi+x_2\sin\theta\sin\varphi-x_3\cos\theta\cos\varphi-x_4\cos\theta\sin\varphi=0, \quad (8)$$

$$\tfrac{\partial L_x}{\partial\varphi}=x_1\cos\theta\sin\varphi-x_2\cos\theta\cos\varphi+x_3\sin\theta\sin\varphi-x_4\sin\theta\cos\varphi+x_5k\sin\varphi-x_6k\cos\varphi=0, \quad (9)$$

and

$$A_{11}=\tfrac{\partial^2L_x}{\partial\theta^2}=x_1\cos\theta\cos\varphi+x_2\cos\theta\sin\varphi+x_3\sin\theta\cos\varphi+x_4\sin\theta\sin\varphi,$$

$$A_{12}=\tfrac{\partial^2L_x}{\partial\theta\partial\varphi}=-x_1\sin\theta\sin\varphi+x_2\sin\theta\cos\varphi+x_3\cos\theta\sin\varphi-x_4\cos\theta\cos\varphi,$$

$$A_{22}=\tfrac{\partial^2L_x}{\partial\varphi^2}=x_1\cos\theta\cos\varphi+x_2\cos\theta\sin\varphi+x_3\sin\theta\cos\varphi+x_4\sin\theta\sin\varphi+x_5k\cos\varphi+x_6k\sin\varphi,$$

and $\det H=\det\left(A_{ij}\right)=0$. (10)

From (8), (9) and (10) we get either

$$\Gamma^1{}_h(\theta,\varphi)=\left\{\begin{array}{l}\left(x_1=-\mu x_2, x_2=x_2, x_3=-\lambda\mu x_2, x_4=\lambda x_2, x_5=x_5, x_3=\mu x_5-\left(\tfrac{1+\mu}{k\cos\theta}\right)x_2\right)\\ \mid \lambda=\tan\theta, \mu=\tan\varphi, \cos\theta\neq0, \cos\varphi\neq0\end{array}\right\},$$

or

$$\Gamma^2{}_h(\theta,\varphi)=\left\{\left(x_1=0, x_2=0, x_3=0, x_4=x_4, x_5=x_5, x_6=-\tfrac{x_4}{k}\right)\middle|\ \theta=\mp\tfrac{\pi}{2}, \varphi=0\right\},$$

or

$\Gamma^3{}_h(\theta,\varphi) = \left\{ \left(x_1 = 0, x_2 = 0, x_3 = x_3, x_4 = x_4, x_5 = \frac{x_3}{k}, x_6 = \frac{x_4}{k} \right) \middle| \ \theta = \mp \frac{\pi}{2}, \varphi \in R \bmod 2\pi \right\}$.

Therefore, $\Gamma_h = \Gamma^1{}_h(\theta,\varphi) \cup \Gamma^2{}_h(\theta,\varphi) \cup \Gamma^3{}_h(\theta,\varphi)$. So h has TRFS property.

Remark. If $f: S^1 \to R^2$ and $g: S^1 \to R^3$ then, by Theorem 1, $f \times g: S^1 \times S^1 \to R^5$ has TRFS property and using Theorem 2, $k: S^1 \times S^1 \to R^6$ also has TRFS property. But in this case focal set of k is different then above result.

REFERENCES

[1] S. Carter and R. Ezentas, *Immersion with totally reducible focal set*, Journal of Geometry, **45**, (1992) 1-7.

[2] B.Y. Chen, *Differential geometry of semiring of immersions*, I: General Theory. Bull. Inst. Math. Acad. Sinica, **21**, (1993) 1-34.

[3] F.Decruyenaere, F. Dillen, L. Verstraelen, L. Vrancken, *The semiring of immersions of manifolds*, Beitrage Algebra Geom., **34**, (1993) 209-215.

[4] F.Decruyenaere, F. Dillen, I. Mihai, L. Verstraelen, *Tensor products of spherical and equivariant immersions*, Bull. Belg. Math. Soc., Simon Stevin **1**, (1994) 643-648.

[5] A.M. Flegmann, *Parallel rank of a submanifold of Euclidean space*, Math. Proc. Camb. Phil. Soc., **106**, (1989) 89-93.

[6] J. Milnor, *Morse Theory*, Princeton Univ. Press, Princeto1963.

[7] R.S. Palais and C.L. Terng, *Critical point theory and submanifolds geometry*, Lecture Notes in Maths. 1353, Springer-Verlag, Berlin, 1988.

[8] C.L. Terng, *Submanifolds with flat normal bundle*, Math. Ann, **277**, (1987) 95-111.