A STUDY ON A PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS AND RESTRICTIVE TAYLOR APPROXIMATIONS

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ABSTRACT

In this paper we solved parabolic partial differential equation using restrictive Taylor's approximations. We use the restrictive Taylor approximation to approximate the exponential matrix exp(xA). The advantage is that has the exact value at certain point.We shall develop a new approach for an explicit method to solve the parabolic partial differential equation. The results of numerical testing show that the numerical method based on the restrictive Taylor approximation discussed in the present paper produce good results.

Keywords: Restrictive Taylor approximation, Finite difference scheme, Parabolic partial differential equations

ÖZET

Bu makalede Parabolik kısmi türevli diferansiyel denklem kısıtlanmış Taylor yaklaşımı kullanılarak çözüldü. Kısıtlanmış Taylor yaklaşımı çözümde üstel matris fonksiyonuna yaklaşmak için kullanıldı. Belirli noktalarda fonksiyonun tam değerini vermesi yöntemin bir avantajını oluşturmaktadır. Bu şekilde parabolik kısmi türevli diferansiyel denklemin açık çözümü için yeni bir yaklaşım geliştirildi. Nümerik sonuçlar ile yöntemin diğer yöntemlere göre iyi sonuçlar verdiği gösterildi.

Anahtar Kelimeler: Kısıtlanmış Taylor yaklaşımı, Sonlu farklar yaklaşımı, Parabolik kısmi diferansiyel denklemler

1.INTRODUCTION

It is well known that parabolic partial diffeential equation in one dimensions, feature in the mathematical modelling of many phenomena. They arise for example, in the study of heat transfer and control theory.These kind of problems have been investigated by many researchers. This paper presents an investigation of the use of restrictive Taylor approximation in solving parabolic partial differential equation.

The Taylor series relates the value of a differentiable function at any point to its first and higher order derivatives at a reference point , and consequently the first (or higher) order derivatives at the reference point can be obtained in terms of the sampled values of the funcions. In certain cases , it may be difficult to find analytical solutions of complicated differential and partial differential equations describing the physical systems. In such cases, numerical solutions can be obtained by replacing derivatives in the equation by approximations based on the Taylor series. The most commonly used approximations of derivatives are the forward difference, backward difference and the central difference approximations. All these approximations are widely used to solve differential and partial differential equations.(1,2)

The purpose of this paper is to present a very efficient finite difference method based on restrictive Taylor approximation for solving the diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial u^2}, \quad 0''x''L, \quad 0''t''T \qquad 1.1$$

with the initial condition

$$u(x,0)=g(x), \quad 0''x''L \qquad 1.2$$

and the boundary conditions

$$u(0,t)=u(L,t)=0, \quad 0''t''T \qquad 1.3$$

An approximation of the function f(x) is by using Taylor's expansion, which approximate f(x) by a polinomial of degree n and truncation error is of order (n+1).

$$f(x)= P_{n,f(x)}(x) + R_{n+1}(x) \qquad 1.4$$

This can be written as

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n+1}(x) \qquad 1.5$$

where

$$R_{n+1}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} \qquad 1.6$$

and ξ between a and x.(3)

2.THE FINITE DIFFERENCE SCHEMES

The domain [0,L]x[0,T] is divided into an MxN mesh with the spatial step size h=L/M in x direction and the time step size k=T/N, respectively. Grid points (x_i,t_j) are defined by x_i=ih, i=0,1,2,...,M, t_j=jk, j=0,1,2,...,N, in which M and N are integers. Using the initial condition

$$u(x,0)=f(x), \quad 0<x<L, \qquad 2.1$$

Eq.(1.1) is solved approximately, commencing with initial values $u(ih,0)=f(x_i)$, i=0,1,2,...,M, and boundary values $u(0,jk)=g_0(t_{j+1})$, $u(L,jk)=g_1(t_{j+1})$, for j=0,1,2,...,N.

3.RESTRICTIVE TAYLOR'S APPROXIMATION

Consider a function f(x) defined in a neighborhood of the point a, and it has derivatives up to order (n+1) in the neighborhood. Hassan et al.(2) used these derivatives to construct the function

$$RT_{n,f(x)}(x,a) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{\varepsilon f^{(n)}(a)}{n!}(x-a)^n \qquad 3.1$$

$RT_{n,f(x)}(x)$ is called restrictive Taylor's approximation for the function f(x) at the point a. The parameter ε is to be determent, such that $RT_{n,f(x)}(x_0)=f(x_0)$. It means that this approximation is exact at two points a and x_0. Let us put

$$f(x)= P_{n,f(x)}(x) + \Re_{n+1}(x) \qquad 3.2$$

where $\Re_{n+1}(x)$ is the remainder term of restrictive Taylor's series.
The error for approximation is given by

$$\Re_{n+1}(x)= \frac{\varepsilon(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi) - \frac{n(\varepsilon-1)(x-a)^{n+1}}{(x-\xi)(n+1)!} f^{n}(\xi) \qquad 3.3$$

where $\xi \in [a,x]$ and ε is a restrictive parameter.(6)

3.1 Restrictive Taylor's Approximation of The Exponential Matrix

The exponential matrix exp(xA) can be formally defined by the convergent power series

$$\exp(xA) = \sum_{n=0}^{\infty} \frac{x^n}{n!} A^n, \qquad A^0=I \qquad 3.4$$

where A is an (N-1)x(N-1) matrix. The term ε in Eq. (3.1) can be reduced to the square restrictive matrix Λ in the case of restrictive Taylor's approximation for matrix function, where

$$\Lambda = \begin{pmatrix} \varepsilon_1 & & & & & & & 0 \\ & \varepsilon_2 & & & & & & \\ & & \varepsilon_3 & & & & & \\ & & & . & & & & \\ & & & & . & & & \\ & & & & & . & & \\ & & & & & & \varepsilon_{N-2} & \\ 0 & & & & & & & \varepsilon_{N-1} \end{pmatrix}_{(N-1)x(N-1)}$$

For example, for the exponential matrix exp(xA) it can be given by

$$RT_{2,\exp(xA)} = I + rA + \frac{r^2}{2}\Lambda A^2 \qquad 3.5$$

4.RESTRICTIVE TAYLOR'S APPROXIMATION FOR DIFFUSION EQUATION

Let's consider diffusion equation (1.1) with the initial and boundary conditions(1.2-1.3). The open rectangular domain is covered by a rectangular grid with spacing h and k in the x, t direction respectively, and the grid point (x,t) denoted by (ih,jk)=$u_{i,j}$, where i=0(1)N, j is a non-negative integer. The exact solution of a grid representation of Eq.(1.1) is given by(6):

$$u_{i,j+1}=\exp(kD^2_x)u_{i,j} \qquad 4.1$$

and approximation of the partial derivative D^2_x at the grid point (ih,jk) will take the form(5):

$$D_x^{\,2}u=\frac{1}{h^2}(u_{i-1,j}-2u_{i,j}+u_{i+1,j}) \qquad 4.2$$

the result of making this aproximation is to replace Eq.(4.1) by the following equation

$$U^{j+1}=\exp(rA)U^j, \qquad r=\frac{k}{h^2} \qquad 4.3$$

where

$$U^j=(u_{1,j},u_{2,j},...u_{N-1,j})^T, \qquad \text{Nh=L} \qquad 4.4$$

and

$$A=\begin{pmatrix} -2 & 1 & & & & & 0 \\ 1 & -2 & 1 & & & & \\ & . & . & . & & & \\ & & . & . & . & & \\ & & & . & . & . & \\ & & & & . & . & . \\ & & & & 1 & -2 & 1 \\ 0 & & & & & 1 & -2 \end{pmatrix}_{(N-1)x(N-1)}.$$

we use the Eq.(3.3) to approximate the exponential matrix in Eq.(4.3), then

$$U^{j+1}=(I+rA+\frac{r^2}{2}\Lambda A^2)U^j=BU^j, \qquad 4.5$$

or in the scalar form which is given as below where j is a non negative integer.

$$\text{I+rA+}\frac{r^2}{2}\Lambda A^2=$$

$$
I+r\begin{pmatrix} -2 & -1 & & & & & 0 \\ 1 & -2 & 1 & & & & \\ & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & \cdot & \cdot & \cdot \\ & & & & 1 & -2 & 1 \\ 0 & & & & & 1 & -2 \end{pmatrix} + \frac{r^2}{2}\begin{pmatrix} \varepsilon_1 & & & & & & & 0 \\ & \varepsilon_2 & & & & & & \\ & & \varepsilon_3 & & & & & \\ & & & \cdot & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & & & \varepsilon_{N-1} & \\ 0 & & & & & & & \varepsilon_N \end{pmatrix}
$$

$$
x\begin{pmatrix} 5 & -4 & 1 & & & & & 0 \\ -4 & 6 & -4 & 1 & & & & \\ 1 & -4 & 6 & -4 & 1 & & & \\ & 1 & -4 & 6 & -4 & 1 & & \\ & & \cdot & \cdot & \cdot & \cdot & & \\ & & & 1 & -4 & 6 & -4 & 1 \\ & & & & 1 & -4 & 6 & -4 \\ 0 & & & & & 1 & -4 & 5 \end{pmatrix}
$$

$$
= \begin{pmatrix} 1-2r+\frac{5}{2}r^2\varepsilon_1 & r-2r^2\varepsilon_1 & \frac{1}{2}r^2\varepsilon_1 & & & & 0 \\ r-2r^2\varepsilon_2 & 1-2r+3r^2\varepsilon_2 & r-2r^2\varepsilon_2 & \frac{1}{2}r^2\varepsilon_2 & & & \\ \frac{1}{2}r^2\varepsilon_3 & r-2r^2\varepsilon_3 & 1-2r+3r^2\varepsilon_3 & r-2r^2\varepsilon_3 & \frac{1}{2}r^2\varepsilon_3 & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & \frac{1}{2}r^2\varepsilon_{N-3} & r-2r^2\varepsilon_{N-3} & 1-2r+3r^2\varepsilon_{N-3} & r-2r^2\varepsilon_{N-3} & \frac{1}{2}r^2\varepsilon_{N-3} \\ & & & \frac{1}{2}r^2\varepsilon_{N-2} & r-2r^2\varepsilon_{N-2} & 1-2r+3r^2\varepsilon_{N-2} & r-2r^2\varepsilon_{N-2} \\ 0 & & & & \frac{1}{2}r^2\varepsilon_{N-1} & r-2r^2\varepsilon_{N-1} & 1-2r+\frac{5}{2}r^2\varepsilon_{N-1} \end{pmatrix}
$$

This explicit method is used in the following. Equation (1.1) is solved numerically using:

$$
u_{i,j}=(r^2/2\bullet_i)u_{i-2,j}+(r-2r^2\bullet_i)u_{i-1,j}+(1-2r+3r^2\bullet_i)u_{i,j}+(r-2r^2\bullet_i)u_{i+1,j}+(r^2/2\bullet_i)u_{i+2,j},\ i=3(1)N-3 \qquad 4.6
$$

applied for i=3,4,...,N-3 for each j=1,2,...,M. Then for each j=0,1,...,M values at points $u_{1,j+1}$, $u_{2,j+1}$ are calculated using the following formula;

$$u_{1,j+1}= (1-2r+5r^2/2\bullet_1)u_{1,j}+(r-2r^2\bullet_1)u_{2,j} + (r^2/2\bullet_1)u_{3,j} \quad 4.7$$

$$u_{2,j+1}=(r-2r^2\bullet_2)u_{1,j}+(1-2r+3r^2\bullet_2)u_{2,j}+(r-2r^2\bullet_2)u_{3,j}+(r^2/2\bullet_2)u_{2,j} \quad 4.8$$

while values at points $u_{N-2,j+1}$, $u_{N-1,j+1}$ are calculated using:

$$u_{N-2,j+1}=(r^2/2\bullet_{N-2})u_{N-4,j}+(r-2r^2\bullet_{N-4})u_{N-3,j}+(1-2r+3r^2\bullet_{N-2})u_{N-2,j}+(r-2r^2\bullet_{N-2})u_{N-1,j} \quad 4.9$$

$$u_{N-1,j+1}=(r^2/2\bullet_{N-1})u_{N-3,j}+(r-2r^2\bullet_{N-1})u_{N-2,j}+(1-2r+5r^2/2\bullet_{N-1})u_{N-1,j} \quad 4.10$$

Application of the Gerschgorin's circle theorem (2) to the matrix $I+rA+\frac{r^2}{2}\Lambda A^2$ in the case of the restrictive Taylor approximation shows that the stability condition is $0\leq r \leq 2/3\,\varepsilon$.

In order to verify theoretical predictions, numerical tests were carried out on a one dimensional time-dependent diffusion equation:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial u^2}, \quad 0''x''\neq, \quad t\geq 0 \quad 4.11$$

$$u(x,0)=\sin x, \quad 0''x''\neq \quad 4.12$$

$$u(0,t)=u(\neq,t)=0, \quad t\geq 0. \quad 4.13$$

5.THE CRANK-NICOLSON SCHEME.

If we replace the spatial derivative with their values at the n and n+1 times levels and then substitute centered difference forms for all derivatives, we get the Cranc-Nicolson formula:

$$-ru_{i-1,j+1}+(2+2r)u_{i,j+1}-ru_{i+1,j+1} = -ru_{i-1,j}+(2-2r)u_{i,j}+ru_{i+1,j} \quad 5.1$$

where $r=k/h^2$.

This procedure is unconditionaly von Neumann stable for all r>0.(2) The modified equivalent partial differential equation for the Cranc-Nicolson formula shows that equation (5.1) has a truncation error which is always $O(h^2)$. So this scheme is second order accurate for all r>0.(5)

The accuracy of RT and CN method are compared in Table 1 for various values of the time t. Table 1 give the absolute error when h=≠/10, k=0.01, ε =0.1681599901. The time of calculation 100 steps of Restrictive Taylor method is 1.05 second while that of Cranc-Nicolson method is 2.65 second.

Table1 Results for u with T=1.0 , h=≠/10, r=1/$≠^2$

x	RT(2). Method	Cr-Nic Method	RT(2).- Error	Cr-Nic -Error	Analitical Solutions
≠/10	0.1136809992	0.1146158157	0.00000000	0.0009348165	0.1136809992
2≠/10	0.2162341101	0.2180122368	$0.1\ 10^{-15}$	0.0017781267	0.2162341101
3≠/10	0.2976207197	0.3000681012	0.0000000000	0.0024473814	0.2976207197
4≠/10	0.3498741397	0.3527512092	$0.1\ 10^{-15}$	0.0028770695	0.3498741397
5≠/10	0.3678794411	0.3709045710	0.0000000000	0.0030251298	0.3678794411
6≠/10	0.3498741397	0.3527512092	0.0000000000	0.0028770695	0.3498741397
7≠/10	0.2976207197	0.3000681012	0.0000000000	0.0024473814	0.2976207197
8≠/10	0.2162413110	0.2180122368	$0.72008713\ 10^{-5}$	0.0017781267	0.2162341101
9≠/10	0.1137373377	0.1146158157	0.0000563385	0.0000934816	0.1136809992

5.CONCLUSIONS

In this article a numerical method was applied to the one dimensional parabolic partial differential equation. The proposed numerical scheme solved this model quite satisfactorily. Using the restrictive Taylor approximation for one dimensional parabolic partial differential equation describe our model well. The explicit finite difference scheme are very simple to implement and economical to use. They are very efficient and they need less CPU time than the implicit finite difference methods.

A comparison with the implicit scheme for the test problem clearly demonstrates that this technique are computationally superior. The numerical test obtained by using these methods give acceptable results.

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