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Research Article

On the class of I-γ-open cover and I-St-γ-open cover

Prasenjit Bal

Department of Mathematics, ICFAI University Tripura, Kamalghat, West Tripura, India-799210

Abstract

Inspired by Pratulananda Das' recent efforts, we develop and investigate a new class of ideal-open covers that are formed after the interplay of the existing ideal-open covers with the star-operator. Interdependencies between specific sorts of open coverings have been detected and in order to grasp the differences between the new and older classes of ideal open covers, several constructive examples are illustrated. Our finding also establish some strong prerequisite for certain of P. Das' findings. In addition, the nature of I-dense subsets of the classes of ideal-open-covers are investigated in this paper.

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1. Introduction

To begin, let's review the definition of a star operator: $St(A, \mathcal{U})$ represents the set $\bigcup\{U \in \mathcal{U} : A \cap U \neq \emptyset\}$ if $\mathcal{U}$ is a collection of subsets of a set X and $A \subseteq X$. For a point $x \in X$, $St(x, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : x \in U\}$ [10]. In this aspect, it's worth noting that Kočinac in [16] broadened the concept of classical selection principles to star-selection principles by incorporating the theory of St-operator. In recent papers, various topological features and other selection principles connected to this operator can be identified [14–17, 23]. Selection principles connected to open covers have a long and fruitful history and readers can gain about current accomplishments in this field by consulting the works of [1–7, 18, 24–26], where many additional references can be discovered.

We recall that the asymptotic density is defined as

$$d(K) = \lim_{n\to\infty} \frac{|K(n)|}{n}$$

as long as the limit exists; where $K \subseteq \mathbb{N}$ (set of all Natural Numbers), $K(n) = \{k \in K : k \leq n\}$ and $|K(n)|$ is the order of the set $K(n)$. H. Fast [12] generalized the concept of convergence of real sequences towards the statistical convergence by using the idea of asymptotic density. In a metric space (X, ρ), the sequence $\{x_n\}_{n\in\mathbb{N}}$ of points converges to l, if for any $\epsilon > 0$, the set $\{k \in \mathbb{N} : \rho(x_k, l) \geq \epsilon\}$ has asymptotic density zero. The investigations on this convergence and its topological properties can be found in [8, 13, 22].

Email address: balprasenjit177@gmail.com
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P. Kystyrko and T. Šalat introduced another beautiful generalization of statistical convergence in [19] by putting the members of a proper ideal I of the set of natural numbers $\mathbb{N}$ in place of the sets of asymptotic density zero in the definition of statistical convergence. This convergence is called ideal convergence. To study some properties of sequences in topological spaces ideal convergence has been used in [9, 20, 21].

We take inspiration from [9] and apply the St-operator in order to get a variation of I-γ-open covers in topological spaces. During our investigation, it is found that a strong condition can be incorporated in Lemma 2.1 of [9] and the result is presented here. More over we discuss interrelation between various versions of γ-covers which have nearer structures and some of their topological properties are also investigated in this paper. Throughout the paper no specific separation axiom is considered otherwise stated and the symbol 'I' will represent a proper Ideal of $\mathbb{N}$ (the set of all natural numbers).

2. Preliminaries

Some existing definitions and results are mentioned in this section for ready references.

If X is a non-empty set, then a family I of subsets of X is said to be an ideal in X if (i) $\emptyset \in I$, (ii) $A, B \in I$ implies $A \cup B \in I$ and (iii) $A \in I$, $B \subseteq A$ implies $B \in I$. I is called non-trivial if $I \neq \emptyset$ and $X \notin I$. A proper ideal I is called admissible ideal or free ideal if $\{x\} \in I$ for each $x \in X$ [11].

A non-empty family $\mathcal{F}$ of subsets of X is called a filter on X if (i) $\emptyset \notin \mathcal{F}$, (ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$ and (iii) $A \in \mathcal{F}$, $A \subseteq B$ implies $B \in \mathcal{F}$. If I is a proper non-trivial ideal then the family of sets $\mathcal{F}(I) = \{M \subseteq X : M = X \setminus A, A \in I\}$ is a filter in X which is called dual filter of I [11].

A family $\mathcal{U}$ of subsets in a space X is called a cover if $\cup\mathcal{U} = X$. If every element of $\mathcal{U}$ is open then $\mathcal{U}$ is called an open cover [11].

An open cover $\mathcal{U}$ of X is a γ-cover if it is infinite, and each $x \in X$ belongs to all but finitely many elements of $\mathcal{U}$ [11].

Let I be a proper ideal on $\mathbb{N}$. A countable cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of X is said to be an I-γ-cover if for each $x \in X$, the set $\{n \in \mathbb{N} : x \notin U_n\}$ belongs to I. [9]

Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be a cover of a topological space (X, τ). A subset $\mathcal{V}$ of the cover $\mathcal{U}$ will be called I-dense in $\mathcal{U}$ if the set $M = \{m_1 < m_2 < m_3 < \dots\}$ of indices of elements of $\mathcal{V}$ belongs to $\mathcal{F}(I)$ [9].

An I-dense subset of an I-γ cover of a topological space (X, τ) is also an I-γ cover of that topological space. (Lemma 2.2 of [9]).

Let $(\mathcal{U}_n : n \in \mathbb{N})$, $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$ be a countable sequence of I-γ covers of X. Then $(\mathcal{V}_n : n \in \mathbb{N})$, defined by

$$\mathcal{V}_n = \{U_{1,m} \cap U_{2,m} \cap \dots \cap U_{n,m} : m \in \mathbb{N}\} \setminus \{\emptyset\}$$

is also a sequence of I-γ covers of X. (Lemma 2.3 of [9]).

3. Some results on I-γ covers

According to Lemma 2.1 of [9], an open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of a topological space X is an I-γ-cover if and only if for each finite set $F \subseteq X$ the set $\{n \in \mathbb{N} : F \nsubseteq U_n\} \in I$. We claim that the subset F of X need not to be finite in the above result.

Theorem 3.1. *In a topological space X, if $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is an I-γ-cover and $F \subseteq X$ is such that $I_F = \{n \in \mathbb{N} : F \nsubseteq U_n\}$ is finite then $I_F \in I$.*

Proof. Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be an I-γ-cover and $F \subseteq X$ be such that $I_F = \{n \in \mathbb{N} : F \nsubseteq U_n\}$ is finite.

Suppose $k \in I_F = \{n \in \mathbb{N} : F \nsubseteq U_n\}$ is arbitrary.

$\implies F \nsubseteq U_k$

$\implies F \setminus U_k \neq \emptyset$
$\implies x \notin U_k \quad \forall x \in F \setminus U_k$
$\implies k \in \{n \in \mathbb{N} : x \notin U_n\} = I_x \quad \forall x \in F \setminus U_k$
$\implies k \in \bigcap_{x \in F \setminus U_k} I_x$

Now, for all $x \in X, I_x = \{n \in \mathbb{N} : x \notin U_n\} \in I$ [$\because \mathcal{U}$ is an I-γ-cover]

But $\bigcap_{x \in F \setminus U_k} I_x \subseteq I_x$

.

$\therefore$ By the subset property of ideal, $k \in \bigcap_{x \in F \setminus U_k} I_x \in I \quad \forall k \in I_F$

$$\implies \bigcup_{k \in I_F} \{k\} \subseteq \bigcup_{k \in I_F} \Big\{ \bigcap_{x \in F \setminus U_k} I_x \Big\}$$

$$\implies I_F \subseteq \bigcup_{k \in I_F} \Big\{ \bigcap_{x \in F \setminus U_k} I_x \Big\}$$

But I_F is finite and by the finite union property of ideal $\bigcup_{k \in I_F} \Big\{ \bigcap_{x \in F \setminus U_k} I_x \Big\} \in I$

and by the subset property of ideal $I_F \in I$. □

Theorem 3.2. *Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be an open cover of a space X. If for each $F \subseteq X$ and for each $p \in \{n \in \mathbb{N} : F \nsubseteq U_n\} = I_F$, finiteness of $F \setminus U_p$ implies that $I_F \in I$, then $\mathcal{U}$ is an I-γ-cover.*

Proof. Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be an open cover of the space X. Let for each $F \subseteq X$ and for each $p \in \{n \in \mathbb{N} : F \nsubseteq U_n\} = I_F$, finiteness of $F \setminus U_p$ implies that $I_F \in I$.

Let $x \in X$ be arbitrary. So $\{x\} \subseteq X$ and $\{x\} \setminus U_p$ is finite for each $p \in I_{\{x\}}$. Then by our assumption $I_{\{x\}} \in I$ for all $x \in X$.
$\implies \{n \in \mathbb{N} : \{x\} \nsubseteq U_n\} \in I$ for each $x \in X$.
$\implies \{n \in \mathbb{N} : x \notin U_n\} \in I$ for each $x \in X$.
Hence $\mathcal{U}$ is an I-γ-cover. □

Theorem 3.3. *In a topological space (X, τ), if $A \subseteq X$ and if $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is an I-γ cover of (X, τ), then $\mathcal{U}_A = \{A \cap U_n : U_n \in \mathcal{U}\}$ is an I-γ cover for (A, τ_A), where τ_A denotes the sub-space topology.*

Proof. Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be an I-γ cover of (X, τ) and $A \subseteq X$. Then (A, τ_A) is a subspace of (X, τ). Also $\{n \in \mathbb{N} : x \notin U_n\} \in I$ for all $x \in X$ as well as $x \in A$. Obviously $\mathcal{U}_A = \{A \cap U_n : U_n \in \mathcal{U}\}$ is a cover of A. Now let $y \in A \subseteq X$ be arbitrary. Then
$\{n \in \mathbb{N} : y \notin U_n\} \in I$.
$\implies \{n \in \mathbb{N} : y \in U_n\} \in \mathcal{F}(I)$.
But $\{n \in \mathbb{N} : y \in U_n\} = \{n \in \mathbb{N} : y \in A \cap U_n\}[\because y \in A]$.
$\implies \{n \in \mathbb{N} : y \in A \cap U_n\} \in \mathcal{F}(I)$.
$\implies \{n \in \mathbb{N} : y \notin A \cap U_n\} \in I$.
Hence $\mathcal{U}_A$ is an I-γ cover of A. □

Example 3.4. Subcover of an I-γ cover may not be an I-γ cover.

Let $E = \{2, 4, 6, \dots\}$ and $I_E = \mathcal{P}(E)$. Obviously I_E is a ideal on the set $\mathbb{N}$ of natural numbers. Suppose $X = \{a, b\}$ and $\tau = \{\emptyset, \{a\}, X\}$. Clearly (X, τ) is a topological space. Consider the open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ where

$$U_n = \begin{cases} \{a\} & \text{if } n \in 2\mathbb{N} \\ X & \text{if } n \notin 2\mathbb{N} \end{cases} .$$

Here $\{n \in \mathbb{N} : x \notin U_n\} \in I_E$ for all $x \in X$. $\therefore$ $\mathcal{U}$ is an I_E-γ cover.

Also $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ where

$$V_n = \begin{cases} U_n & \text{if } n = 1 \\ U_{2n} & \text{otherwise} \end{cases}.$$

is a subcover of $\mathcal{U}_n$. But $\{n \in \mathbb{N} : b \notin V_n\} = \mathbb{N} \setminus \{1\} \notin I_E$. Thus $\mathcal{V}$ is not an I_E-γ cover.

Theorem 3.5. *If $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ and $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ are two I-γ covers of a topological space (X, τ) then $\mathcal{W} = \mathcal{U} \sqcup \mathcal{V} = \{W_n = U_n \cup V_n : n \in \mathbb{N}\}$ is also an I-γ cover.*

Proof. Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ and $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ be two I-γ covers of a topological space (X, τ). Thus for all $x \in X$, $\{n \in \mathbb{N} : x \notin U_n\} \in I$ and $\{n \in \mathbb{N} : x \notin V_n\} \in I$.

$\Longrightarrow$ For all $x \in X$, $\{n \in \mathbb{N} : x \in U_n\} \in \mathcal{F}(I)$ and $\{n \in \mathbb{N} : x \in V_n\} \in \mathcal{F}(I)$.

But for all $x \in X$, $\{n \in \mathbb{N} : x \in U_n\} \subseteq \{n \in \mathbb{N} : x \in U_n \cup V_n\} \in \mathcal{F}(I)$.

$\Longrightarrow$ $\{n \in \mathbb{N} : x \notin U_n \cup V_n\} \in I$. Hence $\mathcal{W}$ is an I-γ cover. □

4. On I-St-γ covers

Definition 4.1. A countable cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is said to be an I-St-γ-cover if for each $x \in X$, the set $\{n \in \mathbb{N} : x \notin St(U_n, \mathcal{U})\}$ belongs to I.

Proposition 4.2. *In a topological space X, every I-γ-cover is an I-St-γ-cover.*

Proof. Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be an I-γ-cover of a topological space X. Then for every $x \in X$, $\{n \in \mathbb{N} : x \notin U_n\} \in I$.

Let $p \in \{n \in \mathbb{N} : x \notin St(U_n, \mathcal{U})\}$ for some $x \in X$. Therefore, $x \notin St(U_p, \mathcal{U})$. But $U_p \subseteq St(U_p, \mathcal{U})$. Therefore, $x \notin U_p$. So $p \in \{n \in \mathbb{N} : x \notin U_n\}$.

Thus, $\{n \in \mathbb{N} : x \notin St(U_n, \mathcal{U})\} \subseteq \{n \in \mathbb{N} : x \notin U_n\}$ and $\{n \in \mathbb{N} : x \notin U_n\} \in I$.

Thus, $\{n \in \mathbb{N} : x \notin St(U_n, \mathcal{U})\} \in I$. [By subset property of I].

Therefore, $\mathcal{U}$ is an I-St-γ-cover. □

Corollary 4.3. *An I-dense subset of an I-γ cover of a topological space (X, τ) is an I-St-γ cover of that topological space.*

Proof. The proof is a direct consequence of Proposition 4.2 of this article and Lemma 2.2 in [9]. Hence omitted. □

Corollary 4.4. *Let $(\mathcal{U}_n : n \in \mathbb{N})$, $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$ be a countable sequence of I-γ covers of X. Then $(\mathcal{V}_n : n \in \mathbb{N})$, defined by*

$$\mathcal{V}_n = \{U_{1,m} \cap U_{2,m} \cap \cdots \cap U_{n,m} : m \in \mathbb{N}\} \setminus \{\emptyset\}$$

is a sequence of I-St-γ covers of X.

Proof. The proof is a direct consequence of Proposition 4.2 of this article and Lemma 2.3 in [9]. Hence omitted. □

Example 4.5. The converse of the Proposition 4.2 need not be true. Indeed there exists an I-St-γ cover of a topological space which is not an I-γ cover.

We take $I = I_{fin}$, the finite set ideal on $\mathbb{N}$.

Let $X = S_1((0,0))$ in $\mathbb{R}^2$, the open circle with radius 1 and center at the origin. $\tau = \{S_r((0,0)) : 0 \leq r \leq 1 \text{ and } r \in \mathbb{R}\} \cup \{\emptyset\}$ is a topology on X. Consider the countable open cover $\mathcal{U} = \{U_n = S_{\frac{1}{n}}((0,0)) : n \in \mathbb{N}\}$ of X.

So, for every $n \in \mathbb{N}$, $St(U_n, \mathcal{U}) = X$.

Thus, for every $x \in X$, $\{n \in \mathbb{N} : x \notin St(U_n, \mathcal{U})\} = \emptyset \in I$.

Therefore, $\mathcal{U}$ is an I-St-γ cover of the space X.

Now, $(\frac{1}{2}, 0) \in X$ and $(\frac{1}{2}, 0) \notin S_{\frac{1}{m}}((0,0)) = U_m$ for all $m \geq 2$.

So, $\{n \in \mathbb{N} : (\frac{1}{2}, 0) \notin U_n\} = \{2, 3, 4, ...\} \notin I$.

Thus $\mathcal{U}$ is not an I-γ cover of the space X.

According to P. Das [9], for an admissible ideal I, every γ-cover of a space X is an I-γ-cover for that space but the converse is not true. Thus we have the relation chart for these classes of open covers as shown in figure 1.

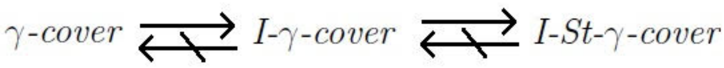

Figure 1. Relation between the variations of γ cover.

Theorem 4.6. *In a topological space X, if $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is an I-St-γ-cover and $F \subseteq X$ is such that $I'_F = \{n \in \mathbb{N} : F \nsubseteq St(U_n, \mathcal{U})\}$ is finite then $I'_F \in I$.*

Proof. Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be an I-St-γ-cover and $F \subseteq X$ is such that $I'_F = \{n \in \mathbb{N} : F \nsubseteq St(U_n, \mathcal{U})\}$ is finite.

Now, let $k \in I'_F = \{n \in \mathbb{N} : F \nsubseteq St(U_n, \mathcal{U})\}$

$\implies F \nsubseteq St(U_k, \mathcal{U})$

$\implies F \setminus St(U_k, \mathcal{U}) \neq \emptyset$

$\implies x \notin St(U_k, \mathcal{U}) \quad \forall x \in F \setminus St(U_k, \mathcal{U})$

$\implies k \in \{n \in \mathbb{N} : x \notin St(U_n, \mathcal{U})\} = I'_x \quad \forall x \in F \setminus St(U_k, \mathcal{U})$

$$\implies k \in \bigcap_{x \in F \setminus St(U_k, \mathcal{U})} I'_x$$

Now, for all $x \in X$, $I'_x = \{n \in \mathbb{N} : x \notin St(U_n, \mathcal{U})\} \in I$.[$\because \mathcal{U}$ is an I-St-γ-cover].

$$\text{But} \bigcap_{x \in F \setminus St(U_k, \mathcal{U})} I'_x \subseteq I_x$$

$$\therefore \text{ By the subset property of ideal, } k \in \bigcap_{x \in F \setminus St(U_k, \mathcal{U})} I'_x \in I \quad \forall k \in I'_F$$

$$\implies \bigcup_{k \in I'_F} \{k\} \subseteq \bigcup_{k \in I'_F} \Big\{ \bigcap_{x \in F \setminus St(U_k, \mathcal{U})} I'_x \Big\}$$

$$\implies I'_F \subseteq \bigcup_{k \in I'_F} \Big\{ \bigcap_{x \in F \setminus St(U_k, \mathcal{U})} I'_x \Big\}$$

But I'_F is finite and by the finite union property of ideal $\bigcup_{k \in I'_F} \Big\{ \bigcap_{x \in F \setminus St(U_k, \mathcal{U})} I'_x \Big\} \in I$

and by the subset property of ideal $I'_F \in I$. □

Theorem 4.7. *Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be an open cover of a space X. If for each $F \subseteq X$ and for each $p \in \{n \in \mathbb{N} : F \nsubseteq St(U_n, \mathcal{U})\} = I'_F$, finiteness of $F \setminus St(U_p, \mathcal{U})$ implies that $I'_F \in I$ then $\mathcal{U}$ is an I-St-γ-cover.*

Proof. Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be an open cover of the topological space X. Let for each $F \subseteq X$ and for each $p \in \{n \in \mathbb{N} : F \nsubseteq St(U_n, \mathcal{U})\} = I'_F$, finiteness of $F \setminus St(U_p, \mathcal{U})$ implies that $I'_F \in I$.

Now, $\forall x \in X$, $\{x\} \subseteq X$ and $\{x\} \setminus St(U_p, \mathcal{U})$ is finite for each $p \in \{n \in \mathbb{N} : \{x\} \nsubseteq St(U_n, \mathcal{U})\} = I'_{\{x\}}$.

$\therefore$ By our assumption, $I'_{\{x\}} \in I \quad \forall x \in X$.

$\implies \{n \in \mathbb{N} : \{x\} \nsubseteq St(U_n, \mathcal{U})\} \in I$ for every $x \in X$.

$\implies \{n \in \mathbb{N} : x \notin St(U_n, \mathcal{U})\} \in I$ for every $x \in X$.

$\therefore \mathcal{U}$ is an I-St-γ-cover. □

Proposition 4.8. *An I-St-γ cover of a topological space in which every pair of distinct open sets are disjoint is an I-γ cover of that space.*

Proof. The theorem follows automatically, so it's proof is omitted. □

Example 4.9. In a topological space (X,τ), if $A \subseteq X$ and if $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is an I-St-γ cover of (X,τ), then $\mathcal{U}_A = \{A \cap U_n : U_n \in \mathcal{U}\}$ may not be an I-St-γ cover for (A,τ_A) (τ_A denotes the sub-space topology).

Let $E = \{2,4,6,\dots\}$ and $I_E = \mathcal{P}(E)$. Obviously I_E is a ideal on the set $\mathbb{N}$ of natural numbers. Suppose $X = [0,3)$. Then $\mathcal{B} = \{\emptyset, [0,1), [1,2), [2,3)\}$ is a base for a suitable topology on the set X and let the topology be τ. Consider the open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ where

$$U_n = \begin{cases} [0,2) & \text{if } n \in \mathbb{N} \setminus 2\mathbb{N} \\ [1,3) & \text{if } n \in 2\mathbb{N} \end{cases}.$$

Here, $St(U_n,\mathcal{U}) = X$ for all $n \in \mathbb{N}$.

Hence $\{n \in \mathbb{N} : x \notin U_n\} = \emptyset \in I_E$ for all $x \in X$. $\therefore$ $\mathcal{U}$ is an I_E-St-γ cover.

Now consider $A = [0,1) \cup [2,3) \subseteq X$. (A,τ_A) is a subspace topology. Here $\mathcal{U}_A = \{V_n = A \cap U_n : U_n \in \mathcal{U}\}$ where

$$V_n = \begin{cases} [0,1) & \text{if } n \in \mathbb{N} \setminus 2\mathbb{N} \\ [2,3) & \text{if } n \in 2\mathbb{N} \end{cases}.$$

$$St(V_n,\mathcal{U}_A) = \begin{cases} [0,1) & \text{if } n \in \mathbb{N} \setminus 2\mathbb{N} \\ [2,3) & \text{if } n \in 2\mathbb{N} \end{cases}.$$

Here, $2.5 \in A$, $\{n \in \mathbb{N} : 2.5 \notin St(V_n,\mathcal{U}_A)\} = \mathbb{N} \setminus 2\mathbb{N} \notin I_E$. Therefore $\mathcal{U}_A$ is not an I_E-St-γ cover for (A,τ_A).

The above property makes a huge difference between I-γ covers and I-St-γ covers.

Example 4.10. Subcover of an I-St-γ cover may not be an I-St-γ cover.

Let $A = \{3,6,9,\dots\}$ and $I_3 = \mathcal{P}(A)$. Obviously I_3 is a ideal on the set $\mathbb{N}$ of natural numbers. Suppose $X = \{a,b,c\}$ and τ is the discrete topology on X. Consider the open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ where

$$U_n = \begin{cases} \{a,b\} & \text{if } n \in \{(m-2) : m \in 3\mathbb{N}\} \\ \{b,c\} & \text{if } n \in \{(m-1) : m \in 3\mathbb{N}\} \\ \{c\} & \text{if } n \in 3\mathbb{N} \end{cases}.$$

Here

$$St(U_n,\mathcal{U}) = \begin{cases} X & \text{if } n \in \{(m-2) : m \in 3\mathbb{N}\} \\ X & \text{if } n \in \{(m-1) : m \in 3\mathbb{N}\} \\ \{b,c\} & \text{if } n \in 3\mathbb{N} \end{cases}.$$

$\therefore$ $\{n \in \mathbb{N} : x \notin St(U_n,\mathcal{U})\} \in I_3$ for all $x \in X$. $\therefore$ $\mathcal{U}$ is an I_3-St-γ cover.

Consider the subcover $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ of $\mathcal{U}_n$. where

$$V_n = \begin{cases} U_{3n+1} & \text{if } n \in \mathbb{N} \setminus 2\mathbb{N} \\ U_{3n} & \text{if } n \in 2\mathbb{N} \end{cases}.$$

Also

$$St(V_n,\mathcal{V}) = \begin{cases} U_{3n+1} & \text{if } n \in \mathbb{N} \setminus 2\mathbb{N} \\ U_{3n} & \text{if } n \in 2\mathbb{N} \end{cases}.$$

But $\{n \in \mathbb{N} : c \notin St(V_n,\mathcal{V})\} = \mathbb{N} \setminus 2\mathbb{N} \notin I_3$. Thus $\mathcal{V}$ is not an I_3-St-γ cover.

Theorem 4.11. *If $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ and $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ are two I-St-γ covers of a topological space (X,τ) then $\mathcal{W} = \mathcal{U} \sqcup \mathcal{V} = \{W_n = U_n \cup V_n : n \in \mathbb{N}\}$ is also an I-St-γ cover.*

Proof. Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ and $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ be two I-St-γ covers of a topological space (X, τ). Thus for all $x \in X$, $\{n \in \mathbb{N} : x \notin St(U_n, \mathcal{U})\} \in I$ and $\{n \in \mathbb{N} : x \notin St(V_n, \mathcal{V})\} \in I$.

$\implies$ For all $x \in X$, $\{n \in \mathbb{N} : x \in St(U_n, \mathcal{U})\} \in \mathcal{F}(I)$ and $\{n \in \mathbb{N} : x \in St(V_n, \mathcal{V})\} \in \mathcal{F}(I)$.

But For all $x \in X$, $\{n \in \mathbb{N} : x \in St(U_n, \mathcal{U})\} \subseteq \{n \in \mathbb{N} : x \in St((U_n \cup V_n), \mathcal{W})\} \in \mathcal{F}(I)$.

$\implies$ $\{n \in \mathbb{N} : x \notin St(W_n, \mathcal{W})\} \in I$. Hence $\mathcal{W}$ is an I-St-γ cover. □

5. On ideal based subsets of I-St-γ covers

It is basically understood that an infinite subset of a γ cover is also a γ cover and an infinite subset of an I-γ cover may not be an I-γ cover. In Lemma 2.2 of [9], it has been established that an I-dense subset of an I-γ cover is always an I-γ cover. Now we want to investigate the nature of I-dense subset of an I-St-γ cover.

Example 5.1. *I-dense subset of an I-St-γ-cover may not be a cover at all.*

Let $A = \{2, 3, 5, 6, 8, 9, \dots\}$ and consider an ideal $I = \mathcal{P}(A)$ of the set of all natural number $\mathbb{N}$ ($\mathcal{P}(A)$ is the power set of A). I is obviously a proper Ideal of $\mathbb{N}$.

Obviously, $A \in I$.

$\implies$ $\mathbb{N} \setminus A \in \mathcal{F}(I)$.

i.e. $\{1, 4, 7, \dots\} \in \mathcal{F}(I)$.

Now we construct a topological space on $X = [0, 4)$. Let $\mathcal{B} = \{\emptyset, [0, 1), [1, 2), [2, 3), [3, 4)\}$. Clearly, $\mathcal{B}$ forms a base for a suitable topology on X. Let τ be the topology generated by the base $\mathcal{B}$.

Let us consider $\mathcal{U} = \{U_1 = [0, 3), U_2 = [1, 2), U_3 = [2, 4), U_4 = U_1, U_5 = U_2, U_6 = U_3, U_7 = U_1, U_8 = U_2, U_9 = U_3, \dots\}$. We want to show that $\mathcal{U}$ is an I-St-γ-cover with respect to the mentioned ideal I.

So we have $St(U_1, \mathcal{U}) = St(U_4, \mathcal{U}) = St(U_7, \mathcal{U}) = \cdots = X$;

$St(U_2, \mathcal{U}) = St(U_5, \mathcal{U}) = St(U_8, \mathcal{U}) = \cdots = [0, 3)$;

$St(U_3, \mathcal{U}) = St(U_6, \mathcal{U}) = St(U_9, \mathcal{U}) = \cdots = X$.

Ultimately, $\{n \in \mathbb{N} : x \notin St(U_n, \mathcal{U})\}$ is either $\emptyset$ or $\{2, 5, 8, \dots\}$ and both of them belongs to the ideal I.

Let $\mathcal{V} = \{U_1, U_4, U_7, \dots\}$. $\mathcal{V}$ is an I-dense subset of $\mathcal{U}$ since $\{1, 4, 7, \dots\} \in \mathcal{F}(I)$. But $\bigcup \mathcal{V} = [0, 3) \neq X$. i.e. $\mathcal{V}$ is not a cover at all. So we can conclude that I-dense subset of an I-St-γ-cover may not be a cover at all.

Since I-dense subset of an I-St-γ cover may not be an I-St-γ cover, we want to impose a criteria on the subset of I-St-γ cover so that a subset becomes an I-St-γ cover. In this regard we introduce the notion of *I-St-dense* subset.

Definition 5.2. Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be a cover of a topological space (X, τ) and $\mathcal{V} = \{U_{m_1}, U_{m_2}, U_{m_3}, \dots\}$ be a subset of $\mathcal{U}$ where $m_1 < m_2 < m_3 < \dots$. We will say $\mathcal{V}$ is I-St dense in $\mathcal{U}$ if the set $\{m_i \in \mathbb{N} : U_{m_i} \in \mathcal{V}$ and $St(U_{m_i}, \mathcal{V}) = X\} \in \mathcal{F}(I)$.

Firstly we verify how different is this I-St-denseness in comparison to I-denseness.

Theorem 5.3. *Every I-St-dense subset of an open cover in a topological space (X, τ) is an I-dense subset of that specific open cover.*

Proof. Let $\mathcal{V} = \{U_{m_1}, U_{m_2}, U_{m_3}, \dots\}$, where $m_1 < m_2 < m_3 < \dots$ be an I-St-dense subset of an open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ in a topological space (X, τ).

$\implies$ $\{m_i \in \mathbb{N} : U_{m_i} \in \mathcal{V}$ and $St(U_{m_i}, \mathcal{V}) = X\} \in \mathcal{F}(I)$.

But $\{m_i \in \mathbb{N} : U_{m_i} \in \mathcal{V}$ and $St(U_{m_i}, \mathcal{V}) = X\} \subseteq \{m_i \in \mathbb{N} : U_{m_i} \in \mathcal{V}\}$

$\therefore$ By the superset property of the dual filter, $\{m_i \in \mathbb{N} : U_{m_i} \in \mathcal{V}\} \in \mathcal{F}(I)$. i.e. $\mathcal{V}$ is an I-dense subset of $\mathcal{U}$ in (X, τ). Hence the theorem. □

Example 5.4. I-dense subset of an open cover in a topological space (X, τ) may not be an I-St-dense subset for that open cover.

Consider the Ideal I, a topological space (X, τ), an open cover $\mathcal{U}$ and the subset $\mathcal{V}$ of the open cover $\mathcal{U}$ constructed in example 5.1. It is already shown in the example 5.1 that $\mathcal{V} = \{U_1, U_4, U_7, \dots\}$ is I-dense in $\mathcal{U}$ in (X, τ). Now for every $U_i \in \mathcal{V}$, $St(U_i, \mathcal{V}) = [0, 3) \neq X$. $\therefore$ $\{m_i \in \mathbb{N} : U_{m_i} \in \mathcal{V}$ and $St(U_{m_i}, \mathcal{V}) = X\} = \emptyset \notin \mathcal{F}(I)$. $\therefore$ $\mathcal{V}$ is not an I-St-dense subset for that open cover $\mathcal{U}$. Hence I-dense subset of an open cover in a topological space (X, τ) may not be an I-St-dense subset for that open cover.

Relation between I-dense subsets and I-st-dense subsets is shown in figure 2.

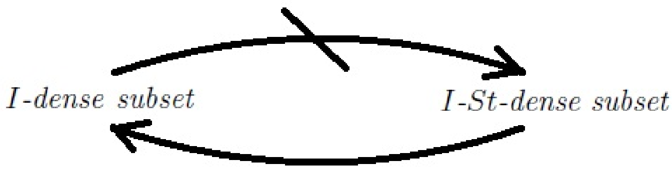

Figure 2. Relation between I-dense subsets and I-St-dense subsets.

Example 5.5. I-St-dense subset of an I-St-γ cover may not be an I-St-γ cover at all.

Let $A = \{3, 4, 7, 8, 11, 12, \dots\}$ and consider an ideal $I = \mathcal{P}(A)$ of the set of all natural number $\mathbb{N}$. I is obviously a proper Ideal of $\mathbb{N}$.

Obviously, $A \in I$.

$\implies \mathbb{N} \setminus A \in \mathcal{F}(I)$.

i.e. $\{1, 2, 5, 6, 9, 10, \dots\} \in \mathcal{F}(I)$.

Now we construct a topological space on $X = [0, 5)$. Let $\mathcal{B} = \{\emptyset, [0, 1), [1, 2), [2, 3), [3, 4), [4, 5)\}$. Clearly, $\mathcal{B}$ forms a base for a suitable topology on X. Let τ be the topology generated by the base $\mathcal{B}$.

Let us consider $\mathcal{U} = \{U_1 = [0, 3), U_2 = [2, 5), U_3 = [1, 2), U_4 = [3, 4), U_5 = U_1, U_6 = U_2, U_7 = U_3, U_8 = U_4, U_9 = U_1, U_{10} = U_2, U_{11} = U_3, U_{12} = U_4, \dots\}$. We want to show that $\mathcal{U}$ is an I-St-γ-cover with respect to the mentioned ideal I.

So we have,

$St(U_1, \mathcal{U}) = St(U_5, \mathcal{U}) = St(U_9, \mathcal{U}) = \cdots = X$;

$St(U_2, \mathcal{U}) = St(U_6, \mathcal{U}) = St(U_{10}, \mathcal{U}) = \cdots = X$;

$St(U_3, \mathcal{U}) = St(U_7, \mathcal{U}) = St(U_{11}, \mathcal{U}) = \cdots = [0, 3)$;

$St(U_4, \mathcal{U}) = St(U_8, \mathcal{U}) = St(U_{12}, \mathcal{U}) = \cdots = [2, 5)$.

Ultimately, $\{n \in \mathbb{N} : x \notin St(U_n, \mathcal{U})\}$ is either $\emptyset$ or $\{3, 7, 11, \dots\}$ or $\{4,8,12,\dots\}$ and all of them belong to the ideal I.

Let $\mathcal{V} = \{V_1 = U_1, V_2 = U_2, V_3 = U_3, V_4 = U_5, V_5 = U_6, V_6 = U_7, V_7 = U_9, V_8 = U_{10}, V_9 = U_{11}, \dots\}$ be a subset of $\mathcal{U}$.

$\{m_i \in \mathbb{N} : U_{m_i} \in \mathcal{V}$ and $St(U_{m_i}, \mathcal{V}) = X\}$

$= \{1, 2, 5, 6, 9, 10, \dots\} \in \mathcal{F}(I)$.

$\therefore$ $\mathcal{V}$ is I-St-dense in $\mathcal{U}$.

Now,

$St(V_1, \mathcal{V}) = St(V_4, \mathcal{V}) = St(U_7, \mathcal{V}) = \cdots = X$;

$St(V_2, \mathcal{V}) = St(V_5, \mathcal{V}) = St(U_8, \mathcal{V}) = \cdots = X$;

$St(V_3, \mathcal{V}) = St(V_6, \mathcal{V}) = St(U_9, \mathcal{V}) = \cdots = [0, 3)$.

This implies that the set

$\{k \in \mathbb{N} : 3 \notin St(U_{m_k}, \mathcal{V})\}$

$= \{k \in \mathbb{N} : 3 \notin St(V_k, \mathcal{V})\}$

$= \{3, 6, 9, \dots\} \notin I$

$\therefore \mathcal{V}$ is not an I-St-γ cover for the topological space (X, τ).
Hence, I-St-dense subset of an I-St-γ cover may not be an I-St-γ cover at all.

Variations of γ covers under the variation of I-dense subsets are shown in figure 3.

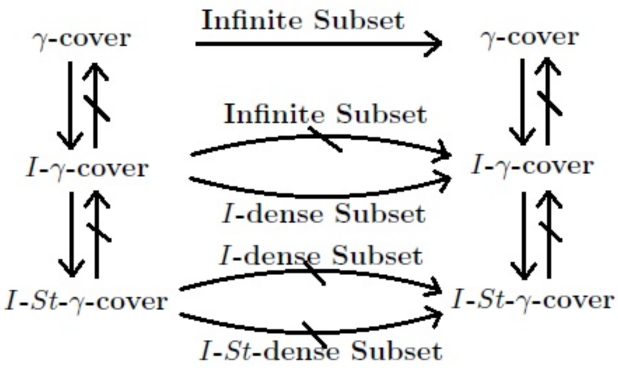

Figure 3. Variations of γ covers under the variation of I-dense subsets.

Problem 5.6. What condition can be imposed on the subset of an I-St-γ cover to make it an I-St-γ cover?

6. Conclusion

(1) In a topological space X, if $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is an I-γ-cover and $F \subseteq X$ be such that $I_F = \{n \in \mathbb{N} : F \nsubseteq U_n\}$ is finite then $I_F \in I$.
(2) For an open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of a space X, if for each $F \subseteq X$ and for each $p \in \{n \in \mathbb{N} : F \nsubseteq U_n\} = I_F$, finiteness of $F \setminus U_p$ implies that $I_F \in I$, then $\mathcal{U}$ is an I-γ-cover.
(3) In a topological space X, every I-γ-cover is an I-St-γ-cover. But converse may not be true.
(4) An open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of a topological space X is an I-St-γ-cover of X if for each finite set $F \subseteq X$ the set $\{n \in \mathbb{N} : F \nsubseteq St(U_n, \mathcal{U})\} \in I$.
(5) An I-St-γ cover of a topological space in which every pair of distinct open sets are disjoint is an I-γ cover of that space.
(6) If $\mathcal{U}$ is an I-γ cover for a topological space (X, τ) then the cover formed by using $\mathcal{U}$ for the subspace (A, τ_A) of the space (X, τ) is also an I-γ cover. But this result does not hold for I-St-γ covers.
(7) Subcover of an I-γ cover may not be an I-γ cover.
(8) Subcover of an I-St-γ cover may not be an I-St-γ cover.
(9) I-dense subset of an I-St-γ-cover may not be a cover at all.
(10) Every I-St-dense subset of an open cover in a topological space (X, τ) is an I-dense subset of that specific open cover. But converse may not be true.
(11) I-St-dense subset of an I-St-γ cover may not be an I-St-γ cover at all.

Data Availability: In this article no dataset has been generated or analysed. So data sharing is not applicable here.

Conflict of Interest: On behalf of all authors, the corresponding author states that there is no conflict of interest.

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References

[1] P. Bal and S. Bhowmik, *Star-selection principle: Another new direction*, J. Indian Math. Soc. **84** (1-2), 01-06, 2017.

[2] P. Bal and S. Bhowmik, *On R-star-Lindelöf spaces*, Palest. J. Math. **6** (2), 480-486, 2017.

[3] P. Bal and S. Bhowmik, *Some new star-selection principles in topology*, Filomat, **31** (13), 4041-4050, 2017.

[4] P. Bal, S. Bhowmik and D. Gauld, *On selectively star-Lindelöf properties*, J. Indian Math. Soc. **85** (3-4), 291-304, 2018.

[5] P. Bal and Lj.D.R. Kočinac, *On selectively star-ccc spaces*, Topology Appl. **281**, 107184, 2020.

[6] M. Bonanzinga and F. Maesano, *Some properties defined by relative versions of star-covering properties*, Topology Appl. **306**, 107-923, 2022.

[7] M. Bonanzinga, D. Giacopello and F. Maesano, *Some properties defined by relative versions of star-covering properties II*, submitted.

[8] J. Connor, *The statistical and strong P-Cesaro convergences of sequences*, Analysis, **8**, 22-32, 1988.

[9] P. Das, *Certain types of open covers and selection principles using ideals*, Houston J. Math. **39** (2), 637-650, 2013.

[10] E.K. van Douwen, G.M. Reed, A.W. Roscoe and I.J. Tree, *Star covering properties*, Topology Appl. **39** (1), 71-103, 1991.

[11] R. Engelking, *General topology*, Sigma Series in Pure Mathematics, Revised and complete ed. Berlin: Heldermann, 1989.

[12] H. Fast, *Sur la convergence statistique*, Colloq. Math. **2**, 241 - 244, 1951.

[13] J.A. Fridy, *On statistical convergence*, Analysis, **5**, 301-313, 1985.

[14] Lj.D.R. Kočinac, *Star selection principles : A survey*, Khayyam J. Math. **1**, 82-106, 2015.

[15] Lj.D.R. Kočinac, *Star-Menger and related spaces II*, Filomat, **13**, 129-140, 1999.

[16] Lj.D.R. Kočinac, *Star-Menger and related spaces*, Publ. Math. Debrecen **55**, 421-431, 1999.

[17] Lj.D.R. Kočinac and S. Singh, *On the set version of selectively star-ccc spaces*, J. Math. **2020**, 9274503, 2020.

[18] Lj.D.R. Kočinac and S Özbçağ, *More on selective covering properties in bitopological space*, J. Math. **2021**, 5558456, 2021.

[19] P. Kostyrko, T. Šalat and W. Wilczynski, *I- convergence*, Real Anal. Exchange **26**, 669-686, 2001.

[20] B.K. Lahiri and P. Das, *I and I* convergence of nets*, Real Anal. Exchange **32**, 431-442, 2008.

[21] B.K. Lahiri and P. Das, *I and I* convergence in topological spaces*, Math. Bohemica **130**, 153-160, 2005.

[22] G. Di Maio and Lj.D.R. Kočinak, *Statistical convergence in topology*, Topology Appl. **156**, 28-45, 2008.

[23] M. Sakai, *Star covering versions of the Menger property*, Topology Appl. **176**, 2234, 2014.

[24] Y.K. Song and W.F. Xuan, *A note on selectively star-ccc spaces*, Topology Appl. **263**, 343-349, 2019.

[25] Y.K. Song and W.F. Xuan, *More on selectively star-ccc spaces*, Topology Appl. **268**, 106905, 2019.

[26] W.F. Xuan and Y.K. Song, *A study of selectively star-ccc spaces*, Topology Appl. **273**, 107103, 2020.