Degenerate Clifford Algebras and Their Reperesentations

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Abstract

In this study, we give an imbedding theorem for a degenerate Clifford algebra into non-degenerate one. By using the representations of non-degenerate Clifford algebra we develop a method for the representations of the degenerate Clifford algebras. We give some explicit constructions for lower dimensions.

Keywords: *Degenerate Clifford algebra, representation of Clifford algebras, quadratic form, nilpotent ideal.*

Dejenere Clifford Cebirleri ve Temsilleri

Özet

Bu çalışmada dejenere Clifford cebirlerinden dejenere olmayan Clifford cebirlerine bir gömme teoremi verdik. Dejenere olmayan Clifford Cebirlerinin temsillerini kullanarak dejenere Clifford cebirlerinin temsilleri için bir metod geliştirdik. Düşük boyutlar için bazı açık temsilleri verdik.

Anahtar kelimeler: *Dejenere Clifford cebri, Clifford cebirlerinin temsili, kuadratik form, nilpotent ideal.*

1 Introduction

It is well known that a non-degenerate real Clifford algebra is isomorphic to a matrix algebra $\mathbb{F}(n)$ or direct sum $\mathbb{F}(n) \oplus \mathbb{F}(n)$ of matrix algebras, where $\mathbb{F}=\mathbb{R},\mathbb{C}$, or $\mathbb{H}$ (see [1,2, 3]). Their structures and representations are being studied by mathematicians and physicists. On the other hand, a degenerate real Clifford algebra can not be isomorphic

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to full matrix algebras as they contain nilpotent ideal. There are some remarks about the degenerate Clifford algebra and degenerate spin groups in [4]. Exterior algebras and dual numbers are special cases of the degenerate Clifford algebras.

Definition 1 *The Clifford algebra $Cl(V,Q)$ associated to a vector space V over F with quadratic form Q is defined as the quotient algebra*

$$Cl(V,Q) = T(V)/I(Q)$$

where T(V) is tensor algebra $T(V) = F \oplus V \oplus (V \otimes V) \oplus \cdots$ and I (Q) is the two-sided ideal in T(V) generated by the elements $\{v \otimes v - Q(v) \cdot 1\}$.

As an example when $Q = 0$ then the Clifford algebra $Cl(V,Q)$ is just the Grassmann algebra (also known as the exterior algebra) $\Lambda(V)$.

A non-degenerate quadratic form Q on the real vector space $V=\mathbb{R}^n$ is given by

$$Q(x) = x_1^1 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 \quad (n = p+q).$$

The Clifford algebra on $\mathbb{R}^n$ associated to Q is denoted by $Cl_{p,q}$. The table of non-degenerate real Clifford algebras is as follows (see [2, 3]):

$(p-q)(\mathrm{mod}\,8)$	$p+q$	$Cl_{p,q}$
0,2	$2m$	$\mathbb{R}(2^m)$
1	$2m+1$	$\mathbb{R}(2^m) \oplus \mathbb{R}(2^m)$
3,7	$2m+1$	$\mathbb{C}(2^m)$
4,6	$2m+2$	$\mathbb{H}(2^m)$
5	$2m+3$	$\mathbb{H}(2^m) \oplus \mathbb{H}(2^m)$

The explicit isomorphism $\Phi_{p,q}$ from the Clifford algebra $Cl_{p,q}$ to the related matrix algebra is given in [5] for all p,q.

Definition 2 *Let A be a real algebra and W a real vector space, then an algebra homomorphism $\rho : A \to End(W)$ is called a representation of the algebra A.*

For example, the canonical representation of the matrix group $\mathbb{R}$(n) on $\mathbb{R}^n$ is ordinary product of a matrix with a vector. The canonical representation of the algebra of complex numbers $\mathbb{C}$ on $\mathbb{R}^2$ is given $\rho : \mathbb{C} \to \mathbb{R}(2) \cong \mathrm{End}(\mathbb{R}^2)$

$$\rho(x+iy) = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}$$

for all $x+iy \in \mathbb{C}$. One can obtain the canonical representation of the algebra $\mathbb{C}$(n), n by n matrices with complex entries, on the vector space $\mathbb{R}^{2n}$ as follows: Let $A = [z_{ij}]_{n \times n} \in \mathbb{C}(n)$ be a matrix of complex numbers z_{ij}. Using the canonical representation of $\mathbb{C}$ we obtain an algebra homomorphism

$$\rho_n : \mathbb{C}(n) \to \mathbb{R}(2n) \cong \mathrm{End}(\mathbb{R}^{2n})$$

whereby

$$\rho_n(A) = \begin{bmatrix} \rho(z_{11}) & \rho(z_{12}) & \cdots & \rho(z_{1n}) \\ \rho(z_{21}) & \rho(z_{22}) & \cdots & \rho(z_{2n}) \\ \vdots & \vdots & \cdots & \vdots \\ \rho(z_{n1}) & \rho(z_{n2}) & \cdots & \rho(z_{nn}) \end{bmatrix}$$

Similarly the canonical representation of the quaternion algebra $\mathbb{H}$ on $\mathbb{R}^4$ is given by $\rho : \mathbb{H} \to \mathbb{R}(4) \cong \mathrm{End}(\mathbb{R}^4)$,

$$\rho(x + iy + ju + kv) = \begin{bmatrix} x & -y & -u & -v \\ y & x & -v & u \\ u & v & x & -y \\ v & -u & y & x \end{bmatrix}$$

for all $x + iy + ju + kv \in \mathbb{H}$. Likewise the complex case using the canonical representation $\rho : \mathbb{H} \to \mathbb{R}(4)$ we obtain an algebra homomorphism

$$\rho_n : \mathbb{H}(\mathrm{n}) \to \mathbb{R}(4\mathrm{n}) \cong \mathrm{End}(\mathbb{R}^{4n})$$

whereby

$$\rho_n(A) = \begin{bmatrix} \rho(q_{11}) & \rho(q_{12}) & \cdots & \rho(q_{1n}) \\ \rho(q_{21}) & \rho(q_{22}) & \cdots & \rho(q_{2n}) \\ \vdots & \vdots & \cdots & \vdots \\ \rho(q_{n1}) & \rho(q_{n2}) & \cdots & \rho(q_{nn}) \end{bmatrix}$$

which is the canonical representation of $\mathbb{H}(\mathrm{n})$, the algebra of $n \times n-$matrices with quaternion entries. By using the canonical representations of matrix algebras one can obtain representation of non-degenerate Clifford algebras Clp,q for all p,q. The representations of the Clp,q are being studied by various authors (see [6, 7, 8]). There are various applications of the matrix representations of Clifford algebras ([5, 9, 10]). For the explicit expression of Dirac operator the representation of the Clifford algebra is also critical (see [11]).

2 Degenerate Clifford algebras on $\mathbb{R}^n$

In this section we investigate Clifford algebras associated to a degenerate quadratic form Q on $\mathbb{R}^n$. From Sylvester theorem it is enough to consider the following degenerate quadratic form

$$Q(x) = x_1^1 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 \quad (n = p + q + r, r > 0).$$

The Clifford algebra associated to the degenerate quadratic form $Q(x)$ is denoted by $Cl_{p,q,r}$. We can determine the degenerate Clifford algebra $Cl_{p,q,r}$ as follows:

Let

$$\{e_1, e_2, \ldots, e_p, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_q, \theta_1, \ldots, \theta_r\}$$

be an orthonormal basis for $\mathbb{R}^n$ with respect to the degenerate quadratic form

$Q(x)$ .Then, we have $xy = -yx$ for the basis elements x and y and $x \neq y$. Also, the following identities hold:

$$\begin{array}{lll} e_i^2 = Q(e_i)\cdot 1 = 1 & \text{for} & 1 \le i \le p \\ \varepsilon_j^2 = Q(\varepsilon_j)\cdot 1 = -1 & \text{for} & 1 \le j \le q \\ \theta_k^2 = Q(\theta_k)\cdot 1 = 0 & \text{for} & 1 \le k \le r \end{array}$$

Definition 3 *An ideal $\Im$ of the algebra A is called nilpotent if its power $\Im^k$ is $\{0\}$ for some positive integer k. Such least k is called order of $\Im$.*

All elements of the form

$$\sum_I a_I \theta_I + \sum_{J,K} b_{JK} e_J \theta_K + \sum_{L,M} c_{LM} \varepsilon_L \theta_M + \sum_{P,R,S} d_{PRS} e_P \varepsilon_R \theta_S$$

constitute an ideal of $Cl_{p,q,r}$, where $a_I, b_{JK}, c_{LM}, d_{PRS}$ are real numbers and we use multiple index notation, i.e., $\theta_I = \theta_{i_1} \ldots \theta_{i_k}$ for $I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, r\}$ and $i_1 < i_2 < \cdots < i_k$. We denote this ideal by $\Im$, some authors call it the Jacobson radical [4]. The set $\{\theta_I, e_J\theta_K, \varepsilon_L\theta_M, e_P\varepsilon_R\theta_S\}$ is a basis for the ideal $\Im$ and also note that the all elements are nilpotent. Then, $\Im$ is nilpotent ideal (see 13.28 Theorem in [12] on page 479).

Nilpotent ideals are important to the characterization of an algebra.

Theorem 4 *Let A be an algebra with unit. Then, A is semi-simple if and only if there is no nilpotent ideal different from zero.*

$Cl_{p,q,r}$ can not be a matrix algebra for $r > 0$, because $\Im$ is nilpotent ideal. By the Theorem 4 the Clifford algebra $Cl_{p,q,r}$ is not semi-simple algebra. On the other hand, $Cl_{p,q,r}$ can be embedded into the matrix algebras.

2.1 Embeddings of degenerate Clifford algebras into matrix algebras

Now let us explain a theorem concerning the degenerate Clifford algebras.

Theorem 5 *The degenerate Clifford algebra $Cl_{p,q,r}$ can be embedded into the non-degenerate Clifford algebra $Cl_{p+r,q+r}$.*

Proof It is enough to give a one to one algebra homomorphism from $Cl_{p,q,r}$ to $Cl_{p+r,q+r}$. It is possible to extend it to whole algebra when the homomorphism is defined on the basis elements. This homomorphism is defined on the basis elements in the following way:

$$
\begin{array}{rcl}
\Psi_{p,q,r} : Cl_{p,q,r} & \to & Cl_{p+r,q+r} \\
e_1 & \mapsto & e_1 \\
\vdots & & \vdots \\
e_p & \mapsto & e_p \\
\varepsilon_1 & \mapsto & \varepsilon_1 \\
\vdots & & \vdots \\
\varepsilon_q & \mapsto & \varepsilon_q \\
\theta_1 & \mapsto & e_{p+1} + \varepsilon_{q+1} \\
\vdots & & \vdots \\
\theta_r & \mapsto & e_{p+r} + \varepsilon_{q+r}
\end{array}
$$

Moreover, there are following equalities:

$$
\Psi_{p,q,r}(e_i^2) = \Psi_{p,q,r}(1) = 1 = [\Psi_{p,q,r}(e_i)]^2 = e_i^2
$$
$$
\Psi_{p,q,r}(\varepsilon_i^2) = \Psi_{p,q,r}(-1) = -1 = [\Psi_{p,q,r}(\varepsilon_i)]^2 = \varepsilon_i^2
$$

$$
\begin{array}{rcl}
\Psi_{p,q,r}(\theta_i^2) & = & \Psi_{p,q,r}(0) = 0 \\
[\Psi_{p,q,r}(\theta_i)]^2 & = & [e_{p+i} + \varepsilon_{q+i}]^2 = e_{p+i}^2 + e_{p+i}\varepsilon_{q+i} + \varepsilon_{q+i}e_{p+i} + \varepsilon_{q+i}^2 \\
 & = & 1 + e_{p+i}\varepsilon_{q+i} - e_{p+i}\varepsilon_{q+i} - 1 = 0
\end{array}
$$

From this theorem, degenerate Clifford algebra $Cl_{p,q,r}$ can be embedded into the matrix algebras such as $\mathbb{R}(m)$, $\mathbb{C}(m)$, $\mathbb{H}(m)$, $\mathbb{R}(m) \oplus \mathbb{R}(m)$, $\mathbb{H}(m) \oplus \mathbb{H}(m)$.

Remark 6 *There is a general embedding theorem of an algebra into an endomorphism algebra (see [13] Theorem 2.11 on page 49). In this case, the dimension of the representation space became larger and it is not useful for explicit constructions.*

Let us give some examples in lower dimensions:

Example 7 The full degenerate Clifford algebra $Cl_{0,0,2}$ which is equivalent to the Grassmann algebra $\Lambda(\mathbb{R}^2)$ can be embedded into the non-degenerate Clifford algebra $Cl_{2,2}$ in the following way:

$$
\begin{array}{rcl}
f : Cl_{0,0,2} = \Lambda(\mathbb{R}^2) & \to & Cl_{2,2} \\
\theta_1 & \mapsto & e_1 + \varepsilon_1 \\
\theta_2 & \mapsto & e_2 + \varepsilon_2
\end{array}
$$

Since $Cl_{2,2}$ is isomorphic to the matrix algebra $\mathbb{R}(4)$, one can also give an imbedding from $Cl_{0,0,2}$ into the matrix algebra $\mathbb{R}(4)$ as follows:

$$\begin{array}{ccc} Cl_{0,0,2} & \to & Cl_{2,2} \cong \mathbb{R}(4) \\ \theta_1 & \mapsto & e_1+\varepsilon_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \\ \theta_2 & \mapsto & e_2+\varepsilon_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix} \end{array}$$

Example 8 Degenerate Clifford algebra $Cl_{1,0,1}$ can be embedded into the non-degenerate Clifford algebra $Cl_{2,1}$. We know that $Cl_{2,1}$ is isomorphic to the matrix algebra $\mathbb{R}(2) \oplus \mathbb{R}(2)$,. The embedding *f* can be given on the basis elements as follows:

$$\begin{array}{ccc} f: Cl_{1,0,1} & \to & Cl_{2,1} \\ e_1 & \mapsto & e_1 \\ \theta_1 & \mapsto & e_2+\varepsilon_1 \end{array}$$

The map *f* defined above can be seen that it is an one to one algebra homomorphism. If we use the composition of the homomorphism *f* and the isomorphism $\Psi_{2,1}$, then we get an one to one algebra homomorphism as follows:

$$\begin{array}{ccc} Cl_{1,0,1} & \to & Cl_{2,1} \cong \mathbb{R}(2) \oplus \mathbb{R}(2) \\ e_1 & \mapsto & e_1 = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \\ \theta_1 & \mapsto & e_2+\varepsilon_1 = \left(\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \right) \end{array}$$

where $e_1\theta_1 = \left(\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \right)$.

Any $x \in Cl_{1,0,1}$ can be written as $x = x_0 \cdot 1 + x_1e_1 + x_2\theta_1 + x_3e_1\theta_1$. By writing the values of $1, e_1, \theta_1, e_1\theta_1$ we get the following identity:

$$\begin{aligned} x = \; & x_0 \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) + x_1 \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \\ & + x_2 \left(\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \right) + x_3 \left(\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \right) \end{aligned}$$

$$= \left(\begin{bmatrix} x_0+x_2+x_3 & x_1-x_2-x_3 \\ x_1+x_2+x_3 & x_0-x_2-x_3 \end{bmatrix}, \begin{bmatrix} x_0-x_2+x_3 & x_1-x_2+x_3 \\ x_1+x_2-x_3 & x_0+x_2-x_3 \end{bmatrix} \right)$$

Hence, the degenerate Clifford algebra $Cl_{1,0,1}$ is isomorphic to the following subalgebra of the matrix algebra $\mathbb{R}(2) \oplus \mathbb{R}(2)$.

$$\left\{ \left(\begin{bmatrix} x_0+x_2+x_3 & x_1-x_2-x_3 \\ x_1+x_2+x_3 & x_0-x_2-x_3 \end{bmatrix}, \begin{bmatrix} x_0-x_2+x_3 & x_1-x_2+x_3 \\ x_1+x_2-x_3 & x_0+x_2-x_3 \end{bmatrix} \right) \middle| x_i \in R, i=0,1,2,3 \right\}$$

Example 9 Let $Q(x) = -x_1^2$ be a degenerate quadratic form on $V = \mathbb{R}^2$ and $Cl_{0,1,1}$ its Clifford algebra. The degenerate Clifford algebra $Cl_{0,1,1}$ is embedded into non-degenerate Clifford algebra $Cl_{1,2}$. Moreover, we know that the Clifford algebra $Cl_{1,2}$ isomorphic to the matrix algebra $\mathbb{C}(2)$. The embedding map from $Cl_{0,1,1}$ to $Cl_{1,2}$ is defined on basis elements as follows:

$$\begin{array}{ccl} Cl_{0,1,1} & \to & Cl_{1,2} \cong \mathbb{C}(2) \\ \varepsilon_1 & \mapsto & \varepsilon_1 \\ \theta_1 & \mapsto & e_1 + \varepsilon_2 \end{array}$$

It can be seen that this map is a one to one algebra homomorphism. By composing this homomorphism and the isomorphism $\Psi_{1,2}$ we get the following homomorphism:

$$\begin{array}{ccl} Cl_{0,1,1} & \to & Cl_{1,2} \cong \mathbb{C}(2) \\ \varepsilon_1 & \mapsto & \varepsilon_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ \theta_1 & \mapsto & e_1 + \varepsilon_2 = \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix} \end{array}$$

where $\varepsilon_1\theta_1 = \begin{bmatrix} -1 & i \\ i & 1 \end{bmatrix}$. Any $x \in Cl_{0,1,1}$ can be written as

$$x = x_0 \cdot 1 + x_1\varepsilon_1 + x_2\theta_1 + x_3\varepsilon_1\theta_1 .$$

Then, by writing the values of $1, \varepsilon_1, \theta_1, \varepsilon_1\theta_1$ we have the following identity:

$$\begin{aligned} \mathrm{x} &= x_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + x_1 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + x_2 \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix} + x_3 \begin{bmatrix} -1 & i \\ i & 1 \end{bmatrix} \\ &= \begin{bmatrix} x_0 - x_3 + x_2 i & -x_1 + x_2 + x_3 i \\ x_1 + x_2 + x_3 i & x_0 + x_3 - x_2 i \end{bmatrix} \end{aligned}$$

Hence,

$$Cl_{0,1,1} \cong \left\{ \begin{bmatrix} x_0 - x_3 + x_2 i & -x_1 + x_2 + x_3 i \\ x_1 + x_2 + x_3 i & x_0 + x_3 - x_2 i \end{bmatrix} \middle| x_i \in R, i=0,1,2,3 \right\} \subset \mathbb{C}(2).$$

Example 10 Let $Q(x) = x_1^2 - x_2^2$ be a degenerate quadratic form on $V = \mathbb{R}^3$ and $Cl_{1,1,1}$ its Clifford algebra. In similar way it is possible to be embedded into a suitable matrix algebra. As above examples the degenerate Clifford algebra $Cl_{1,1,1}$ is embedded into non-degenerate Clifford algebra $Cl_{2,2}$. The embedding is defined on the basis elements as follows:

$$\begin{array}{ccc} Cl_{1,1,1} & \to & Cl_{2,2} \cong \mathbb{R}(4) \\ e_1 & \mapsto & e_1 \\ \varepsilon_1 & \mapsto & \varepsilon_1 \\ \theta_1 & \mapsto & e_2 + \varepsilon_2 \end{array}$$

By composing this homomorphism and the isomorphism $\Psi_{2,2}$ we get a homomorphism from $Cl_{1,1,1}$ to the matrix algebra $\mathbb{R}(4)$ as follows:

$$\begin{array}{ccc} Cl_{1,1,1} & \to & Cl_{2,2} \cong \mathbb{R}(4) \\ e_1 & \mapsto & e_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ \varepsilon_1 & \mapsto & \varepsilon_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ \theta_1 & \mapsto & e_2 + \varepsilon_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix} \end{array}$$

We have

$$e_1\varepsilon_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},\ e_1\theta_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix},\ \varepsilon_1\theta_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix},$$

$$e_1\varepsilon_1\theta_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}.$$

Then, any element x in $Cl_{1,1,1}$ can be written in the following way:

$$x = x_0 \cdot 1 + x_1 e_1 + x_2 \varepsilon_1 + x_3 \theta_1 + x_4 e_1\varepsilon_1 + x_5 \varepsilon_1\theta_1 + x_6 e_1\theta_1 + x_7 e_1\varepsilon_1\theta_1$$

We know the values of $e_1, \varepsilon_1, \theta_1, e_1\varepsilon_1, \varepsilon_1\theta_1, e_1\theta_1, e_1\varepsilon_1\theta_1$. If these values are used, then we get

$$
\begin{aligned}
x &= x_0\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}+x_1\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}+x_2\begin{bmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&1&0\end{bmatrix}\\
&= +x_3\begin{bmatrix}0&0&0&0\\0&0&0&0\\2&0&0&0\\0&-2&0&0\end{bmatrix}+x_4\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}+x_5\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&2&0&0\\2&0&0&0\end{bmatrix}\\
&\quad +x_6\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&-2&0&0\\2&0&0&0\end{bmatrix}+x_7\begin{bmatrix}0&0&0&0\\0&0&0&0\\2&0&0&0\\0&2&0&0\end{bmatrix}.
\end{aligned}
$$

Then,

$$
x=\begin{bmatrix}x_0+x_4 & x_1-x_2 & 0 & 0\\ x_1+x_2 & x_0-x_4 & 0 & 0\\ 2x_3+2x_7 & 2x_5-2x_6 & x_0+x_4 & x_1-x_2\\ 2x_5+2x_6 & -2x_3+2x_7 & x_1+x_2 & x_0-x_4\end{bmatrix}.
$$

Hence, the degenerate Clifford algebra $Cl_{1,1,1}$ is isomorphic to the following subalgebra of the matrix algebra $\mathbb{R}(4)$.

$$
\left\{\begin{bmatrix}x_0+x_4 & x_1-x_2 & 0 & 0\\ x_1+x_2 & x_0-x_4 & 0 & 0\\ 2x_3+2x_7 & 2x_5-2x_6 & x_0+x_4 & x_1-x_2\\ 2x_5+2x_6 & -2x_3+2x_7 & x_1+x_2 & x_0-x_4\end{bmatrix}\middle| x_i\in R, i=0,1,\ldots,7\right\}
$$

If we continue in similar way, we have the following imbeddings:

$$
\begin{array}{ll}
Cl_{0,3,1}\to Cl_{1,4}\cong \mathbb{H}(2)\oplus\mathbb{H}(2) & Cl_{1,1,1}\to Cl_{2,2}\cong \mathbb{R}(4)\\
Cl_{0,4,1}\to Cl_{1,5}\cong \mathbb{H}(4) & Cl_{1,2,1}\to Cl_{2,3}\cong \mathbb{C}(4)\\
Cl_{0,1,2}\to Cl_{2,3}\cong \mathbb{C}(4) & Cl_{1,3,1}\to Cl_{2,4}\cong \mathbb{H}(4)\\
Cl_{0,2,2}\to Cl_{2,4}\cong \mathbb{H}(4) & Cl_{1,4,1}\to Cl_{2,5}\cong \mathbb{H}(4)\oplus\mathbb{H}(4)\\
Cl_{0,3,2}\to Cl_{2,5}\cong \mathbb{H}(4)\oplus\mathbb{H}(4) & Cl_{1,0,2}\to Cl_{3,2}\cong \mathbb{R}(4)\oplus\mathbb{R}(4)
\end{array}
$$

We can easily determine the degenerate Clifford algebras $Cl_{p,q,r}$ for the higher values of p,q,r in the similar way.

3 Spinor Representations of Degenerate Clifford Algebras

A representation of degenerate Clifford algebra can be given as follows: Let

$$\rho : Cl_{p+r,q+r} \to End(V)$$

be the spinor representation of $Cl_{p+r,q+r}$ which is widely known, and

$\Psi_{p,q,r} : Cl_{p,q,r} \to Cl_{p+r,q+r}$ be the imbedding defined above, then the composition

$$\rho \circ \Psi_{p,q,r} : Cl_{p,q,r} \to End(V)$$

is an algebra homomorphism, so it is a representation of the degenerate Clifford algebra $Cl_{p,q,r}$ and we call it spinor representation of $Cl_{p,q,r}$.

Now we give some example in lower dimension.

Example 11

$$\begin{array}{ccc} Cl_{0,0,2} & \to & \mathbb{R}(4) \\ \theta_1 & \mapsto & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \\ \theta_2 & \mapsto & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix} \end{array}$$

Example 12

$$\begin{array}{ccc} Cl_{1,0,1} & \to & \mathbb{R}(2) \\ e_1 & \mapsto & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ \theta_1 & \mapsto & \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \end{array}$$

Example 13

$$\begin{array}{ccc} Cl_{0,1,1} & \to & \mathbb{R}(4) \\ \varepsilon_1 & \mapsto & \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\ \theta_1 & \mapsto & \begin{bmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \end{array}$$

Example 14

$$
\begin{array}{rcl}
Cl_{1,1,1} & \to & \mathbb{R}(4) \\
e_1 & \mapsto & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\
\varepsilon_1 & \mapsto & \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\
\theta_1 & \mapsto & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix}
\end{array}
$$

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