

Variational iteration and homotopy perturbation method for solving Lorenz system

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Abstract

In this paper, homotopy perturbation method and variational iteration method are implemented to give approximate and analytical solutions of nonlinear ordinary differential equation systems such as Lorenz system. Homotopy perturbation is compared the variational iteration method for Lorenz system. The variational iteration method is predominant than the other non-linear methods, such as perturbation method. The main property of the VIM method is in its flexibility and ability to solve nonlinear equations accurately and conveniently. In this method, in general Lagrange multipliers are constructed by correction functionals for the systems. Multipliers can be identified by the variational theory. Some plots are presented to compare of the VIM and HPM.

Keywords: *Variational iteration method; Homotopy perturbation method; Lorenz system*

Özet

Bu çalışmada, Lorenz sistemi gibi lineer olmayan adi diferensiyel denklem sistemlerinin yaklaşık analitik çözümlerini elde edebilmek için homotopy perturbation ve varyasyonel iterasyon yöntemleri uygulandı. Homotopy perturbation yöntemi varyasyonel iterasyon yöntemi ile mukayese edildi. Varyasyonel iterasyon yöntemi perturbation yöntemi olarak bilinen diğer non lineer yöntemlerden daha üstündür. VIM yönteminin temel özelliği lineer olmayan denklemleri doğru ve uygun çözebilecek esneklikte olmasıdır. Bu yöntemde genelde Lagrange çarpanları sistemler için düzeltme fonksiyoneli ile elde edildi. Çarpanlar varyasyonel teori ile belirlendi. VIM ve HPM yöntemlerini karşılaştırmak için bir kaç tane grafik sunuldu.

Anahtar kelimeler: *Varyasyonel iterasyon yöntemi, Homotopy perturbation yöntemi; Lorenz sistemi*

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1. Introduction

Dynamics of chaotic dynamical systems as Lorenz system is examined at the study [2]. The components of the basic tree-component model are proportional to the convective velocity, the temperature difference between descending and ascending flows, and the mean convective heat flow is denoted respectively by $x(t)$, $y(t)$ and $z(t)$. These quantities satisfy

$$\begin{cases} \frac{dx}{dt} = s(y - x) \\ \frac{dy}{dt} = rx - y - xz \\ \frac{dz}{dt} = xy - bz \end{cases} \quad (1)$$

with the initial conditions:

$$x(0) = M_1, \quad y(0) = M_2, \quad z(0) = M_3.$$

s, b and the so-called bifurcation parameter r are real constants.. Throughout this paper, we set $s = 10, b = 8/3$ and vary the parameter r . It is well-known that chaos sets in around the critical parameter value $r = 20.5$ [24, 25].

The motivation of this paper is to extend the application of the analytic homotopy-perturbation method (HPM) and variational iteration method [3, 6–14, 19, 21] to solve the Lorenz system (1). The homotopy perturbation method (HPM) was first proposed by Chinese mathematician He [16–22]. The variational iteration method, which was proposed originally by He [13] in 1999, has been proved by many authors to be a powerful mathematical tool for various kinds of nonlinear problems.

2. Variational iteration method

According to the variational iteration method [10], we consider the following differential equation:

$$Lu + Nu = g(x), \quad (2)$$

where L is a linear operator, N is a non-linear operator, and $g(x)$ is an inhomogeneous term. Then, we can construct a correct functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda \{Lu_n(s) + N\tilde{u}_n(s) - g(s)\} ds, \quad (3)$$

where λ is a general Lagrangian multiplier [9–11], which can be identified optimally via variational theory. The second term on the right is called the correction and \tilde{u}_n is considered as a restricted variation, i.e., $\delta\tilde{u}_n = 0$.

3. Homotopy perturbation method

To illustrate the homotopy perturbation method (HPM) for solving non-linear differential equations, He [21, 22] considered the following non-linear differential equation:

$$A(u) = f(r), \quad r \in \Omega \quad (4)$$

subject to the boundary condition

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma \quad (5)$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, Γ is the boundary of the domain Ω and $\frac{\partial}{\partial n}$ denotes differentiation along the normal vector drawn outwards from Ω . The operator A can generally be divided into two parts M and N . Therefore, (3) can be rewritten as follows:

$$M(u) + N(u) = f(r), \quad r \in \Omega \quad (6)$$

He [22,23] constructed a homotopy $v(r, p) : \Omega \times [0, 1] \rightarrow \Re$ which satisfies

$$H(v, p) = (1 - p)[M(v) - M(u_0)] + p[A(v) - f(r)] = 0, \quad (7)$$

which is equivalent to

$$H(v, p) = M(v) - M(u_0) + pM(v_0) + p[N(v) - f(r)] = 0, \quad (8)$$

where $p \in [0, 1]$ is an embedding parameter, and u_0 is an initial approximation of (4). Obviously, we have

$$H(v, 0) = M(v) - M(u_0) = 0, \quad H(v, 1) = A(v) - f(r) = 0. \quad (9)$$

The changing process of p from zero to unity is just that of $H(v, p)$ from $M(v) - M(u_0)$ to $A(v) - f(r)$. In topology, this is called deformation and $M(v) - M(u_0)$ and $A(v) - f(r)$ are called homotopic. According to the homotopy perturbation method, the parameter p is used as a small parameter, and the solution of Eq. (7) can be expressed as a series in p in the form

$$v = v_0 + pv_1 + p^2v_2 + p^3v_3 + \dots \quad (10)$$

When $p \rightarrow 1$, Eq. (7) corresponds to the original one, Eqs. (6) and (10) become the approximate solution of Eq. (6), i.e.,

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + v_3 + \dots \quad (11)$$

The convergence of the series in Eq. (11) is discussed by He in [21, 22].

4. Applications

In this section, we will apply the homotopy perturbation method to nonlinear ordinary differential systems (1).

4.1 Homotopy perturbation method to Lorenz system

According to homotopy perturbation method, we derive a correct functional as follows:

$$\begin{aligned} (1-p)(\dot{v}_1 - \dot{x}_0) + p(\dot{v}_1 + s(v_1 - v_2)) &= 0, \\ (1-p)(\dot{v}_2 - \dot{y}_0) + p(\dot{v}_2 - rv_1 + v_2 + v_1v_3) &= 0, \\ (1-p)(\dot{v}_3 - \dot{z}_0) + p(\dot{v}_3 - v_1v_2 + bv_3) &= 0, \end{aligned} \quad (12)$$

where “dot” denotes differentiation with respect to t , and the initial approximations are as follows:

$$\begin{aligned} v_{1,0}(t) = x_0(t) = x(0) = M_1, \\ v_{2,0}(t) = y_0(t) = y(0) = M_2, \\ v_{3,0}(t) = z_0(t) = z(0) = M_3. \end{aligned} \quad (13)$$

And

$$\begin{aligned} v_1 &= v_{1,0} + pv_{1,1} + p^2v_{1,2} + p^3v_{1,3} + \dots, \\ v_2 &= v_{2,0} + pv_{2,1} + p^2v_{2,2} + p^3v_{2,3} + \dots, \\ v_3 &= v_{3,0} + pv_{3,1} + p^2v_{3,2} + p^3v_{3,3} + \dots, \end{aligned} \quad (14)$$

where $v_{i,j}, i, j = 1, 2, 3, \dots$ are functions yet to be determined. Substituting Eqs.(13) and (14) into Eq. (12) and arranging the coefficients of “p” powers, we have

$$\begin{aligned} (\dot{v}_{1,1} + s(M_1 - M_2))p + (\dot{v}_{1,2} + s(v_{1,1} - v_{2,1}))p^2 + (\dot{v}_{1,3} + s(v_{1,2} - v_{2,2}))p^3 + \dots = 0, \\ (\dot{v}_{2,1} - rM_1 + M_2 + M_1M_3)p + (\dot{v}_{2,2} - rv_{1,1} + v_{2,1} + M_1v_{3,1} + M_3v_{1,1})p^2 \\ + (\dot{v}_{2,3} - rv_{1,2} + v_{2,2} + M_1v_{3,2} + M_3v_{1,2} + v_{1,1}v_{3,1})p^3 + \dots = 0, \\ (\dot{v}_{3,1} - M_1M_2 + bM_3)p + (\dot{v}_{3,2} - M_1v_{2,1} - M_2v_{1,1} + bv_{3,1})p^2 \\ + (\dot{v}_{3,3} - M_1v_{2,2} - M_2v_{1,2} - v_{1,1}v_{2,1} + bv_{3,2})p^3 + \dots = 0, \end{aligned} \quad (15)$$

In order to obtain the unknowns $v_{i,j}(t), i, j = 1, 2, 3$, we must construct and solve the following system which includes nine equations with nine unknowns, considering the initial conditions

$$\begin{aligned}
 v_{i,j}(0) &= 0, i, j = 1, 2, 3, \\
 \dot{v}_{1,1} + s(M_1 - M_2) &= 0, \\
 \dot{v}_{1,2} + s(v_{1,1} - v_{2,1}) &= 0, \\
 \dot{v}_{1,3} + s(v_{1,2} - v_{2,2}) &= 0, \\
 \dot{v}_{2,1} - rM_1 + M_2 + M_1M_3 &= 0, \\
 \dot{v}_{2,2} - rv_{1,1} + v_{2,1} + M_1v_{3,1} + M_3v_{1,1} &= 0, \\
 \dot{v}_{2,3} - rv_{1,2} + v_{2,2} + M_1v_{3,2} + M_3v_{1,2} + v_{1,1}v_{3,1} &= 0, \\
 \dot{v}_{3,1} - M_1M_2 + bM_3 &= 0, \\
 \dot{v}_{3,2} - M_1v_{2,1} - M_2v_{1,1} + bv_{3,1} &= 0, \\
 \dot{v}_{3,3} - M_1v_{2,2} - M_2v_{1,2} - v_{1,1}v_{2,1} + bv_{3,2} &= 0.
 \end{aligned} \tag{16}$$

From Eq. (11), if the three terms approximations are sufficient, we will obtain:

$$\begin{aligned}
 x(t) &= \lim_{p \rightarrow 1} v_1(t) = \sum_{k=0}^3 v_{1,k}(t), \\
 y(t) &= \lim_{p \rightarrow 1} v_2(t) = \sum_{k=0}^3 v_{2,k}(t), \\
 z(t) &= \lim_{p \rightarrow 1} v_3(t) = \sum_{k=0}^3 v_{3,k}(t),
 \end{aligned} \tag{17}$$

therefore

$$\begin{aligned}
 x(t) &= M_1 + s(M_2 - M_1)t \\
 &+ \frac{1}{2} [rM_1 - M_2 - M_1M_3 - sM_2 + sM_1] t^2 \\
 &+ \frac{1}{6} \left[rs(M_2 - M_1) - (rM_1 - M_2 - M_1M_3) - M_1(M_1M_2 - bM_3) \right. \\
 &\quad \left. - sM_3(M_2 - M_1) - s(rM_1 - M_2 - M_1M_3 - sM_2 + sM_1) \right] t^3 \\
 y(t) &= M_2 + (rM_1 - M_2 - M_1M_3)t \\
 &+ \frac{1}{2} [rsM_2 - rsM_1 - rM_1 + M_2 + M_1M_3 - M_1^2M_2 + bM_1M_3 - sM_2M_3 + sM_1M_3] t^2 \\
 &+ \frac{1}{6} \left[rs(rM_1 - M_2 - M_1M_3 - sM_2 + sM_1) - rs(M_2 - M_1) + (rM_1 - M_2 - M_1M_3) \right. \\
 &\quad \left. + M_1(M_1M_2 - bM_3) + sM_3(M_2 - M_1) - (rM_1 - M_2 - M_1M_3)M_1^2 \right. \\
 &\quad \left. - M_1M_2(M_2 - M_1)s - M_3(rM_1 - M_2 - M_1M_3 - sM_2 + sM_1)s \right. \\
 &\quad \left. - 2s(M_1M_2 - bM_3)(M_2 - M_1) \right] t^3 \\
 z(t) &= M_3 + (M_1M_2 - bM_3)t \\
 &+ \frac{1}{2} [rM_1^2 - M_1M_2 - M_1^2M_3 - bM_1M_2 - sM_1M_2 + sM_2^2 + b^2M_3] t^2 \\
 &+ \frac{1}{6} \left[rsM_1(M_2 - M_1) - (rM_1 - M_2 - M_1M_3) - M_1(M_1M_2 - bM_3) \right. \\
 &\quad \left. - M_3(M_2 - M_1)s + M_2(rM_1 - M_2 - M_1M_3 - sM_2 + sM_1)s \right. \\
 &\quad \left. + 2s(M_2 - M_1)(rM_1 - M_2 - M_1M_3) - bM_1(rM_1 - M_2 - M_1M_3) \right. \\
 &\quad \left. + sM_2(M_2 - M_1) - bM_1(M_1M_2 - bM_3) \right] t^3,
 \end{aligned} \tag{18}$$

Here

$M_1 = 0, M_2 = 1$ and $M_3 = 0$ for the three-component model.

A few first approximations for $x(t)$, $y(t)$ and $z(t)$ are calculated and presented below:

Three terms approximations:

$$\begin{aligned} x(t) &= 10t - 55.t^2 + 526.6666667t^3, \\ y(t) &= 1 - t + 103t^2 - 410.1666667t^3, \\ z(t) &= 5t^2 - 26.11111111t^3, \end{aligned} \quad (19)$$

Four terms approximations:

$$\begin{aligned} x(t) &= 10t - 55.t^2 + 526.6666667t^3 - 2342.083334t^4, \\ y(t) &= 1 - t + 103t^2 - 410.1666667t^3 + 2789.208335t^4, \\ z(t) &= 5t^2 - 26.11111111t^3 + 420.3240740t^4, \end{aligned} \quad (20)$$

Five terms approximations:

$$\begin{aligned} x(t) &= 10t - 55.t^2 + 526.6666667t^3 - 2342.083334t^4 + 10262.58334t^5, \\ y(t) &= 1 - t + 103t^2 - 410.1666667t^3 + 2789.208335t^4 - 10053.16112t^5, \\ z(t) &= 5t^2 - 26.11111111t^3 + 420.3240740t^4 - 2751.256172t^5, \end{aligned} \quad (21)$$

Six terms approximations:

$$\begin{aligned} x(t) &= 10t - 55.t^2 + 526.6666667t^3 - 2342.083334t^4 + 10262.58334t^5 - 33859.5741t^6, \\ y(t) &= 1 - t + 103t^2 - 410.1666667t^3 + 2789.208335t^4 - 10053.16112t^5 + 35360.57242t^6, \\ z(t) &= 5t^2 - 26.11111111t^3 + 420.3240740t^4 - 2751.256172t^5 + 20773.21108t^6, \end{aligned} \quad (22)$$

The results obtained by homotopy perturbation method with three, four, five and six terms approximations for $x(t)$, $y(t)$ and $z(t)$ are plotted in Figure 1.

In this next section, we will apply the variational iteration method to nonlinear ordinary differential systems (1).

4.2 Variational iteration method to Lorenz system

According to the variational iteration method, we derive a correct functional as follows:

$$\begin{aligned} x_{n+1}(t) &= x_n(t) + \int_0^t \lambda_1 \{x'_n(\xi) - s(\tilde{y}_n(\xi) - \tilde{x}_n(\xi))\} d\xi, \\ y_{n+1}(t) &= y_n(t) + \int_0^t \lambda_2 \{y'_n(\xi) - r\tilde{x}_n(\xi) + \tilde{y}_n(\xi) + \tilde{x}_n(\xi)\tilde{z}_n(\xi)\} d\xi, \\ z_{n+1}(t) &= z_n(t) + \int_0^t \lambda_3 \{z'_n(\xi) - \tilde{x}_n(\xi)\tilde{y}_n(\xi) + b\tilde{z}_n(\xi)\} d\xi, \end{aligned} \quad (23)$$

where λ_1 , λ_2 and λ_3 are general Lagrange multipliers,

$\tilde{x}_n(\xi)$, $\tilde{x}_n(\xi)\tilde{y}_n(\xi)$, $\tilde{x}_n(\xi)\tilde{z}_n(\xi)$, $\tilde{y}_n(\xi)$ and $\tilde{z}_n(\xi)$ denote restricted variations, i.e.

$$\delta\tilde{x}_n(\xi) = \delta\tilde{x}_n(\xi)\tilde{y}_n(\xi) = \delta\tilde{x}_n(\xi)\tilde{z}_n(\xi) = \delta\tilde{y}_n(\xi) = \delta\tilde{z}_n(\xi) = 0$$

Making the above correction functionals stationary, we can obtain following stationary conditions:

$$\begin{aligned} \lambda_1'(\xi) &= 0, \\ 1 + \lambda_1(\xi)\Big|_{\xi=t} &= 0, \\ \lambda_2'(\xi) &= 0, \\ 1 + \lambda_2(\xi)\Big|_{\xi=t} &= 0, \\ \lambda_3'(\xi) &= 0, \\ 1 + \lambda_3(\xi)\Big|_{\xi=t} &= 0, \end{aligned} \tag{24}$$

The Lagrange multipliers, therefore, can be identified as

$$\lambda_1 = \lambda_2 = \lambda_3 = -1. \tag{25}$$

Substituting Eq. (25) into the correction functional Eq. (23) results in the following iteration formula:

$$\begin{aligned} x_{n+1}(t) &= x_n(t) - \int_0^t \{x_n'(\xi) - s(y_n(\xi) - x_n(\xi))\} d\xi, \\ y_{n+1}(t) &= y_n(t) - \int_0^t \{y_n'(\xi) - rx_n(\xi) + y_n(\xi) + x_n(\xi)z_n(\xi)\} d\xi, \\ z_{n+1}(t) &= z_n(t) - \int_0^t \{z_n'(\xi) - x_n(\xi)y_n(\xi) + bz_n(\xi)\} d\xi, \end{aligned} \tag{26}$$

We start with initial approximations $x_0(t) = M_1$, $y_0(t) = M_2$ and $z_0(t) = M_3$. By the above iteration formula, we can obtain a few first terms being calculated:

$$\begin{aligned} x_1(t) &= M_1 + s(M_2 - M_1)t, \\ y_1(t) &= M_2 + (rM_1 - M_2 - M_1M_3)t, \\ z_1(t) &= M_3 + (M_1M_2 - bM_3)t, \end{aligned} \tag{27}$$

$$\begin{aligned}
 x_2(t) &= M_1 + s(M_2 - M_1)t \\
 &\quad + \left[rsM_1 - sM_2 - sM_1M_3 - s^2M_2 + s^2M_1 \right] t^2 \\
 y_2(t) &= M_2 + (rM_1 - M_2 - M_1M_3)t \\
 &\quad + \left[rsM_2 - rsM_1 - rM_1 + M_2 + M_1M_3 - M_1^2M_2 + bM_1M_3 - sM_2M_3 + sM_1M_3 \right] t^2 \\
 &\quad + \left[-sM_2^2M_1 + bM_2M_3 + sM_1^2M_2 - sbM_1M_3 \right] t^3 \\
 z_2(t) &= M_3 + (M_1M_2 - bM_3)t \\
 &\quad + \left[rM_1^2 - M_1M_2 - M_1^2M_3 - bM_1M_2 - sM_1M_2 + sM_2^2 + b^2M_3 \right] t^2 \\
 &\quad + \left[srM_2M_1 - sM_2^2 - sM_2M_1M_3 - srM_1^2 + sM_1M_2 + sM_1^2M_3 \right] t^3,
 \end{aligned} \tag{28}$$

Continuing in this manner, we can find the rest of components. A five terms approximation to the solutions are considered

$$\begin{aligned}
 x(t) &\approx x_4, \\
 y(t) &\approx y_4, \\
 z(t) &\approx z_4.
 \end{aligned} \tag{29}$$

This was done with the standard parameter values given above and initial values $M_1 = 0$, $M_2 = 1$ and $M_3 = 0$ for the three-component model.

A few first approximations for $x(t)$, $y(t)$ and $z(t)$ are calculated and presented below:

$$\begin{aligned}
 x_1(t) &= 10t, \\
 y_1(t) &= 1 - t, \\
 z_1(t) &= 0,
 \end{aligned} \tag{30}$$

$$\begin{aligned}
 x_2(t) &= 10t - 110t^2, \\
 y_2(t) &= 1 - 1t + 206t^2, \\
 z_2(t) &= 10t^2 - 10t^3,
 \end{aligned} \tag{31}$$

$$\begin{aligned}
 x_3(t) &= 10t + 3160t^3, \\
 y_3(t) &= 1 - 1t - 2461t^3 - 100t^4 + 1200t^5 - 1100t^6, \\
 z_3(t) &= -126.6666667t^3 + 2196.666667t^4 - 22660t^5,
 \end{aligned} \tag{32}$$

$$\begin{aligned}
 x_4(t) &= 10t - 6320t^3 - 110t^2 - 56210t^4 - 1000t^5 + 12000t^6 - 11000t^7, \\
 y_4(t) &= 1 - t + 4922t^3 + 67541t^4 - 3433.333333t^5 - 17666.66667t^6 + 206t^2 \\
 &\quad + 627966.6668t^7 - 6941466.668t^8 + 71605600t^9, \\
 z_4(t) &= 243.3333334t^3 - 3092.222223t^4 + 57012.22220t^5 + 10t^2 \\
 &\quad + 59426.66667t^6 - 7764760t^7 - 327000t^8 + 3792000t^9 - 3476000t^{10},
 \end{aligned} \tag{33}$$

$$\begin{aligned}
 x_5(t) &= 10t + 15800t^3 + 281050t^4 + 1241510t^5 - 84333.33333t^6 \\
 &\quad - 230666.6667t^7 + 6389666.668t^8 - 69414666.68t^9 \\
 &\quad + 716056000t^{10}, \\
 y_5(t) &= 1. - .2730823200E + 12 * t^{15} - .457100047E + 11 * t^{11} - .4387306540E + 12 * t^{12} \\
 &\quad - 2648335556. * t^{13} - .5257700000E + 11 * t^{16} + .8342400000E + 11 * t^{17} \\
 &\quad + .2846838133E + 12 * t^{14} - .3823600000E + 11 * t^{18} - t - 12305t^3 - 337205t^4 \\
 &\quad - 1207445.999t^5 + 192155.5555t^6 - 2314433.333t^7 \\
 &\quad + 47558800t^8 - 295092033.5t^9 + 2651872121t^{10}, \\
 z_5(t) &= -.1618108334E + 12 * t^{15} - 727036000.5t^{11} - 71037332.39t^{12} \\
 &\quad - .6316975059E + 11 * t^{13} + .9356233333E + 12 * t^{16} - .7876616000E + 12 * t^{17} \\
 &\quad - .4010279376E + 13 * t^{14} - 633.3333335t^3 + 4477.777777t^4 \\
 &\quad - 243132.9629t^5 - 1561885.926t^6 - 3707054.443t^7 \\
 &\quad - 680557713.3t^8 - 3798460943t^9 + 176579000.0t^{10},
 \end{aligned}
 \tag{34}$$

These results obtained by HPM, three, four ,five and six terms approximations for $x(t)$, $y(t)$ and $z(t)$ are calculated and presented follow.

These results are plotted in Figure 1.

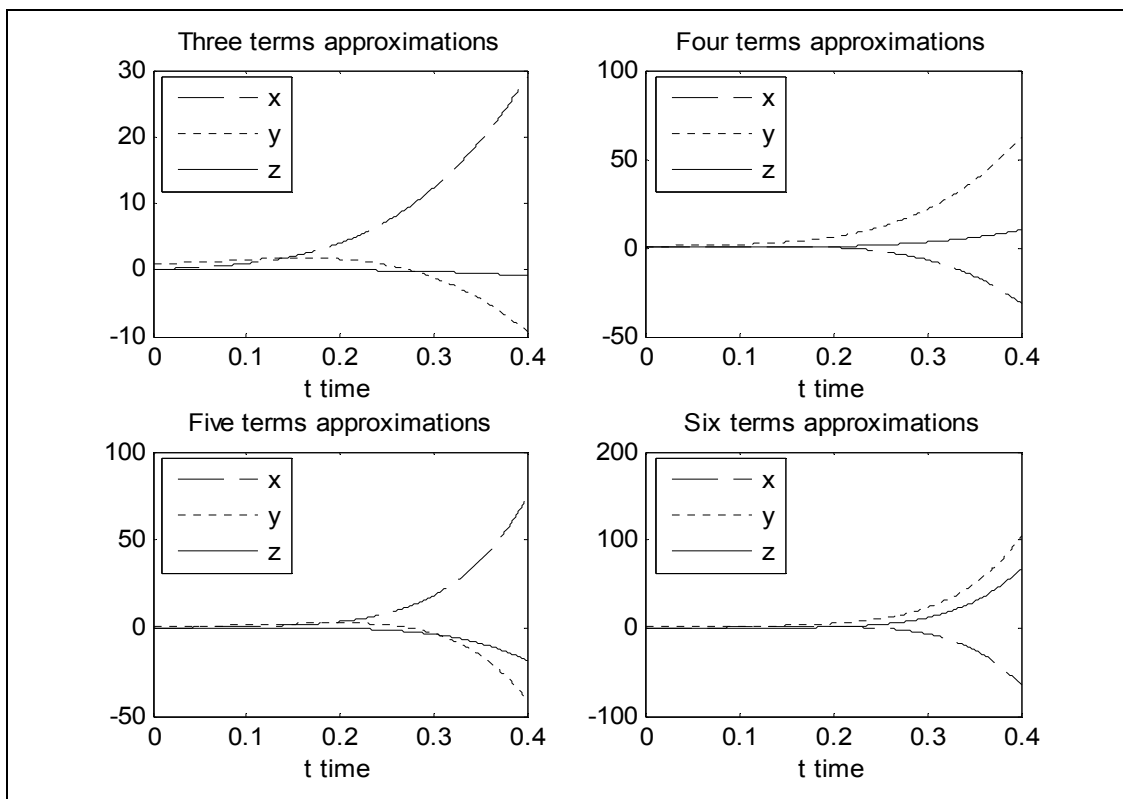


Figure. 1. Plots of three, four five and six terms HPM approximations for Lorenz system
The HPM was tested by comparing the results with the results of the VIM.

These results are plotted in Figure 2.

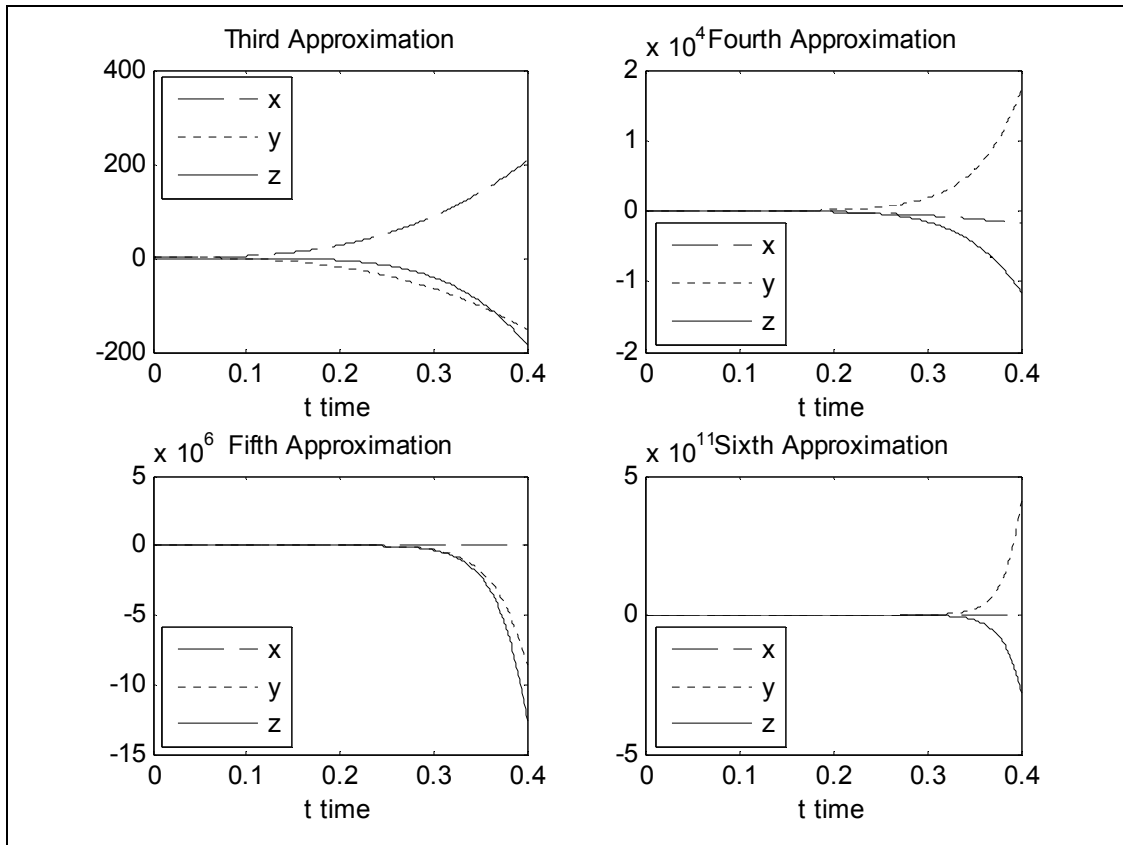


Figure. 2. Plots of third, fourth, fifth and sixth approximations obtained by VIM for the Lorenz system

5. Conclusions

In this paper, HPM and VIM was used for finding the solutions of nonlinear ordinary differential equation systems such as Lorenz system. VIM is predominant than the other non-linear methods, such as perturbation method. We apply He's HPM to calculate certain integrals. It is easy and very beneficial tool for calculating certain difficult integrals or in deriving new integration formula. An interesting state about VIM is that with the fewest number of iterations or even in some cases, once, it can converge to correct results. Some plots are presented to compare of the VIM and HPM

The computations associated with the examples in this paper were performed using Maple 7 and Matlab 7

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